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GENERATORS FOR THE SEMIMODULE OF VARIETIES OF A FREE MODULE

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Throughout this paper all rings are commutative with identity and all modules are unital modules. Furthermore, the symbols R, M and F represent a ring, R-module and free R-module, respectively. For any submodule N of M, we define $(N : M) = \{r \in R : rM \subseteq N\}$. A submodule P of M is called *prime* if $P \neq M$ and whenever $r \in R$, $m \in M$ and $rm \in P$, then $m \in P$ or $rM \subseteq P$. It is well known (and easy to check) that a submodule P of M is prime if and only if $\mathfrak{p} = (P : M)$ is a prime ideal of R and the $R/\mathfrak{p}$-module M/P is torsion-free. For this fact and other basic properties of prime submodules, see, for example, [**1**], [**3**] and [**7**].

The (possibly empty) collection of prime submodules of M will be denoted by $\operatorname{spec} M$. For any submodule N of M, we set $V(N) = \{P \in \operatorname{spec} M : N \subseteq P\}$ and call $V(N)$ the *variety* of N. The collection of all such varieties $V(N)$, N a submodule of M, is denoted by $\zeta(M)$. In [**4**] we proved that, in general, $\zeta(M)$ is closed under arbitrary intersections but is not closed under finite unions. However, $\zeta(R)$ is a semiring with addition given by intersection and multiplication given by union, and in [**5**] we proved that $\zeta(M)$ is a $\zeta(R)$-semimodule with respect to the following addition and multiplication:

$$V(N_1) + V(N_2) = V(N_1 + N_2) \quad \text{and} \quad V(\mathfrak{a})V(N) = V(\mathfrak{a}N),$$

for all submodules N_1, N_2 and N of M and ideals $\mathfrak{a}$ of R. For a broader study of semimodules, see, for example, [**2**]. The module M is called *Zariski-finite* if $\zeta(M)$ is a finitely generated $\zeta(R)$-semimodule, i.e., there exist a positive integer k and submodules N_i, $1 \leq i \leq k$, of M such

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that

$$\begin{aligned}\zeta(M) &= \zeta(R)V(N_1) + \cdots + \zeta(R)V(N_k)\\ &= \{V(\mathfrak{a}_1)V(N_1) + \cdots + V(\mathfrak{a}_k)V(N_k) :\\ &\qquad \mathfrak{a}_i, \dots, \mathfrak{a}_k, \text{ are ideals of } R\}\\ &= \{V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k) : \mathfrak{a}_i, \dots, \mathfrak{a}_k, \text{ are ideals of } R\}.\end{aligned}$$

Zariski-finite modules were introduced and studied in [**6**].

In general, free R-modules are not Zariski-finite. Before we prove this fact, given an index set I and submodules N_i, $i \in I$, of M, we let $\langle V(N_i) : i \in I\rangle$ denote the collection of varieties in $\zeta(M)$ of the form $V(\sum_{i\in J} \mathfrak{a}_i N_i)$ where J is a finite subset of I and $\mathfrak{a}_i$, $i \in J$, ideals of R. Then the varieties $V(N_i)$, $i \in I$, *generate* $\zeta(M)$ if $\zeta(M) = \langle V(N_i) : i \in I\rangle$ and in this case we call the varieties $V(N_i)$, $i \in I$, *generators* of $\zeta(M)$. Of course, $\zeta(M)$ is generated by $\{V(N) : N \text{ is a submodule of } M\}$, but we want to find more interesting generating sets!

Our first main result shows that if R is a domain and M is torsion-free and contains nonzero elements x, y such that $Rx \cap Ry = 0$, then the cardinality of every generating set of $\zeta(M)$ must be greater than or equal to the cardinality of R. We also show that if R is a Noetherian ring with only one minimal prime ideal $\mathfrak{p}$ and if F is of finite rank $n \geq 2$, then $\zeta(F)$ is generated by the varieties of the height 1 $\mathfrak{p}$-prime submodules of F. Furthermore, if in this case $\zeta(F)$ is generated by $\{V(N_j) : j \in J\}$, then for each $j \in J$, there must exist a height 1 $\mathfrak{p}$-prime submodule P and an ideal $\mathfrak{c}_j$ of R such that $V(N_j) = V(\mathfrak{c}_j P)$.

1. General modules. Note that if k is a positive integer, N and N_i, $1 \leq i \leq k$, are submodules of M and $\mathfrak{a}_i$, $1 \leq i \leq k$, ideals of R such that $V(N) = V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k)$, then in general the "coefficients" $\mathfrak{a}_i$, $1 \leq i \leq k$, will not be unique. Our first two results examine this situation. If $\mathfrak{a}$ is an ideal of R, then $\sqrt{\mathfrak{a}}$ will denote the prime radical of $\mathfrak{a}$. Recall that $\sqrt{\mathfrak{a}} = \{r \in R : r^n \in \mathfrak{a} \text{ for some positive integer } n\}$. If $r \in R$, $m \in M$ and $P \in \operatorname{spec} M$, then $rm \in P$ if and only if $r^n m \in P$ for some (or any) positive integer n.

Proposition 1.1. *Let M be any R-module, let k be a positive integer, let N_i, $1 \leq i \leq k$, be submodules of M and let $\mathfrak{a}_i, \mathfrak{b}_i$,*

$1 \leq i \leq k$, be ideals of R such that $\sqrt{\mathfrak{a}_i} = \sqrt{\mathfrak{b}_i}$, $1 \leq i \leq k$. Then $V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k) = V(\mathfrak{b}_1 N_1 + \cdots + \mathfrak{b}_k N_k)$.

Proof. Let $P \in \operatorname{spec} M$. Then

$$\begin{aligned} P \in V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k) &\Longleftrightarrow \mathfrak{a}_i N_i \subseteq P, \quad 1 \leq i \leq k \\ &\Longleftrightarrow \sqrt{\mathfrak{a}_i} N_i \subseteq P, \quad 1 \leq i \leq k, \end{aligned}$$

and the result follows. $\square$

In particular, if, in Proposition 1.1, R is a Noetherian domain and $\mathfrak{a}_i$ is a nonzero proper ideal of R for some $1 \leq i \leq k$, then the ideals $\mathfrak{a}_i^m$, $m \geq 1$, are distinct, see, for example, [**8**, p. 216], so that $V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k) = V(\mathfrak{a}_1^{m_1} N_1 + \cdots + \mathfrak{a}_k^{m_k} N_k)$ for all positive integers m_i, $1 \leq i \leq k$, gives an infinite number of expressions for $V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k)$. However, as we show next in all these expressions, there is a unique maximal choice for the ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$. Recall that, for any submodule N of M, the *prime radical* $\operatorname{rad} N$ of N is defined to be the intersection of all prime submodules of M containing N, and in case there are no such prime submodules $\operatorname{rad} N$ is defined to be M.

Proposition 1.2. *Let k be a positive integer, let N and N_i, $1 \leq i \leq k$, be submodules of M, and let $\mathfrak{a}_i$, $1 \leq i \leq k$, be ideals of R such that $V(N) = V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k)$. Let $\mathfrak{c}_i = (\operatorname{rad} N : N_i)$, $1 \leq i \leq k$. Then $\mathfrak{c}_i$, $1 \leq i \leq k$, are ideals of R such that*

(i) *$V(N) = V(\mathfrak{c}_1 N_1 + \cdots + \mathfrak{c}_k N_k)$ and*

(ii) *$\mathfrak{b}_i \subseteq \mathfrak{c}_i$, $1 \leq i \leq k$, for all ideals $\mathfrak{b}_i$, $1 \leq i \leq k$, of R such that $V(N) = V(\mathfrak{b}_1 N_1 + \cdots + \mathfrak{b}_k N_k)$.*

Proof. (i) Let $P \in V(N)$. Then $N \subseteq P$ and $P \in \operatorname{spec} M$. Clearly $\operatorname{rad} N \subseteq P$ and hence $\mathfrak{c}_i N_i \subseteq \operatorname{rad} N \subseteq P$, $1 \leq i \leq k$. Thus $\mathfrak{c}_1 N_1 + \cdots + \mathfrak{c}_k N_k \subseteq P$ and $P \in V(\mathfrak{c}_1 N_1 + \cdots + \mathfrak{c}_k N_k)$. Conversely, let $Q \in V(\mathfrak{c}_1 N_1 + \cdots + \mathfrak{c}_k N_k)$. Note that $\mathfrak{a}_i N_i \subseteq \operatorname{rad} N$ so that $\mathfrak{a}_i \subseteq \mathfrak{c}_i$, $1 \leq i \leq k$. Hence $\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k \subseteq Q$ and $Q \in V(N)$.

(ii) Clear. $\square$

In view of these results, we shall concentrate in the sequel on the submodules N_i rather than the ideals $\mathfrak{a}_i$. The next result is illuminating for what comes later. For any set X, $|X|$ will denote the cardinality of X.

Theorem 1.3. *Let R be a domain, and let M be a torsion-free R-module which contains nonzero elements x, y such that $Rx \cap Ry = 0$. Let I be an index set, and let N_i, $i \in I$, be submodules of M such that $V(N_i)$, $i \in I$, generate $\zeta(M)$. Then $|I| \geq |R|$.*

Proof. For each element $r \in R$, let P_r be the submodule of M containing the element $x + ry$ such that $P_r/R(x + ry)$ is the torsion submodule of the R-module $M/R(x + ry)$. Clearly $y \notin P_r$ so that $P_r \neq M$ and P_r is a prime submodule of M for each $r \in R$. Let $r \in R$. Then $V(P_r) = V(\sum_{j \in J} \mathfrak{a}_j N_j)$ for some finite subset J of I and nonzero ideals $\mathfrak{a}_j$, $j \in J$, of R. Since $P_r \in V(P_r)$, it follows that, for each $j \in J$, $\mathfrak{a}_j N_j \subseteq P_r$ and, since $(P_r : M) = 0$, then $N_j \subseteq P_r$. If $N_j = 0$, $j \in J$, then $V(P_r) = V(0)$, which gives the contradiction $0 \in V(P_r)$, i.e., $P_r = 0$. Hence there exists $j(r) \in J$ such that $0 \neq N_{j(r)} \subseteq P_r$.

Finally note that $P_r \cap P_s = 0$ for all distinct r, s in R because $Rx \cap Ry = 0$. In particular, $j(r) \neq j(s)$ for all $r \neq s$ in R and the mapping $j : R \to I$ is an injection. $\square$

Corollary 1.4. *Let R be a domain. Then a free R-module F is Zariski-finite if and only if F has rank ≤ 1 or R is a finite field and F has finite rank.*

Proof. The necessity follows by Theorem 1.3. Conversely, if F has rank 1, then $F = Rf$ for some $f \in F$. In this case, for every submodule G of F, there exists an ideal $\mathfrak{g}$ of R such that $G = \mathfrak{g}f$ and hence $V(G) = V(\mathfrak{g}f) = V(\mathfrak{g})V(Rf)$. Thus F is Zariski-finite. On the other hand, if R is a finite field and F has finite rank, then F, and hence also $\operatorname{spec} F$, is finite so that clearly F is Zariski-finite. $\square$

Let R be any ring, and let M be an R-module. Given submodules N, L of M, we set $N \sim L$ if $\operatorname{rad} N = \operatorname{rad} L$. Clearly, $N \sim L$ if and only if $V(N) = V(L)$. It is elementary to check that $\sim$ is an equivalence relation on the lattice of submodules of M. The next result is also elementary, but we shall give its proof for completeness.

Lemma 1.5. *Let N_i, L_i, $1 \le i \le k$, be submodules of M such that $N_i \sim L_i$, $1 \le i \le k$, for some positive integer k. Then $\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k \sim \mathfrak{a}_1 L_1 + \cdots + \mathfrak{a}_k L_k$ for any ideals $\mathfrak{a}_i$, $1 \le i \le k$, of R.*

Proof. Let $P \in V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k)$. For each $1 \le i \le k$, $\mathfrak{a}_i N_i \subseteq P$ and hence $\mathfrak{a}_i M \subseteq P$ or $N_i \subseteq P$. But $N_i \subseteq P$ implies $L_i \subseteq P$ for any $1 \le i \le k$. Thus $\mathfrak{a}_i L_i \subseteq P$, $1 \le i \le k$, and hence $P \in V(\mathfrak{a}_1 L_1 + \cdots + \mathfrak{a}_k L_k)$. $\square$

Corollary 1.6. *Let N_i, L_i, $i \in I$, be submodules of M, and let $\mathfrak{a}_i$, $i \in I$, be ideals of R such that $N_i \sim \mathfrak{a}_i L_i$ for all $i \in I$. If $\{V(N_i) : i \in I\}$ generates $\zeta(M)$, then $\{V(L_i) : i \in I\}$ generates $\zeta(M)$.*

Proof. Let N be any submodule of M. There exist a finite subset J of I and ideals $\mathfrak{b}_j$, $j \in J$, of R such that $V(N) = V(\sum_{j \in J} \mathfrak{b}_j N_j)$, i.e., $N \sim \sum_{j \in J} \mathfrak{b}_j N_j$. By Lemma 1.5, $N \sim \sum_{j \in J} \mathfrak{b}_j \mathfrak{a}_j L_j$, i.e., $V(N) = V(\sum_{j \in J} \mathfrak{b}_j \mathfrak{a}_j L_j)$. It follows that $\zeta(M)$ is generated by $\{V(L_i) : i \in I\}$. $\square$

Note that in Corollary 1.6 if $\mathfrak{a}_i = R$, $i \in I$, then $\{V(N_i) : i \in I\}$ generates $\zeta(M)$ if and only if $\{V(L_i) : i \in I\}$ generates $\zeta(M)$. This fact will be used to give a further corollary of Lemma 1.5. First recall that a submodule S of M is called *semiprime* if S is an intersection of prime submodules of M. Clearly S is a semiprime submodule of M if and only if $S \ne M$ and $S = \operatorname{rad} S$. Moreover, if M is a finitely generated module, then every proper submodule N is contained in a maximal submodule P, say, of M and $P \in V(N)$. In this case $\operatorname{rad} N$ is a semiprime submodule of M.

Corollary 1.7. *Let M be a finitely generated R-module which is not cyclic. Then $\{V(S) : S$ is a semiprime submodule of $M\}$ generates $\zeta(M)$.*

Proof. Recall that $\zeta(M)$ is generated by $\{V(N) : N$ is a submodule of $M\}$. If N is a proper submodule of M, then $\operatorname{rad} N$ is a semiprime submodule of M and $N \sim \operatorname{rad} N$. To account for $V(M)$, note that there exist a positive integer k and elements $m_i \in M$, $1 \le i \le k$, such that $M = Rm_1 + \cdots + Rm_k$. Since $M \ne Rm_i$, it follows that $\operatorname{rad} Rm_i$ is a semiprime submodule of M for each $1 \le i \le k$ and $M = \operatorname{rad} Rm_1 + \cdots + \operatorname{rad} Rm_k$. Thus, $\zeta(M)$ is generated by $\{V(N) : N$ is a proper submodule of $M\}$. Apply Corollary 1.6. □

In [**6**] it was shown that in case M is cyclic, $\zeta(M)$ is generated by $\{V(M)\}$.

2. Free modules. In the sequel $\mathfrak{w}$ will always denote the prime radical of the zero ideal of the ring R. Recall that $\mathfrak{w}$ is the intersection of all prime ideals of R and consists of all nilpotent elements of R. An element c of R is called *regular* if $cr \ne 0$ for every nonzero element $r \in R$. For any ideal $\mathfrak{a}$ of R, we set $C(\mathfrak{a}) = \{c \in R : c + \mathfrak{a}$ is a regular element of the ring $R/\mathfrak{a}\}$. For example, if $\mathfrak{p}$ is any prime ideal of R, then $C(\mathfrak{p}) = R \backslash \mathfrak{p}$.

Recall that F is a free R-module. For each element x of F, we set $S(x) = \{y \in F : cy \in Rx + \mathfrak{w}F$ for some $c \in C(\mathfrak{w})\}$. Note that $S(x)$ is a submodule of F and $Rx \subseteq S(x)$ for each $x \in F$.

Lemma 2.1. *Let F be a free R-module, and let $x \in F$. Then $Rx \sim \mathfrak{a}S(x)$ where $\mathfrak{a}$ is the ideal $(Rx + \mathfrak{w}F : S(x))$ of R.*

Proof. Since $\mathfrak{w}$ is a nil ideal, it follows that $\mathfrak{w}F \subseteq P$ for all $P \in \operatorname{spec} F$. Now $\mathfrak{a}S(x) \subseteq Rx + \mathfrak{w}F$ gives that $V(Rx) = V(Rx + \mathfrak{w}F) \subseteq V(\mathfrak{a}S(x))$. Conversely, suppose that $Q \in V(\mathfrak{a}S(x))$. Then $\mathfrak{a}S(x) \subseteq Q$ implies that $\mathfrak{a}F \subseteq Q$ or $S(x) \subseteq Q$. Suppose that $\mathfrak{a}F \subseteq Q$ and let $y \in S(x)$. Furthermore, let $\{f_i : i \in I\}$ be a basis of F. There exist a finite subset J of I, $x_i, y_i \in R$, $i \in J$, $c \in C(\mathfrak{w})$ and $r \in R$ such that $x = \sum_{i \in J} x_i f_i$, $y = \sum_{i \in J} y_i f_i$ and $cy = rx + w$, where $w = \sum_{i \in J} w_i f_i \in \mathfrak{w}F$ for some $w_i \in \mathfrak{w}$, $i \in J$. Clearly $cy_i = rx_i + w_i$, $i \in J$.

Now $c(x_i y - y_i x) = x_i(rx + w) - (rx_i + w_i)x = x_i w - w_i x \in \mathfrak{w}F$ so that $x_i y - y_i x \in \mathfrak{w}F$ for all $i \in J$. Thus, for each $i \in J$, we have $x_i y \in Rx + \mathfrak{w}F$, i.e., $x_i \in \mathfrak{a}$, $i \in J$. Hence $x \in \mathfrak{a}F \subseteq Q$. In any case,

$Rx \subseteq Q$ and $Q \in V(Rx)$. Thus $V(\mathfrak{a}S(x)) \subseteq V(Rx)$. It follows that $V(Rx) = V(\mathfrak{a}S(x))$, as required. □

Corollary 2.2. *Let x_i, $1 \leq i \leq k$, be elements of F for some positive integer k. For each $1 \leq i \leq k$, let $\mathfrak{a}_i = (Rx_i + \mathfrak{w}F : S(x_i))$. Then $Rx_1 + \cdots + Rx_k \sim \mathfrak{a}_1 S(x_1) + \cdots + \mathfrak{a}_k S(x_k)$.*

Proof. By Lemmas 1.5 and 2.1. □

Lemma 2.3. *Let R be a Noetherian ring, and let F be of finite rank. Then the varieties $V(S(x))$, where $x \in F \backslash \mathfrak{w}F$, generate $\zeta(F)$.*

Proof. Let N be any submodule of F. Then $N = Rx_1 + \cdots + Rx_k$ for some positive integer k and elements $x_i \in N$, $1 \leq i \leq k$. If $x_i \in \mathfrak{w}F$, $1 \leq i \leq k$, then $N \subseteq \mathfrak{w}F$ and hence $V(N) = V(\mathfrak{w}F) = V(0) = V(0S(x))$ for any $x \in F \backslash \mathfrak{w}F$. Otherwise, we can suppose without loss of generality that there exists $1 < m \leq k$ such that $x_i \notin \mathfrak{w}F$, $1 \leq i \leq m$, and $x_i \in \mathfrak{w}F$, $m+1 \leq i \leq k$. It follows that $V(N) = V(Rx_1 + \cdots + Rx_m) = V(\mathfrak{a}_1 S(x_1) + \cdots + \mathfrak{a}_m S(x_m))$, where $\mathfrak{a}_i = (Rx_i + \mathfrak{w}F : S(x_i))$, $1 \leq i \leq m$, by Corollary 2.2. □

Let M be any R-module and let $\mathfrak{p}$ be a prime ideal of R. By a $\mathfrak{p}$-*prime* submodule P of M, if it exists, we mean a prime submodule P such that $\mathfrak{p} = (P : M)$. In Corollary 1.7 we saw that if M is a finitely generated noncyclic R-module, then $\zeta(M)$ is generated by the varieties $V(S)$ of semiprime submodules S of M. Now we show that in Lemma 2.4, for each $x \in F \backslash \mathfrak{w}F$, the submodule $S(x)$ is semiprime in case F has finite rank $n \geq 2$.

In the remainder of this paper we shall assume that R is a Noetherian ring with minimal prime ideals $\mathfrak{p}_i$, $1 \leq i \leq k$, for some positive integer k, and F is a free R-module of finite rank $n \geq 2$. Note that $\mathfrak{w} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$. Let $x \in F$. We have already defined $S(x) = \{y \in F : cy \in Rx + \mathfrak{w}F \text{ for some } c \in C(\mathfrak{w})\}$, and we now set $P_i(x) = \{y \in F : cy \in Rx + \mathfrak{p}_i F \text{ for some } c \in C(\mathfrak{p}_i)\}$.

Lemma 2.4. *With the above notation, for any $x \in F$, $P_i(x)$ is a $\mathfrak{p}_i$-prime submodule of F for each $1 \leq i \leq k$ and $S(x) = P_1(x) \cap \cdots \cap P_k(x)$. In particular, $S(x)$ is a semiprime submodule of F.*

Proof. Let $1 \leq i \leq k$ and let $x \in F$. Note that $P_i(x)$ is a submodule of F. If $F = P_i(x)$, then the $R/\mathfrak{p}_i$-module $F/\mathfrak{p}_i F$ is uniform, which contradicts the fact that $\operatorname{rank} F \geq 2$. Thus, $P_i(x)$ is a proper submodule of F, $\mathfrak{p}_i F \subseteq P_i(x)$ and $F/P_i(x)$ is a torsion-free $R/\mathfrak{p}_i$-module. Hence $P_i(x)$ is a $\mathfrak{p}_i$-prime submodule of F.

Next note that $\mathfrak{w} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$ and hence $C(\mathfrak{p}_1) \cap \cdots \cap C(\mathfrak{p}_k) \subseteq C(\mathfrak{w})$. Let $c \in C(\mathfrak{w})$. Let $r \in R$ such that $cr \in \mathfrak{p}_1$. Then $cr(\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k) \subseteq \mathfrak{w}$ so that $r(\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k) \subseteq \mathfrak{p}_1$ and hence $r \in \mathfrak{p}_1$ since $\mathfrak{p}_i \not\subseteq \mathfrak{p}_1$, $2 \leq i \leq k$. Thus $c \in C(\mathfrak{p}_1)$. Similarly, $c \in C(\mathfrak{p}_i)$ for all $2 \leq i \leq k$. It follows that $C(\mathfrak{w}) = C(\mathfrak{p}_1) \cap \cdots \cap C(\mathfrak{p}_k)$. Furthermore, since $\mathfrak{w}F \subseteq \mathfrak{p}_i F$ for every $1 \leq i \leq k$, we now have $S(x) \subseteq P_1(x) \cap \cdots \cap P_k(x)$.

Let $y \in P_1(x) \cap \cdots \cap P_k(x)$. For each i, $1 \leq i \leq k$, there exists $c_i \in C(\mathfrak{p}_i)$ such that $c_i y \in Rx + \mathfrak{p}_i F$ and there exists $d_i \in C(\mathfrak{p}_i) \cap (\cap_{j \neq i} \mathfrak{p}_j)$ and hence $c_i d_i y \in Rx + \mathfrak{w}F$. Let $c = c_1 d_1 + \cdots + c_k d_k$. Then $c \in C(\mathfrak{w})$ and $cy \in Rx + \mathfrak{w}F$. Thus, $y \in S(x)$. It follows that $P_1(x) \cap \cdots \cap P_k(x) \subseteq S(x)$. Therefore, $S(x) = P_1(x) \cap \cdots \cap P_k(x)$ and $S(x)$ is a semiprime submodule of F. □

Let P be any prime submodule of the R-module M. Then we say that P has *height* n, where n is a nonnegative integer, provided there exists a chain $P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n$ of prime submodules P_i, $0 \leq i \leq n$, of M but no longer such chain. We now investigate the submodules $P_i(x)$ where $1 \leq i \leq k$ and x is any element of the free R-module F. Note that if $\mathfrak{p}$ is any prime ideal of R, then $\mathfrak{p}F$ is a prime submodule of F.

Lemma 2.5. *Let $1 \leq i \leq k$ and let $x \in F$. Then the following statements are equivalent.*

(i) $P_i(x) = \mathfrak{p}_i F$.

(ii) $x \in \mathfrak{p}_i F$.

(iii) $P_i(x)$ *is a height* 0 $\mathfrak{p}_i$*-prime submodule of* F.

Proof. This is an easy consequence of the fact that $\mathfrak{p}_i F \subseteq P_i(x)$. □

Lemma 2.6. *Let* $1 \leq i \leq k$. *Then* P *is a height* 1 $\mathfrak{p}_i$*-prime submodule of* F *if and only if* $P = P_i(x)$ *for some* $x \in F\backslash \mathfrak{p}_i F$. *In this case* $P = P_i(z)$ *for any* $z \in P\backslash \mathfrak{p}_i F$.

Proof. Suppose first that $P = P_i(x)$ for some $x \in F\backslash \mathfrak{p}_i F$. Then P is a $\mathfrak{p}_i$-prime submodule of F by Lemma 2.4. Moreover, $P \supsetneq \mathfrak{p}_i F$ gives that P has height ≥ 1. Suppose that $P = P_0 \supseteq P_1 \supsetneq P_2$ is a chain of prime submodules of F. Note that $(P_2 : F) \subseteq (P : F) = \mathfrak{p}_i$ so that $(P_2 : F) = \mathfrak{p}_i$, because $\mathfrak{p}_i$ is minimal, and $\mathfrak{p}_i F \subseteq P_2 \subsetneq P_1$. It follows that P/P_1 is $C(\mathfrak{p}_i)$-torsion and hence $P = P_1$. Thus P has height 1. Let $z \in P\backslash \mathfrak{p}_i F$. Then $\mathfrak{p}_i F \subsetneq P_i(z) \subseteq P$ gives $P = P_i(z)$.

Conversely, suppose that P is a height 1 $\mathfrak{p}_i$-prime submodule of F. Then $P \supsetneq \mathfrak{p}_i F$ by Lemma 2.5. Let $x \in P\backslash \mathfrak{p}_i F$. Then $\mathfrak{p}_i F \subsetneq P_i(x) \subseteq P$ and hence $P = P_i(x)$. $\square$

Corollary 2.7. *Let* $1 \leq i \leq k$ *and let* $x, y \in F$. *Then either* $P_i(x) = P_i(y)$ *or* $P_i(x) \cap P_i(y) = \mathfrak{p}_i F$.

Proof. Suppose that $P_i(x) \cap P_i(y) \neq \mathfrak{p}_i F$ and choose $z \in P_i(x) \cap P_i(y)\backslash \mathfrak{p}_i F$. By Lemma 2.6, $P_i(x) = P_i(z) = P_i(y)$. $\square$

Let $x \in F\backslash \mathfrak{w} F$. In Lemma 2.4, we saw that $S(x) = P_1(x) \cap \cdots \cap P_k(x)$. Let $J = \{j : 1 \leq j \leq k, x \notin \mathfrak{p}_j F\}$ and note that J is a nonempty subset of $\{1, \dots, k\}$. Now let $I = \{1, \dots, k\}\backslash J$ and observe that $P_i(x) = \mathfrak{p}_i F$ for all $i \in I$. We define $S^*(x) = \cap_{j \in J} P_j(x)$. Thus $S(x) = (\cap_{i \in I} \mathfrak{p}_i F) \cap S^*(x)$.

Lemma 2.8. *Let* J *be a nonempty subset of* $\{1, \dots, k\}$, *and let* P_j *be a height* 1 $\mathfrak{p}_j$*-prime submodule of* F *for each* $j \in J$. *Then there exists* $x \in F\backslash \mathfrak{w} F$ *such that* $S^*(x) = \cap_{j \in J} P_j$.

Proof. Let $j \in J$. Note that $\prod_{i \neq j} \mathfrak{p}_i \nsubseteq \mathfrak{p}_j$ and $P_j \neq \mathfrak{p}_j F$. Thus $(\prod_{i \neq j} \mathfrak{p}_i) P_j \nsubseteq \mathfrak{p}_j F$ and hence $(\cap_{i \neq j} \mathfrak{p}_i F) \cap P_j \nsubseteq \mathfrak{p}_j F$. Choose $x_j \in (\cap_{i \neq j} \mathfrak{p}_i F) \cap P_j$ and let $x = \sum_{j \in J} x_j$. Then $x \in (\cap_{j \in J} P_j) \cap (\cap_{i \in I} \mathfrak{p}_i F)$, where $I = \{1, \dots, k\}\backslash J$. Moreover, $x \notin \mathfrak{p}_j F$, $j \in J$. Thus, $S^*(x) = \cap_{j \in J} P_j$. $\square$

We shall call a submodule S of F a $*$-*semiprime submodule* of F if $S = S^*(x)$ for some $x \in F \backslash \mathfrak{w}F$. We next show that $*$-semiprime submodules of F have a unique expression as an intersection of height 1 $\mathfrak{p}_i$-prime submodules.

Lemma 2.9. *Let J be a nonempty subset of $\{1, \dots, k\}$, and let P_j be a height 1 $\mathfrak{p}_j$-prime submodule of F for each $j \in J$. Let $i \in \{1, \dots, k\}$ and let P be a height 1 $\mathfrak{p}_i$-prime submodule of F such that $\cap_{j \in J} P_j \subseteq P$. Then $i \in J$ and $P = P_i$.*

Proof. Note first that $\prod_{j \in J} \mathfrak{p}_j \subseteq (\cap_{j \in J} P_j : F) \subseteq (P : F) = \mathfrak{p}_i$ so that $\mathfrak{p}_j \subseteq \mathfrak{p}_i$ for some $j \in J$ and hence $j = i$. Moreover,

$$\left(\prod_{s \in J \backslash \{i\}} \mathfrak{p}_s \right) P_i \subseteq \bigcap_{j \in J} P_j \subseteq P$$

gives $P_i \subseteq P$ and hence $P = P_i$. $\square$

Combining Lemmas 2.8 and 2.9 we see that a submodule S of F is a $*$-semiprime submodule if and only if there exist a unique nonempty subset J of $\{1, \dots, k\}$ and unique $\mathfrak{p}_j$-prime submodules P_j, $j \in J$, such that $S = \cap_{j \in J} P_j$. Now we come to our first main theorem.

Theorem 2.10. *Let R be a Noetherian ring and let F be of finite rank $n \geq 2$. Then $\zeta(F)$ is generated by $\{V(S) : S$ is a $*$-semiprime submodule of $F\}$.*

Proof. Let $x \in F \backslash \mathfrak{w}F$. Then $S(x) = \mathfrak{a}F \cap S^*(x)$ for some ideal $\mathfrak{a}$ of R. It can easily be checked that $V(S(x)) = V(\mathfrak{a}F \cap S^*(x)) = V(\mathfrak{a}S^*(x))$, i.e., $S(x) \sim \mathfrak{a}S^*(x)$. By Corollary 1.6 and Lemma 2.1, $\zeta(F)$ is generated by $\{V(S^*(x)) : x \in F \backslash \mathfrak{w}F\}$. $\square$

It is not clear in general how to "simplify" the generating set $\{V(S) : S$ is a $*$-semiprime submodule of $F\}$ for $\zeta(F)$. In the next section we shall look at a special case which includes the case of domains.

3. A special case. We begin this section with the following version of Theorem 2.10 in the special case when R has only one minimal prime ideal and in particular when R is a domain.

Theorem 3.1. *Let R be a Noetherian ring which has only one minimal prime ideal $\mathfrak{p}$, and let F be of finite rank $n \geq 2$. Then $\zeta(F)$ is generated by $\{V(P) : P$ is a height 1 $\mathfrak{p}$-prime submodule of $F\}$ but not by any proper subset.*

Proof. The first part follows by Theorem 2.10. Suppose that m is a positive integer, P and P_i, $1 \leq i \leq m$, are height 1 $\mathfrak{p}$-prime submodules of F and $\mathfrak{a}_i$, $1 \leq i \leq m$, are ideals of R such that $V(P) = V(\mathfrak{a}_1 P_1 + \cdots + \mathfrak{a}_m P_m)$. If $\mathfrak{a}_i \subseteq \mathfrak{p}$, $1 \leq i \leq m$, then $\mathfrak{a}_1 P_1 + \cdots + \mathfrak{a}_m P_m \subseteq \mathfrak{p}F$ so that $\mathfrak{p}F \in V(\mathfrak{a}_1 P_1 + \cdots + \mathfrak{a}_m P_m)$ but $\mathfrak{p}F \notin V(P)$. Thus there exists $1 \leq j \leq m$ such that $\mathfrak{a}_j \nsubseteq \mathfrak{p}$. Then $\mathfrak{a}_j P_j \subseteq P$ gives $P_j \subseteq P$ and hence $P = P_j$. It follows that no proper subset of $\{V(P) : P$ is a height 1 $\mathfrak{p}$-prime submodule of $F\}$ generates $\zeta(F)$. $\square$

In general, if R has minimal prime ideals $\mathfrak{p}_i$, $1 \leq i \leq k$, and $\zeta(F)$ is generated by $\{V(P) : P$ is a height 1 $\mathfrak{p}_i$-prime submodule of F for some $1 \leq i \leq k\}$, then $\zeta(F)$ is not generated by a proper subset. This can be seen by adapting the proof of Theorem 3.1.

Now suppose that R is a Noetherian UFD, for example, R could be the polynomial ring in a finite number of indeterminates over a field or over $\mathbf{Z}$. Let F be of finite rank $n \geq 2$ with basis $\{f_i : i \in I\}$. Let $0 \neq x \in F$. There exist a finite subset J of I and nonzero elements $x_j \in R$, $j \in J$, such that $x = \sum_{j \in J} x_j f_j$. Let d be the greatest common divisor of the elements x_j, $j \in J$. For each $j \in J$, there exists $y_j \in R$ such that $x_j = d y_j$. It can easily be checked that $P(x) = Ry$. We call the submodule Ry a *principal prime* submodule of F. Now Theorem 3.1 gives the following result.

Corollary 3.2. *Let R be a Noetherian* UFD *and let F be of finite rank $n \geq 2$. Then $\zeta(F)$ is generated by $\{V(P) : P$ is a principal prime submodule of $F\}$ but by no proper subset.*

We have already observed in Section 1 that if $V(N) = V(\mathfrak{a}_1 N_1 + \cdots + \mathfrak{a}_k N_k)$ for some positive integer k, ideals $\mathfrak{a}_i$, $1 \leq i \leq k$, of a ring R and submodules N, N_i, $1 \leq i \leq k$, of an R-module M, then the $\mathfrak{a}_i$'s are not unique in general. Even for a Noetherian UFD R, the submodules N_i are not unique. For, let R be a Noetherian UFD and let N be any proper nonzero submodule of a free R-module F of finite rank $n \geq 2$. There exist a positive integer m and elements $x_i \in N$, $1 \leq i \leq m$, such that $N = Rx_1 + \cdots + Rx_m$. Then it can easily be checked that $V(N) = V(P(x_1) + \cdots + P(x_m)) = V(P(x_1)) + \cdots + V(P(x_m))$. This can lead to an infinite number of ways of expressing $V(N)$ as a sum of varieties of principal primes, as the following example shows.

Example 3.1. Let R denote the ring $\mathbf{Z}$ of rational integers, let $F = R \oplus R$ and let $N = R(3,2) + R(3,5)$. Then N is a proper nonzero submodule of F and $V(N) = V(R(3,2)) + V(R(3+3n, 5+2n))$ for all positive integers n. Moreover, $R(3,2)$ and $R(3+3n, 5+2n)$ are principal prime submodules of F for all positive integers $n \not\equiv 2 \pmod 3$.

Proof. It is easy to check that N is a proper nonzero submodule of F and that $N = R(3,2) + R(3+3n, 5+2n)$ for all positive integers n. Thus $V(N) = V(R(3,2) + R(3+3n, 5+2n)) = V(R(3,2)) + V(R(3+3n, 5+2n))$ for all positive integers n.

Since 3 and 2 are coprime, it follows that $R(3,2)$ is a principal prime. Let n be any positive integer such that $n \not\equiv 2 \pmod 3$. Suppose that $3+3n$ and $5+2n$ are not coprime. Let p be any prime such that p divides both $3+3n$ and $5+2n$. Then p divides $3(1+n)$. Thus $p = 3$ or p divides $1+n$. But $p = 3$ implies that $5+2n \equiv 0 \pmod 3$, i.e., $n \equiv 2 \pmod 3$, a contradiction. Thus, p divides $1+n$ and hence p divides $3 = (5+2n) - 2(1+n)$, a contradiction. Thus $3+3n$ and $5+2n$ are coprime and hence $R(3+3n, 5+2n)$ is a principal prime. $\square$

In case the ring R has only one minimal prime ideal, we have the following uniqueness theorem.

Theorem 3.3. *Let R be a Noetherian ring with only one minimal prime ideal $\mathfrak{p}$ and let F be of finite rank $n \geq 2$. Let P_i, $i \in I$, denote the height 1 $\mathfrak{p}$-prime submodules of F. Let N_j, $j \in J$, be submodules of*

F. Then $\zeta(F)$ is generated by $\{V(N_j) : j \in J\}$ if and only if there exist pairwise disjoint finite subsets J_i, $i \in I$, of J such that $P_i \sim \sum_{j \in J_i} N_j$ for all $i \in I$. In this case, for each $j \in J_i$, there exists an ideal $\mathfrak{c}_j$ of R such that $N_j \sim \mathfrak{c}_j P_i$.

Proof. Suppose first that, for each $i \in I$, there exists a finite subset J_i of J such that $P_i \sim \sum_{j \in J_i} N_j$. Let N be any submodule of F. By Theorem 3.1 there exist a finite subset I' of I and ideals $\mathfrak{a}_i$, $i \in I'$, of R such that $N \sim \sum_{i \in I'} \mathfrak{a}_i P_i$. Applying Lemma 1.5, we have $N \sim \sum_{i \in I'} \mathfrak{a}_i \sum_{j \in J_i} N_j = \sum_{i \in I'} \sum_{j \in J_i} \mathfrak{a}_i N_j$. Thus $\zeta(F)$ is generated by $\{V(N_j) : j \in \cup_{i \in I} J_i\}$.

Conversely, suppose that $\{V(N_j) : j \in J\}$ is a set of generators of $\zeta(F)$. Let $i \in I$. Then $V(P_i) = V(\mathfrak{b}_1 N_{j(1)} + \cdots + \mathfrak{b}_m N_{j(m)})$ for some positive integer m, elements $j(s) \in J$, $1 \leq s \leq m$, and ideals $\mathfrak{b}_s$, $1 \leq s \leq m$, of R. Without loss of generality, we can suppose that $\mathfrak{b}_s \not\subseteq \mathfrak{p}$ and $N_{j(s)} \not\subseteq \mathfrak{p}F$ for all $1 \leq s \leq m$. Since $\mathfrak{b}_s N_{j(s)} \subseteq P_i$, it follows that $N_{j(s)} \subseteq P_i$, $1 \leq s \leq m$. Thus $V(P_i) = V(N_{j(1)} + \cdots + N_{j(m)})$, i.e., $P \sim N_{j(1)} + \cdots + N_{j(m)}$. We set $J_i = \{j(1), \ldots, j(m)\}$.

Suppose that i, i' are distinct elements of I. Then $P_i \cap P_{i'} = \mathfrak{p}F$ by Corollary 2.7, and hence $J_i \cap J_{i'}$ is empty. Thus, the finite sets J_i, $i \in I$, are pairwise disjoint.

Finally, suppose that $j \in J_i$ for some $i \in I$. Then $N_j \subseteq P_i$. By Theorem 3.1 there exist a finite subset I'' of I and ideals $\mathfrak{c}_t$, $t \in I''$, such that $N_j \sim \sum_{t \in I''} \mathfrak{c}_t P_t$ and $\mathfrak{c}_t \not\subseteq \mathfrak{p}$, $t \in I''$. Now $\sum_{t \in I''} \mathfrak{c}_t P_t \subseteq P_i$ gives $I'' = \{i\}$ by Corollary 2.7. Thus $N_j \sim \mathfrak{c} P_i$ for some ideal $\mathfrak{c}$ of R, as required. $\square$

Finally we determine when $\zeta(F)$ is generated by $\{V(P) : P$ is a height 1 $\mathfrak{p}_i$-prime submodule of F for some $1 \leq i \leq k\}$, where R is a Noetherian ring with minimal prime ideals $\mathfrak{p}_i$, $1 \leq i \leq k$, and F is of finite rank $n \geq 2$. We already know that this is the case when $k = 1$ (Theorem 3.1). First we prove a preliminary result.

Let R be any ring with pairwise comaximal prime ideals $\mathfrak{q}_i$, $1 \leq i \leq t$, for some positive integer t. For each $1 \leq i \leq t$, let $\hat{\mathfrak{q}}_i = \cap_{j \neq i} \mathfrak{q}_j$. Then the ideals $\{\hat{\mathfrak{q}}_i : 1 \leq i \leq t\}$ are also comaximal, as can be readily seen.

Lemma 3.4. *Let $\mathfrak{q}_i$, $1 \leq i \leq t$, be pairwise comaximal prime ideals of a ring R. Let M be an R-module, and let Q_i be a $\mathfrak{q}_i$-prime submodule of M for each $1 \leq i \leq t$. Then $V(Q_1 \cap \cdots \cap Q_t) = V(\hat{\mathfrak{q}}_1 Q_1 + \cdots + \hat{\mathfrak{q}}_t Q_t)$.*

Proof. Since $\hat{\mathfrak{q}}_1 Q_1 + \cdots + \hat{\mathfrak{q}}_t Q_t \subseteq Q_1 \cap \cdots \cap Q_t$, it follows that $V(Q_1 \cap \cdots \cap Q_t) \subseteq V(\hat{\mathfrak{q}}_1 Q_1 + \cdots + \hat{\mathfrak{q}}_t Q_t)$. Now suppose that $P \in V(\hat{\mathfrak{q}}_1 Q_1 + \cdots + \hat{\mathfrak{q}}_t Q_t)$. Let $\mathfrak{p} = (P : M)$. By the remark immediately preceding this lemma, there exists $1 \leq i \leq t$ such that $\hat{\mathfrak{q}}_i \not\subseteq \mathfrak{p}$. Then $\hat{\mathfrak{q}}_i Q_i \subseteq P$ gives $Q_i \subseteq P$ and hence $Q_1 \cap \cdots \cap Q_t \subseteq P$. Thus $P \in V(Q_1 \cap \cdots \cap Q_t)$. It follows that $V(\hat{\mathfrak{q}}_1 Q_1 + \cdots + \hat{\mathfrak{q}}_1 Q_1) \subseteq V(Q_1 \cap \cdots \cap Q_t)$. $\square$

Theorem 3.5. *Let R be a Noetherian ring with minimal prime ideals $\mathfrak{p}_i$, $1 \leq i \leq k$, for some positive integer $k \geq 2$. Then the following statements are equivalent.*

(i) *$R = R_1 \oplus \cdots \oplus R_k$ is a direct sum of rings R_i, $1 \leq i \leq k$, each having only one minimal prime ideal.*

(ii) *$\mathfrak{p}_i + \mathfrak{p}_j = R$ for all $1 \leq i < j \leq k$.*

(iii) *For any positive integer $n \geq 2$ and any free R-module F of rank n, $\zeta(F)$ is generated by $\{V(P) : P$ is a height 1 $\mathfrak{p}_i$-prime submodule of F for some $1 \leq i \leq k\}$.*

(iv) *There exists a positive integer $n \geq 2$ and a free R-module F of rank n such that $\zeta(F)$ is generated by $\{V(P) : P$ is a prime submodule of $F\}$.*

Proof. (i) $\Leftrightarrow$ (ii). This is a well-known consequence of the Chinese Remainder Theorem.

(ii) $\Rightarrow$ (iii). By Theorem 2.10 and Lemma 3.4.

(iii) $\Rightarrow$ (iv). Clear.

(iv) $\Rightarrow$ (ii). Suppose that (iv) holds and suppose that $\mathfrak{p}_1 + (\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k) \neq R$. Let $\mathfrak{r} = \sqrt{\mathfrak{p}_1 + (\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k)}$ and let $x \in F \backslash \mathfrak{r} F$. There exist a positive integer m, ideals $\mathfrak{a}_i$, $1 \leq i \leq m$, and distinct prime submodules P_i, $1 \leq i \leq m$, such that $V(S(x)) = V(\mathfrak{a}_1 P_1 + \cdots + \mathfrak{a}_m P_m)$ and $\mathfrak{a}_i P_i \not\subseteq \mathfrak{w} F$, $1 \leq i \leq m$, where again $\mathfrak{w} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$.

Suppose that $P_1 \not\subseteq P_i(x)$, $1 \leq i \leq k$. Then $\mathfrak{a}_1 P_1 \subseteq S(x) \subseteq P_i(x)$, by Lemma 2.4, gives $\mathfrak{a}_1 F \subseteq P_i(x)$ for all $1 \leq i \leq k$. Thus $\mathfrak{a}_1 \subseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k = \mathfrak{w}$, a contradiction. Thus we can suppose without loss of generality that $P_1 \subseteq P_1(x)$. Then $(P_1 : F) = \mathfrak{p}_1$ and $P_1 = \mathfrak{p}_1 F$ or $P_1 = P_1(x)$ by Lemmas 2.5 and 2.6. Next note that $\mathfrak{a}_1 P_1 \subseteq P_i(x)$, $2 \leq i \leq k$, by Lemma 2.4, so that $\mathfrak{a}_1 \subseteq \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k$. Since $\mathfrak{a}_1 P_1 \not\subseteq \mathfrak{w} F$, it follows that $P_1 = P_1(x)$. Note further that $\mathfrak{a}_1 P_1 \subseteq (\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k) P_1(x)$. A similar argument will show that, by rearranging if necessary, $\mathfrak{a}_i P_i \subseteq (\cap_{j \neq i} \mathfrak{p}_j) P_i(x)$, $2 \leq i \leq k$. It follows that $m = k$ and $V(S(x)) = V((\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k) P_1(x) + \cdots + (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{k-1}) P_k(x))$.

Since $x \in F \backslash \mathfrak{r} F$, it follows that there exists a prime ideal $\mathfrak{q}$ of R such that $\mathfrak{p}_1 + (\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k) \subseteq \mathfrak{q}$ but $x \notin \mathfrak{q} F$. Then $\mathfrak{q} F \in \operatorname{spec} F$ and $(\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k) P_1(x) + \cdots + (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{k-1}) P_k(x) \subseteq \mathfrak{q} F$. Thus $S(x) \subseteq \mathfrak{q} F$ and $x \in \mathfrak{q} F$, a contradiction. It follows that $\mathfrak{p}_1 + (\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k) = R$, i.e., $\mathfrak{p}_1 + \mathfrak{p}_i = R$, $2 \leq i \leq k$. Similarly, $\mathfrak{p}_i + \mathfrak{p}_j = R$ for all $2 \leq i < j \leq k$. $\square$

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