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MULTIVARIATE POLYNOMIAL SPLINE SPACES

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ABSTRACT. In this paper we construct bases of certain spaces of multivariate polynomial splines defined on rectangular partitions. These bases are presented by polynomials, truncated power functions and products of these functions. This setting provides a natural generalization of the two-dimensional polynomial spline, proposed by Chui and Wang [**3**], to p variables.

1. Introduction. A standard way of approximating the cause-and-effect relationship is using a single model over the entire range of variables, for example, the models for linear or polynomial functions. In practice, however, it might be more realistic to partition the range of variables into disjoint regions, and to approximate the relationship by a sequence of submodels which are smoothly connected, in some sense, at the boundaries of the neighboring regions. Polynomial spline functions are useful for this purpose.

Polynomial spline functions are generally defined as piecewise polynomials of degree k, whose partial derivatives satisfy certain smoothness conditions. Theoretical as well as applied research on univariate polynomial spline spaces has widely developed in the last two decades, see, for example, [**5**] and [**7**]. For bivariate polynomial splines, progress has also been made in the study of their bases and dimensions, see, for example, [**2, 3, 4, 6**] and a review article by [**8**]. For general multivariate polynomial splines, Alfeld, Schumaker and Sirvent [**1**] have studied the dimension and existence of local bases in triangulation partition. However, only the basis with zero smoothness-degree has been found. Therefore, development of the theory of multivariate polynomial splines, analogous to the well-known theory of univariate polynomial splines, has still remained to be achieved, see also [**5**, p. 362] and Schumaker [**8**, p. 195] for this point. Besides the need for its theoretical

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study, the multivariate polynomial spline also provides a tool for analyzing scattered data of arbitrary dimension p, from the aspect of application.

In this paper we introduce multivariate polynomial spline spaces. In order to derive explicit dimensions and bases, we study only the splines with multivariate rectangular partition. The smoothness conditions in these spline spaces, relatively more flexible than the purely homogeneous ones used by Chui and Wang [**4**], are classified into three types, viz., homogeneous, semi-homogeneous and non-homogeneous. However, all these types of smoothness conditions satisfy a certain monotonicity property, since the multivariate polynomial splines without the monotonicity property may not be able to be represented by the convenient tools of "+" and "−" functions, that are used in this paper. Section 2, Preliminaries, introduces the notations and the general form of the space of multivariate polynomial splines. Section 3 discusses two non-homogeneous multivariate polynomial spline spaces and derives their corresponding bases. Section 4 presents a homogeneous and several semi-homogeneous multivariate polynomial spline spaces.

2. Preliminaries. As noted by Chui and Schumaker [**2**], the study of multivariate polynomial spline space for general partitions is extremely difficult, especially for providing explicit bases, see also [**3**]. We introduce the general form of the multivariate polynomial spline space for rectangular partition in this section. Although the application of spline defined on a special kind of partition may be limited, the polynomial spline defined on rectangular partition for scattered data in a multivariate plane region seems to be reasonable. Consider p meshes $\{\delta^i_{t_i} : t_i = 0, 1, \dots, a_i + 1\}$, with positive integer a_i, $i = 1, \dots, p$, the elements of which are termed as knots. These meshes define a p-dimensional rectangular grid in the space of variables $x_1, \dots, x_p$, consisting of $(a_1 + 1) \cdots (a_p + 1)$ multivariate rectangles as the region set:

$$\{(x_1, \dots, x_p) : \delta^i_{t_i} < x_i \le \delta^i_{t_i+1}, i = 1, \dots, p\},$$
$$t_i = 0, 1, \dots, a_i, i = 1, \dots, p.$$

Here the region $T = \{(x_1, \dots, x_p) : \delta^i_0 < x_i < \delta^i_{a_i+1}, i = 1, \dots, p\}$ is a bounded domain of spline functions. Let P^k denote the collection of all real polynomials of degree k; that is, each $p \in P^k$ has the representation:

$$p(x_1, \dots, x_p) = \sum_{m_1+\cdots+m_p=0}^{k} \beta_{m_1\dots m_p} \prod_{i=1}^{p} x_i^{m_i}.$$

A piecewise multivariate polynomial of degree k has the form

$$\sum_{t_p=0}^{a_p} \cdots \sum_{t_1=0}^{a_1} p^{t_1\dots t_p}(x_1, \dots, x_p) I(\delta^i_{t_i} < x_i < \delta^i_{t_i+1}, i = 1, \dots, p)$$

where $p^{t_1\dots t_p}$ are in P^k and I is the characteristic function. Let S^k denote the collection of all piecewise multivariate polynomials of degree k. The dimension of S^k is $\binom{p+k}{p} \prod_{i=1}^{p}(a_i + 1)$.

The multivariate polynomial spline is then a member in S^k, satisfying some continuity conditions of its partial derivatives and retaining its segmented nature. However, the smoothness conditions need to be forced on a piecewise multivariate polynomial only on the boundaries of multivariate rectangles.

For fixed multi-index $(t_1 \cdots t_i \cdots t_p)$, $i = 1, \dots, p$, denoted by

$$\begin{aligned} l(t_1 \cdots t_i^- \cdots t_p) = (\delta^1_{t_1}, \delta^1_{t_1+1}] \times \cdots \times (\delta^{i-1}_{t_i-1}, \delta^{i-1}_{t_{i-1}+1}] \\ \times \delta^i_{t_i} \times (\delta^{i+1}_{t_{i+1}}, \delta^{i+1}_{t_{i+1}+1}] \times \cdots \times (\delta^p_{t_p}, \delta^p_{t_p+1}], \end{aligned} \tag{2.1}$$

$l(t_1 \cdots t_i^- \cdots t_p)$ is the boundary separating the domains of polynomials $p^{t_1\cdots t_i \cdots t_p}$ and $p^{t_1\dots t_i+1\dots t_p}$. With multivariate rectangle partition, a boundary in (2.1) is the smallest unit for which the partial derivatives of the piecewise multivariate polynomial are required to satisfy the smoothness conditions. Thus, a multivariate polynomial spline is a piecewise multivariate polynomial to which partial derivatives of some order of neighboring multivariate polynomials are smoothly connected.

Let $D = \{(t_1, \dots, t_i^-, \dots, t_p) : i = 1, \dots, p,\ t_i^- = 1, \dots, a_i,\ t_j = 0, 1, \dots, a_j, \text{for } j \neq i\}$ be the index set. Then the collection of all the boundaries is

$$\{l(t_1 \cdots t_i^- \cdots t_p) : (t_1 \cdots t_i^- \cdots t_p) \in D\}.$$

The smoothness condition is then to force the regression function to be smooth on each boundary in some way. Consider a smoothness set as

$$J=\{\gamma(t_1\cdots t_i^-\cdots t_p):\gamma(t_1\cdots t_i^-\cdots t_p)=0,1,\dots,k-1,\\ (t_1\cdots t_i^-\cdots t_p)\in D\}.$$

Let us define

$$S^k(J)=\{f\in S^k: f\in C^{\gamma(t_1\cdots t_i^-\cdots t_p)}\\ \text{on } l(t_1\cdots t_i^-\cdots t_p),\ \text{for } (t_1\cdots t_i^-\cdots t_p)\in D\}$$

to be the space of multivariate polynomial splines of order k and smoothness J. If $f\in S^k(J)$, then for each $(t_1\cdots t_i^-\cdots t_p)$, function f has continuous partial derivatives of order less than or equal to $\gamma(t_1\cdots t_i^-\cdots t_p)$ on the boundary $l(t_1\cdots t_i^-\cdots t_p)$.

The aim of this paper is thus to generalize the rectangular spline spaces, derived by Chui and Wang [**3, 4**] from the bivariate case to any finite dimensional case and from the uniform smoothness condition to some general smoothness conditions. For any smoothness set, J, it is not always possible to provide an explicit representation to the space $S^k(J)$. We therefore restrict to some types of J such that the corresponding spline spaces can be explicitly represented by either a "+" or "− function.

3. Multivariate polynomial spline spaces with monotone smoothness conditions. Two multivariate polynomial spline spaces with monotone smoothness conditions are derived in this section. For simplicity of introducing the smoothness sets, we define a partial ordering by writing $(t_1\cdots t_p)\le(s_1\cdots s_p)$ whenever $t_i\le s_i$, $i=1,\dots,p$. Also, through the end of this paper, we simplify the vector notation $(0,\dots,0,t_1,0,\dots,t_2,0,\dots,t_b,0,\dots,0)$ as $(t_1,t_2,\dots,t_b)$.

Definition 3.1. We re-denote J as J_d and consider it as a nonincreasing smoothness set if, for each i, $1\le i\le p$, $t_i=s_i$ and $(t_1\cdots t_p)\le(s_1\cdots s_p)$ in D, $\gamma(t_1\cdots t_i^-\cdots t_p)\ge\gamma(s_1\cdots s_i^-\cdots s_p)$. We also re-denote J as J_i and consider it as a nondecreasing smoothness set if, for each i, $1\le i\le p$, $t_i=s_i$ and $(t_1\cdots t_p)\le(s_1\cdots s_p)$ in D, $\gamma(t_1\cdots t_i^-\cdots t_p)\le\gamma(s_1\cdots s_i^-\cdots s_p)$.

There are some choices for choosing the monotone smoothness set for application. We introduce here the multivariate spline spaces for some of these choices.

We will find bases of the spaces $S^k(J_d)$ and $S^k(J_i)$ through the bases of S^k. For this purpose, we first show that any piecewise polynomial in S^k can be decomposed into linear combination of either right-hand or left-hand piecewise polynomials. However, basis representation for spline space relies on the types of smoothness set J.

Lemma 3.2. *The following three spaces are the same.*

(a) *The space of piecewise multivariate polynomials of order k, i.e., the space S^k;*

(b) *The space spanned by the piecewise multivariate polynomials*

$$\begin{array}{c} p \\ p^{t_i} I(x_i > \delta^i_{t_i}) \\ p^{t_{i_1} t_{i_2}} I(x_{i_1} > \delta^{i_1}_{t_{i_1}}, x_{i_2} > \delta^{i_2}_{i_2}) \\ \vdots \\ p^{t_1 \cdots t_p} I(x_1 > \delta^1_{t_1}, \dots, x_p > \delta^p_{t_p}) \end{array} \tag{3.1}$$

where all $p^{\cdots}$ are multivariate polynomials in P^k and t_i ranges from 1 *to a_i, $i = 1, \dots, p$;*

(c) *The space spanned by the piecewise multivariate polynomials in* (3.1)*, by replacing all indices "$>$" in indicator function $I(\,)$ by "$<$."*

Proof. We prove only part (b), the proof for part (c) being similar. Let $n_{pq} = \min\{p, q\}$. For $0 \le q \le \sum_{i=1}^p a_i - 1$, denote a piecewise polynomial $f_q = f_q^1 + f_q^2$, where

$$\begin{aligned} f_q^1 = p_0 &+ \sum_{i=1}^{p} \sum_{t_i=1}^{\min\{q, a_i\}} p^{t_i}_{t_i} I(x_i > \delta^i_{t_i}) \\ &+ \sum_{1 \le i_1 < i_2 \le p} \sum_{\substack{t_{i_1} + t_{i_2} = 2 \\ 1 \le t_{i_j} \le a_{i_j}}}^{q} p^{t_{i_1}, t_{i_2}}_{t_{i_1} + t_{i_2}} I(x_{i_1} > \delta^{i_1}_{t_{i_1}}, x_{i_2} > \delta^{i_2}_{t_{i_2}}) \\ &\;\vdots \end{aligned}$$

$$+\sum_{1\le i_1<\ldots<i_{n_{pq}}\le p}\ \sum_{\substack{t_{i_1}+\cdots+t_{i_{n_{pq}}}=q\\ 1\le t_{i_j}\le a_{i_j}}} p_{t_{i_1}+\cdots+t_{i_{n_{pq}}}}^{t_{i_1},t_{i_2},\ldots,t_{i_{n_{pq}}}}$$

(3.2)

$$I(x_{i_j}>\delta^{i_j}_{t_{i_j}}, j=1,\ldots,n_{pq})$$

and

$$f_q^2=\sum_{t_1+\cdots+t_p=q+1}^{a_1+\cdots+a_p} p_q^{t_1\cdots t_p} I(\delta^i_{t_i}<x_i\le\delta^i_{t_i+1}, i=1,\ldots,p),$$

where $p_q^{\cdots}\in P^k$. We further let

$$f_q^{20}=\sum_{t_1+\cdots+t_p=q+1} p_q^{t_1\cdots t_p} I(\delta^i_{t_i}<x_i\le\delta^i_{t_i+1}, i=1,\ldots,p).$$

Consider the decomposition $f_q^{20}=B_q+(f_q^{20}-B_q)$, where

$$\begin{aligned}B_q=&\sum_{i=1}^p p_q^{q+1} I(x_i>\delta^i_{q+1}q+1\le a_i)\\ &+\sum_{1\le i_1<i_2\le p}\ \sum_{\substack{t_{i_1}+t_{i_2}=q+1\\ 1\le t_{i_j}\le a_{i_j}}} p_q^{t_{i_1}t_{i_2}} I(x_{i_j}>\delta^{i_j}_{t_{i_j}}, j=1,2)\\ &\qquad\vdots\\ &+\sum_{1\le i_1<\cdots<i_{n_{pq}}\le p}\\ &\cdot\sum_{\substack{t_{i_1}+\cdots+t_{i_{n_{pq}}}=q+1\\ 1\le t_{i_j}\le a_{i_j}}} p_q^{t_{i_1}\cdots t_{i_{n_{pq}}}} I(x_{i_j}>\delta^{i_j}_{t_{i_j}}, i=1,\ldots,n_{pq}).\end{aligned}$$

We will show that, for each $f\in S^k$ and $0\le q\le\sum_{i=1}^p a_i-1$, there exists a piecewise polynomial f_q of (3.2) such that $f=f_q$. Let $q=0$. Obviously,

$$f=p+\sum_{t_1+\cdots+t_p=1}^{a_1+\cdots+a_p} p_1^{t_1\cdots t_p} I(\delta^i_{t_i}<x_i\le\delta^i_{t_i+1}, i=1,\ldots,p)$$

with $p_1^{t_1\cdots t_p} = p^{t_1\cdots t_p} - p$. By denoting $p_0 = p$, $f = f_1$. Suppose that $f = f_q (= f_q^1 + f_q^2)$. It can be checked that $f_q^{20} - B_q$ and $f_q^2 - f_q^{20}$ are all piecewise multivariate polynomials defined on the region

$$\bigcup_{t_1+\cdots+t_p=q+2}^{a_1+\cdots+a_p} \{(x_1,\ldots,x_p) : \delta_{t_i}^i < x_i \le \delta_{t_i+1}^i, i=1,\ldots,p\}.$$

We now define polynomials $p_{q+1}^{t_1\cdots t_p}$. Let us denote the polynomial $p_q^{t_1\cdots t_p}$ by $p_{q+1}^{t_1\cdots t_p}$ for which $p_q^{t_1\cdots t_p}$ is the polynomial of B_q defined on region $\{(x_1,\ldots,x_p) : \delta_{t_i}^i < x_i, i = 1,\ldots,p\}$. For $(t_1,\ldots,t_p)$ such that $\sum_{i=1}^p t_i \ge q+2$, let $p_{q+1}^{t_1\cdots t_p}$ be such that $f_{q+1}^2 = f_q^2 - B_q$. We can also see that $f_q^1 + B_{q+1} = f_{q+1}^1$. We then have

$$\begin{aligned} f &= f_q^1 + B_q + f_q^2 - B_q \\ &= f_{q+1}^1 + f_{q+1}^2 \\ &= f_{q+1}. \end{aligned}$$

The proof is then done by setting $q = \sum_{i=1}^p a_i - 1$. $\square$

We define a_+ and a_- to be the values $\max\{a,0\}$ and $\min\{a,0\}$, respectively. The formulation of a piecewise multivariate polynomial by those right-hand and left-hand truncated polynomials indicates that some sort of multivariate polynomial splines may be represented explicitly by them. In the rest of this paper, when we say replace "+" function by "−" function, it would mean that we replace a_+ by a_-.

Theorem 3.3. (a) *The following functions form a basis for the space S^k.*

$$\begin{gathered} \prod_{i=1}^p x_i^{m_i}, \\ (x_i - \delta_{t_i}^i)_+^{m_i} \prod_{j\ne i} x_j^{m_j}, \\ (x_{i_1} - \delta_{t_{i_1}}^{i_1})_+^{m_{i_1}} (x_{i_2} - \delta_{t_{i_2}}^{i_2})_+^{m_{i_2}} \prod_{j\ne i_1,i_2} x_j^{m_j}, \\ \vdots \\ \prod_{i=1}^p (x_i - \delta_{t_i}^i)_+^{m_i}, \end{gathered} \tag{3.3}$$

where $0 \le \sum_{i=1}^p m_i \le k$ and t_i ranges from 1 to a_i for $i = 1,\ldots,p$.

(b) *The space of functions of* (3.3), *by replacing all "+" functions by "−" functions, is also a basis of space* S^k.

Proof. We prove only part (a), the proof of (b) being similar. For any finite set $\{y_1, \ldots, y_p\}$, let us denote by $\prod y_i = y_1 \cdots y_p$ and $\bar{y} = y_1, \ldots, y_p$. Define a set of polynomial functions and truncated polynomial functions ψ by

(3.4)
$$\psi_{\pi t_i, \pi m_i}(\bar{x}) = \left(\prod_{i=1}^{p} m_i\right)^{-1} \prod_{i=1}^{p} [(x_i - \delta_{t_i}^{i})_+^{m_i} I(t_i > 0) + x_i^{m_i} I(t_i = 0)],$$
$$0 \le \sum_{i=1}^{p} m_i \le k, \quad t_i = 0, 1, \ldots, a_i, i = 1, \ldots, p.$$

Equation (3.4) also has the number of elements $\binom{k+p}{p} \prod_{i=1}^{p} (a_i + 1)$, and each element can be formulated as a linear combination of functions in (3.3). So the proof of this theorem is obtained if it can be shown that the functions ψ in (3.4) are linearly independent. For $h_i = -, +$, $i = 1, \ldots, p$, let the partial derivative be denoted as

$$\psi^{\pi d_i}(\delta_{u_1}^{h_1}, \ldots, \delta_{u_p}^{h_p}) = \frac{\partial^{\sum_{i=1}^{p} d_i}}{\partial x_1^{d_1} \cdots \partial x_p^{d_p}} \psi_{\pi t_i, \pi m_i}(\bar{x})\big|_{x_i = \delta_{u_i}^{h_i}}, \quad i = 1, \ldots, p.$$

We also define linear function $\lambda_{\pi u_i, \pi d_i}$ on ψ as

(3.5)
$$\lambda_{\pi u_i, \pi d_i}(\psi) = \sum_{h_p = -,+} \cdots \sum_{h_1 = -,+} (-1)^{\sum_{i=1}^{p} \alpha(h_i)} \psi^{\pi d_i}(\delta_{u_1}^{h_1}, \ldots, \delta_{u_p}^{h_p}),$$

where α is the binary function defined by $\alpha(h) = 1$ if $h = +$ and 0 if $h = -$. If we let

$$h_j(x_j) = \sum_{\substack{h_i = -,+ \\ i \ne j}} (-1)^{\sum_{i \ne j} \alpha(h_i)} \psi^{\pi_{i \ne j} d_i}(\delta_{u_1}^{h_1}, \ldots, x_j, \ldots, \delta_{u_p}^{h_p})$$

to have continuous m_jth derivative at δ_{u_j}, then

$$\lambda_{\pi u_i, \pi d_i}(\psi) = -\frac{\partial^{m_j}}{\partial x_j^{m_j}} h_j(x_j)\Big|_{x_j = \delta_{u_j}^{+}} + \frac{\partial^{m_j}}{\partial x_j^{m_j}} h_j(x_j)\Big|_{x_j = \delta_{u_j}^{-}} = 0.$$

By careful inspection, it can be seen that

$$\lambda_{\pi u_i, \pi d_i}(\psi) = \begin{cases} 1 & \text{if } u_i = t_i,\ d_i = m_i,\ i = 1, \dots, p, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Any zero linear combination of (3.4) will have zero coefficients by (3.6). This shows that the set $\{\psi_{\pi s_i, \pi c_i}(\bar{x})\}$ is linearly independent and thus the set of functions in (3.3) is a basis of S^k. $\square$

The number of elements in the set of (3.3) is $\binom{p+k}{p} \prod_{i=1}^{p}(a_i+1)$, which is exactly the dimension of space S^k. From the results of Theorem 3.3, each multivariate regression spline in $S^k(J)$ can be formulated as a linear combination of the elements in the basis of (3.3) or the basis in (b). However, an explicit representation of a multivariate polynomial spline is not always possible for any continuity condition of its partial derivatives. The following theorem provides bases of spline spaces $S^k(J_d)$ and $S^k(J_i)$.

Theorem 3.4. (a) *The following functions form a basis of* $S^k(J_d)$.

$$\prod_{i=1}^{p} x_i^{m_i}, \quad 0 \leq_i \sum_{i=1}^{p} m_i \leq k,$$

$$(x_i - \delta_{t_i}^{i})_+^{m_i} \prod_{j \neq i} x_j^{m_j}, \quad \gamma(t_i^-) + 1 \leq m_i \leq k,$$

$$\begin{aligned} 0 \leq \sum_{i=1}^{p} m_i \leq k, &\Big\{ (x_{i_1} - \delta_{t_{i_1}}^{i_1})_+^{m_{i_1}} (x_{i_2} - \delta_{t_{i_2}}^{i_2})_+^{m_{i_2}} \prod_{j \neq i_1, i_2} x_j^{m_j}, \\ &\gamma(t_{i_1}^-, t_{i_2}) + 1 \leq m_{i_1} \leq k, \gamma(t_{i_1}, t_{i_2}^-) + 1 \leq m_{i_2} \leq k, \\ &\qquad 0 \leq \sum_{i=1}^{p} m_i \leq k \Big\} I(\gamma_0(t_{i_1}, t_{i_2}) \leq k-2) \\ &\qquad \vdots \\ &\Big\{ \prod_{i=1}^{p} (x_i - \delta_{t_i}^{i})_+^{m_i}, \gamma(t_1 \dots t_i^- \dots t_p) + 1 \leq m_i \leq k, \\ &\qquad 0 \leq \sum_{i=1}^{p} m_i \leq k \Big\} I(\gamma_0(t_1, \dots, t_p) \leq k-p) \end{aligned} \tag{3.7}$$

where $\gamma_0(t_1,\dots,t_b)=\sum_{j=1}^{b}\gamma(t_{i_1}\cdots t_{i_j}^{-}\cdots t_{i_b})$ *and* t_{i_j} *ranges from* 1 *to* a_{i_j}.

(b) *The space of functions of* (3.7), *by replacing all "+" functions by "−" functions, is a basis of space* $S^k(J_i)$.

Proof. We prove only part (a), the proof of (b) being similar. First it is obvious that all the functions in (3.7) satisfy the smoothness condition J_d. We therefore need only to show that these functions form a generator of $S^k(J_d)$. The proof is an analogue of the one for proving the linear independence of the set in (3.3). Therefore, we only briefly sketch it here.

Suppose that we have a piecewise multivariate polynomial P in $S^k(J_d)$. Since $S^k(J_d)\subset S^k$, P can be formulated as a linear function of the elements of (3.3), we need only to show that the coefficients associated with the elements that are not part of (3.7) have to be zeros. Consider the linear functions $\lambda_{\pi t_i \pi m_i}$ of (3.5) for these $(t_1,\dots,t_p)$ and $(m_1,\dots,m_p)$ belonging to the complement of set (3.7). Then the smoothness condition J_d on P implies that (3.6) holds for those elements which are not part of (3.7) and then their corresponding coefficients are zeros. Thus, the set in (3.7) is a generator of $S^k(J_d)$. □

Corollary 3.5. $\dim S^k(J_d)=\dim S^k(J_i)$ *and equals*

$$\binom{p+k}{p}+\sum_{b=1}^{p}\sum_{1\le i_1<\cdots<i_b\le p}\;\sum_{t_{i_s}=1,\dots,a_{i_s},s=1,\dots,b}\Bigg[\sum_{j=0}^{k-(\gamma_0(t_{i_1},\dots,t_{i_b})+b)}\binom{b+j-1}{b-1}\binom{p+k-2b-\gamma_0(t_{i_1},\dots,t_{i_b})-j}{p-b}\, I(\gamma_0(t_{i_1},\dots,t_{i_b})\le k-b)\Bigg]. \tag{3.8}$$

Proof. Fix b, $(i_1,\dots,i_b),(t_{i_1},\dots,t_{i_b})$ with $1\le b\le p$, $1\le i_1<\cdots<i_b\le p$, $t_{i_j}=1,\dots,a_{i_j}$ for $j=1,\dots,b$. The polynomials forming a

basis in region $\{\delta_{t_{i_1}}^{i_1} \leq x_{i_1}, \ldots, \delta_{t_{i_b}}^{i_b} \leq x_{i_b}\}$ are

$$(x_{i_1} - \delta_{t_{i_1}}^{i_1})_+^{m_{i_1}} \ldots (x_{i_b} - \delta_{t_{i_b}}^{i_b})_+^{m_{i_b}}$$
$$\prod j \neq i_i, \ldots, i_b x_j^{m_j},$$
$$\gamma(0 \cdots t_{i_1} \cdots t_{i_j}^- \cdots t_{i_b}) + 1 \leq m_{i_j} \leq k, j = 1, \ldots, b, \tag{3.9}$$
$$0 \leq \sum_{i=1}^{b} m_i \leq k.$$

We will compute the number of basis functions in (3.9). First, the number of $\{(m_1, \ldots, m_p) : \sum_{j \neq i_1, \ldots, i_b} m_{i_j} = 0, 1, \ldots, k - \sum_{j=1}^{b} m_{i_j}\}$ is

$$\binom{p - b + k - \sum_{j=1}^{b} m_{i_j}}{p - b}. \tag{3.10}$$

Moreover, the range of the function $\sum_{j=1}^{b} m_{i_j}$ is $\{\gamma_0(t_{i_1}, \ldots, t_{i_b}) + b + e : e = 0, 1, \ldots, k - \gamma_0(t_{i_1}, \ldots, t_{i_b})\}$, and the number of the set $\{(m_{i_1}, \ldots, m_{i_b}) : \sum_{j=1}^{b} m_{i_j} = \gamma_0(t_{i_1}, \ldots, t_{i_b}) + b + e\}$ is

$$\binom{b + e - 1}{b - 1}, \quad \text{for } e = 0, 1, \ldots, k - (\gamma_0(t_{i_1}, \ldots, t_{i_b}) + b). \tag{3.11}$$

Combining (3.10) and (3.11), we have the number of basis functions in (3.9) as

$$\sum_{e=0}^{k-(\gamma_0(t_{i_1}, \ldots, t_{i_b})+b)} \binom{b + e - 1}{b - 1} \binom{p - b + k - (\gamma_0(t_{i_1}, \ldots, t_{i_b}) + b) - e}{p - b}.$$

Considering the fact that the dimension of a polynomial in P_k is $\binom{p+k}{p}$, the dimension of $S^k(J_d)$ is (3.8). $\square$

A simpler way to set the smoothness set for the case where $p = 2$ is to fill up nonnegative integers $\gamma(t_1, t_2)$s in the cells of the following index matrix, where we let $a = a_1$ and $b = a_2$:

$$\begin{bmatrix} & (1,0) & (2,0) & \cdots & (a,0) \\ (0,1) & (1^+,1) & (2^+,1) & \cdots & (a^+,1) \\ & (1,1^+) & (2,1^+) & \cdots & (a,1^+) \\ (0,2) & (1^+,2) & (2^+,2) & \cdots & (a^+,2) \\ & (1,2^+) & (2,2^+) & \cdots & (a,2^+) \\ & \vdots & \vdots & & \vdots \\ (0,b) & (1^+,b) & (2^+,b) & & (a^+,b) \\ & (1,b^+) & (2,b^+) & \cdots & (a,b^+) \end{bmatrix}.$$

In the next section, we consider several special cases of monotone multivariate polynomial spline spaces. These cases include spaces of a homogeneous and several semi-homogeneous smoothness conditions. The bases and dimensions of the corresponding multivariate polynomial spline spaces are also presented.

4. Multivariate polynomial splines under homogeneous and semi-homogeneous smoothing conditions. We first consider certain semi-homogeneous smoothness conditions and then obtain their corresponding spline spaces. Let us denote a knot hyperplane as

$$l_i(t_i) = \{(x_1, \dots, x_p) \in T : x_i = \delta^i_{t_i}\}.$$

Obviously,

$$l_i(t_i) = \bigcup_{\substack{t_j=0,1,\dots,a_j \\ j\neq i}} l(t_1 \cdots t_i^- \cdots t_p).$$

In this setting, the boundary set is

$$N_1 = \{l_i(t_i) : t_i = 1, \dots, a_i, i = 1, \dots, p\}.$$

This design tries to force the polynomial spline a constant smoothness condition on the boundary plane $x_i = \delta^i_{t_i}$ for each i, $i = 1, \dots, p$ and t_i, $t_i = 1, \dots, a_i$. Let the smoothness set be

$$J_1 = \{\gamma_i(t_i) : \text{ for } 1 \le i \le p, 0 \le \gamma_i(t_i) \le k-1, t_i = 1, \dots, a_i\}.$$

The number of smoothness conditions for set J_1 is thus reduced to $\prod_{i=1}^{p} a_i$. We then give a basis for the following multivariate polynomial spline space

$$S^k(J_1) = \{f \in S^k : \text{ for } 1 \le i \le p, 1 \le t_i \le a_i, f \in C^{\gamma_i(t_i)} \text{ on } l_i(t_i)\}.$$

Corollary 4.1. *The set of* (3.3), *by replacing* $\gamma(t_{i_1} \cdots t_{i_j}^- \cdots t_{i_b})$ *by* $\gamma_{i_j}(t_{i_j})$ *and redefining* $\gamma_0(t_{i_1}, \dots, t_{i_b})$ *by* $\sum_{j=1}^{b} \gamma_{i_j}(t_{i_j})$ *is a basis of* $S^k(J_1)$ *with dimension*

$$\binom{p+k}{p} \sum_{b=1}^{p} \sum_{1 \le i_1 < \cdots < i_b \le p} \sum_{t_{i_s}=1,\dots,a_{i_s}, s=1,\dots,b} \Bigg[\sum_{j=0}^{k-(\sum_{l=1}^{b} \gamma_{il}(t_{il})+b)} \binom{b+j-1}{b-1} \binom{p+k-2b-\sum_{l=1}^{b}\gamma_{il}(t_{il})-j}{p-b} I\Bigg(\sum_{l=1}^{b} \gamma_{il}(t_{il}) \le k-b\Bigg)\Bigg].$$

Consider a more uniform condition of smoothness as

$$J_2 = \{\gamma_i : 0 \le \gamma_i \le k-1, 1 \le i \le p\}.$$

This setting forces the polynomial spline to obey the constant smoothness condition on the boundary set $\cup_{t_i=1}^{a_i}\{x_i = \delta_{t_i}^{i}\}$. Consider the space of multivariate polynomial spline space as

$$S^k(J_2) = \{f \in S^k : \text{ for } 1 \le i \le p, f \in C^{\gamma_i} \text{ on } l_i(t_i) \text{ for } 1 \le t_i \le a_i\}.$$

This space sets one smoothness condition on each of the sets $\{x_i = \delta_i(t_i), 1 \le t_i \le a_i\}$. There are only p number of smoothness indices.

Corollary 4.2. *The set of* (3.3), *by replacing* $\gamma_0(t_{i_1} \cdots t_{i_j}^- \cdots t_{i_b})$ *by* γ_{t_i} *and redefining* $\gamma(t_{i_1}, \dots, t_{i_b})$ *by* $\sum_{j=1}^{b} \gamma_{i_j}$, *is a basis of* $S^k(J_2)$ *with* *dimension*

$$\binom{p+k}{p} + \sum_{b=1}^{p} \sum_{1 \le i_1 < \cdots < i_b \le p} \left\{ \prod_{s=1}^{b} a_{i_s} \sum_{j=0}^{k-(\sum_{l=1}^{b} \gamma_{il}+1)} \binom{b+j-1}{b-1} \binom{p+k-2b-\sum_{l=1}^{b}\gamma_{il}-j}{p-b} \right\} I\Big(\sum_{l=1}^{b} \gamma_{il} \le k-b\Big).$$

Consider the case as $a_1 = a_2 = \cdots = a_p = a$. Let us denote a knot hyperplane as

$$l(j) = \bigcup_{i=1}^{p} l_i(j) = \bigcup_{i=1}^{p} \{(x_1, \ldots, x_p) \in T : x_i = \delta_j^i\}, \quad j = 1, \ldots, a.$$

In fact, for each j, $l(j)$ forms a boundary of a multivariate rectangle. This design forces a multivariate polynomial spline to obey a constant smoothness condition on the rectangle boundary $l(j)$.

There are two types of smoothness sets that can be considered; they are:

$$J_3^d = \{\gamma_1, \ldots, \gamma_a : k-1 \ge \gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_a \ge 0\}$$

and

$$J_3^i = \{\gamma_1, \ldots, \gamma_a : 0 \le \gamma_1 \le \gamma_2 \le \cdots \le \gamma_a \le k-1\}.$$

We can thus obtain bases for the multivariate polynomial spline spaces $S^k(J_3^d)$ and $S^k(J_3^i)$.

Corollary 4.3. (a) *The set of* (3.3), *by replacing* $\gamma_0(t_{i_1} \cdots t_{i_j}^{-} \cdots t_{i_b})$ *by* $\gamma_{t_{i_j}}$ *and redefining* $\gamma_0(t_{i_1}, \ldots, t_{i_b})$ *by* $\sum_{l=1}^{p} \gamma_l n_l(t_{i_1}, \ldots, t_{i_b})$ *where* *the counting function* $n_l(t_{i_1}, \ldots, t_{i_b})$ *is defined as* $\sum_{j=1}^{p} I(t_{i_j} = l)$, *is a basis of* $S^k(J_3^d)$ *with dimension*

$$\binom{p+k}{p} + \sum_{b=1}^{p} \sum_{1 \le i_1 < \cdots < i_b \le p} \sum_{t_{i_s}=1,\ldots,a_{i_s}, s=1,\ldots,b} \Bigg[\sum_{j=0}^{k-(\sum_{l=1}^{p} \gamma_l n_l(t_{i_1},\ldots,t_{i_b})+b)} \binom{b+j-1}{b-1} \binom{p+k-2b-\sum_{l=1}^{p} \gamma_l n_l(t_{i_1}, \ldots, t_{i_b})-j}{p-b} I\bigg(\sum_{l=1}^{p} \gamma_l n_l(t_{i_1}, \ldots, t_{i_b}) \le k-b \bigg) \Bigg].$$

(b) *The set of the basis in* (b) *of Theorem* 3.4, *by replacing* $\gamma(t_{i_1} \cdots t_{i_j}^{-} \cdots t_{i_b})$ *by* $\gamma_{t_{i_j}}$ *and redefining* $\gamma_0(t_{i_1}, \ldots, t_{i_b})$ *by* $\sum_{l=1}^{p} \gamma_l n_l(t_{i_1}, \ldots, t_{i_b})$ *is a basis of* $S^k(J_3^i)$ *with the same dimension as that of* $S^k(J_3^d)$.

The most uniform smoothness condition is to set J containing a single element γ, $0 \le \gamma \le k-1$. We denote this spline space as $S^k(\gamma)$ and describe it in detail.

Corollary 4.4. *The set of the following functions is a basis of* $S^k(\gamma)$:

$$\prod_{i=1}^{p} x_i^{m_i}, 0 \le \sum_{i=1}^{p} m_i \le k,$$

$$(x_i - \delta_{t_i}^{i})_+^{m_i} \prod_{j \ne i} x_j^{m_j}, \gamma + 1 \le m_i \le k,$$

$$0 \le \sum_{i=1}^{p} m_i \le k, \bigg\{ (x_{i_1} - \delta_{t_{i_1}}^{i_1})_+^{m_{i_1}} (x_{i_2} - \delta_{t_{i_2}}^{i_2})_+^{m_{i_2}} \prod_{j \ne i_1, i_2} x_j^{m_j}, \gamma + 1 \le m_{i_1}, m_{i_2} \le k, 0 \le \sum_{i=1}^{p} m_i \le k \bigg\}$$

$$I(2(\gamma+1) \le k)$$

$$\vdots$$

$$\left\{\prod_{i=1}^{p}(x_i-\delta^i_{t_i})_+^{m_i}, \gamma+1\le m_i\le k, i=1,\ldots,p, \sum_{i=1}^{p} m_i\le k\right\}$$
$$I(p(\gamma+1)\le k)$$

with dimension

$$\binom{p+k}{p}+\sum_{b=1}^{p}\Bigg\{\binom{p}{b}\Big(\prod_{s=1}^{b}a_{i_s}\Big) \cdot\sum_{j=0}^{k-(b+1)\gamma}\binom{b+j-1}{b-1}\binom{p+k-2b-b\gamma-j}{p-b}\Bigg\}$$
$$I(b(\gamma+1)\le k).$$

Notice that any piecewise multivariate polynomial of degree k with k continuous partial derivatives would actually be a single multivariate polynomial of degree k. The smoothest possible piecewise multivariate polynomial which retains a segmented nature is the multivariate polynomial spline in $S^k(k-1)$. The smoothest multivariate polynomial spline can be formulated as

$$\sum_{m_1+\cdots+m_p=0}^{k}\beta_{m_1\cdots m_p}\prod_{i=1}^{p}x_i^{m_i}+\sum_{i=1}^{p}\sum_{t_i=1}^{a_i}\beta_k^{t_i}(x_i-\delta^i_{t_i})_+^k$$

with dimension $\binom{p+k}{p}+a_1+\cdots+a_p$. Now, let's consider the special case of $p=2$ and uniform smoothness condition. The basis for this bivariate polynomial spline with smoothness j is, if $0\le j<(k-1)/2$,

$$x_1^{m_1}x_2^{m_2}, 0\le m_1+m_2\le k,$$
$$(x_1-\delta^1_t)_+^{m_1}x_2^{m_2}, j+1\le m_1\le k,$$
$$0\le m_1+m_2\le k, \quad \text{for } 1\le t\le a,$$
$$x_1^{m_1}(x_2-\delta^2_t)_+^{m_2}, j+1\le m_2\le k,$$
$$0\le m_1+m_2\le k, \quad \text{for } 1\le t\le b,$$

$$(x_1-\delta^1_{t_1})^{m_1}_+(x_2-\delta^2_{t_2})^{m_2}_+, \quad j+1\le m_1\le k,$$
$$j+1\le m_2\le k, \quad \text{for } 1\le t_1\le a, 1\le t_2\le b, (k-1)/2\le j\le k-1,$$
$$x_1^{m_1}x_2^{m_2}, 0\le m_1+m_2\le k,$$
$$(x_1-\delta^1_t)^{m_1}_+, j+1\le m_1\le k, \quad \text{for } 1\le t\le a,$$
$$(x_2-\delta^2_t)^{m_2}_+, j+1\le m_2\le k, 1\le t\le b,$$

with dimension

$$\begin{aligned}2^{-1}[(k+2)(k+1)+(a+b)(k-j)(k+j+3)&\\ +ab(k^2+4k-3j^2-9j-4)]& \quad \text{if } 0\le j<(k-1)/2\end{aligned}$$

and

$$2^{-1}(k+2)(k+1)+(a+b)(k-j) \quad \text{if } (k-1)/2\le j\le k-1.$$

The basis of bivariate polynomial spline for rectangular partition and uniform smoothness condition, proposed by Chui and Wang [**4**], is different from the one stated above; however, they are equivalent. Based on the above result, it can be argued that the basis formulation of a multivariate polynomial spline is not unique. However, the formulation presented in this paper is a natural generalization of the usual univariate polynomial spline basis. To see this, for example, bivariate polynomial spline basis in this paper consists of all the terms $(x_1-\delta^1_{t_1})^{m_1}_+(x_2-\delta^2_{t_2})^{m_2}_+$ for (m_1,m_2) satisfying $m_1+m_2\le k$, which is equivalent to the fact that a usual basis of univariate basis contains all the terms $(x-\delta)^m_+$ for m satisfying $m\le k$, whereas the one proposed by Chui and Wang [**4**] includes only the term $(x_1-\delta^1_{t_1})^{j+1}_+(x_2-\delta^2_{t_2})^{j+1}_+$.

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