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SOME AMPLIFICATIONS AND A CORRECTION OF IARROBINO'S CONSTRUCTION OF LOCAL PARAMETERS ON $\mathrm{Hilb}^n_{\mathbf{P}^2_k}$

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ABSTRACT. Let k be an algebraically closed field of characteristic 0, $\mathbf{P}^2_k$ the projective plane over k and $\mathrm{Hilb}^n_{\mathbf{P}^2_k} = H^n$ the Hilbert scheme whose k-points $[Z]$ parameterize the 0-dimensional closed subschemes $Z \subseteq \mathbf{P}^2_k$ of length n. It is well known that H^n is a nonsingular variety of dimension $2n$. In his memoir [**8**], Iarrobino describes a procedure for constructing local parameters about any k-point $[Z] \in H^n$. The key step is to construct local parameters in case Z is supported at a single point of $\mathbf{P}^2_k$, which we can assume is the origin of a standard affine patch $\mathrm{Spec}\,(k[x,y]) \subseteq \mathbf{P}^2_k$); in this case, Iarrobino claims to construct a map $\phi : U \to H^n$ with source an open neighborhood of the origin 0 in the $2n$-dimensional affine space $\mathbf{A}^{2n}_k$, such that $0 \mapsto [Z] \in H^n$ and such that ϕ is étale at 0. Unfortunately, Iarrobino's procedure does not work for all Z (the map ϕ is sometimes left undefined); the subscheme defined by the ideal $(x^5, x^4y, y^2) \subseteq k[x,y]$ of colength 9 is an example in which his procedure fails. On the other hand, Iarrobino's procedure works very well for some important classes of ideals, including the ideals of "generic type" and the "fat point" ideals $(x,y)^r$, $r = 1, 2, \ldots$; in these cases, the map $\phi : A^{2n}_k \to H$ is globally defined and an open immersion.

1. Introduction. We fix an algebraically closed ground field k of characteristic 0. Let $\mathrm{Hilb}^n_{\mathbf{P}^2_k} = H^n$ denote the Hilbert scheme parameterizing zero-dimensional closed subschemes of the projective plane $\mathbf{P}^2_k$ having length n; if $Z \subseteq \mathbf{P}^2_k$ is such a closed subscheme, we write $[Z] \in H^n$ for the associated point. It is well known that H^n is irreducible and nonsingular of dimension $2n$ [**5**]; a new proof has recently been given by Haiman [**7**], who notes that H^n "has received particular attention" for these special properties, "neither of which is true for $\mathrm{Hilb}^n(\mathbf{A}^m)$ for general m."

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In his memoir [**8**], Iarrobino describes a procedure for constructing local parameters about any k-point $[Z] \in H^n$. The key step is to construct local parameters in case Z is supported at a single point of $\mathbf{P}_k^2$, which we can assume is the origin of a standard affine patch $\operatorname{Spec}(k[x,y])$; by localizing at the maximal ideal of the origin and then completing, we can write $Z = \operatorname{Spec}(R/I)$, where $R = k[[x,y]]$ is the ring of power series in x and y, and $I \subseteq R$ is an ideal of *colength* n, that is, $\dim_k(R/I) = n$ (Iarrobino studies families of such ideals in [**8**]). Iarrobino shows that I has a distinguished set of polynomial generators $f_0, \dots, f_d$, called "standard generators" (their construction may require an initial linear change of the local coordinates x, y). From these, he produces polynomials $f_0(X), \dots, f_d(X) \in k[x,y,X]$, where $X = \{X_{ij\mu}\}$ is a set of $2n$ indeterminates indexed to correspond to a particular k-basis of the tangent space $TH^n_{[Z]} \approx \operatorname{Hom}_R(I, R/I)$; the polynomials $f_l(X)$ specialize to the standard generators f_l when each $X_{ij\mu}$ is set to zero. Setting $I(X) = (f_0(X), \dots, f_d(X))$, we obtain a map

$$\operatorname{Spec}(k[x,y,X]/I(X)) \xrightarrow{F} \operatorname{Spec}(k[X]) = \mathbf{A}_k^{2n}$$

which Iarrobino shows is quasi-finite and flat over a neighborhood U of the origin 0 of its target.

If the map F is in addition finite over U, it defines a finite and flat family of zero-dimensional subschemes of $\operatorname{Spec}(k[x,y]) \subseteq \mathbf{P}_k^2$ having length n, which by the universal property of H^n corresponds to a map

$$\phi : U \longrightarrow H^n, \quad 0 \longmapsto [Z];$$

one can then show that the induced map on tangent spaces

$$TU_0 \xrightarrow{\phi_0'} TH^n_{[Z]}$$

is an isomorphism, which implies that ϕ is étale in a neighborhood of 0. The indeterminates $X_{ij\mu}$ then correspond to the desired local parameters at $[Z] \in H^n$. (The details of this construction are given in the proof of Proposition 3.1.)

Iarrobino asserts [**8**, Theorem 4.16, p. 72] that the map ϕ is always defined when F is quasi-finite over U, but this is not the case. In Section 5 we give examples of ideals I for which the map F is not finite in any neighborhood U of 0; indeed, the family of subschemes defined by F has members of the wrong length arbitrarily close to $Z = F^{-1}(0)$. Therefore, the map ϕ is not defined in these cases. On the other hand, in several important cases, the map F is globally finite over $\mathrm{Spec}\,(k[X])$ and the induced map

$$\phi : \mathrm{Spec}\,(k[X]) = \mathbf{A}_k^{2n} \longrightarrow H^n$$

is an open immersion, yielding an open neighborhood of $[Z]$ isomorphic to $\mathbf{A}_k^{2n}$. Ideals enjoying this good behavior include the ideals of "generic type" and the "fat point" ideals $(x, y)^r$, $r = 1, 2, \ldots$. There are also "intermediate" cases for which the map ϕ is only defined on a proper open subset $U \subseteq \mathrm{Spec}\,(k[X])$; an example is given in Section 5.

Having provided an overview of the paper's contents, we turn next to a section-by-section outline of its structure. We lay the groundwork on which we build in Sections 2 and 3: the first of these sections summarizes the concepts and notation from [**8**] that are needed for a thorough understanding of Iarrobino's approach to local parameters; the second gives a detailed proof that Iarrobino's construction succeeds when F is finite over U, and presents a free resolution of the ring $\Gamma(F^{-1}(U))$ that is needed in the sequel.

Our main results reside in Section 4, which is devoted to a study of conditions under which the map $\phi : U \to H^n$ is an open immersion and/or globally defined. Theorem 4.2 shows that ϕ is an open immersion provided that F is finite over U and each of the generators $f_l(X)$ has the "pullback uniqueness property," see Definition 4.1; note that this property depends on U. We establish a convenient condition for verifying that these hypotheses hold globally in Proposition 4.5; namely, if each of the $f_l(X)$ has "good form", see Definition 4.3, then the hypotheses of Theorem 4.2 hold for $U = \mathrm{Spec}\,(k[X])$. Consequently, the map ϕ is a globally defined open immersion, Corollary 4.6. Proposition 4.8 then gives a simple condition, depending only on the "type," see Section 2, of the original ideal $I \subseteq R$, for verifying that the $f_l(X)$ all have good form. In particular, this condition holds when I is of "generic type" or a "fat point" ideal, Examples 4.12 and 4.13.

The final section of the paper comprises several examples, some of which have already been mentioned. Examples 5.1, 5.2 and 5.3 show that Iarrobino's procedure does not always lead to a well-defined map $\phi : U \to H^n$. Example 5.5 shows that the condition that all the generators $f_l(X)$ of $I(X)$ have "good form," although sufficient, is not necessary for the conclusion that the map ϕ be a globally defined open immersion. The final example 5.6 shows that the map ϕ is sometimes defined only on a proper subset $U \subset \operatorname{Spec}(k[X])$; in this example, ϕ turns out to be an open immersion on U. We have no example in which the map $\phi : U \to H^n$ is defined and not an open immersion; we end the paper with the conjecture that no such example exists.

We note that the aforementioned paper [**7**] of Haiman includes another approach to local parameters on H^n: for any point $[Z] \in H^n$ that corresponds to a *monomial* ideal $I \subseteq k[x,y]$, Haiman exhibits a list of $2n$ functions that give a k-basis of $\mathfrak{m}_{[Z]}/\mathfrak{m}_{[Z]}^2$. It follows that H^n is smooth at all such $[Z]$, which are the fixed points of a certain torus action on H^n; from this Haiman deduces that H^n is everywhere smooth. In [**3**, Section 2.5], Cheah constructs an explicit basis of the tangent space $TH^n_{[Z]} \approx \operatorname{Hom}_R(I, R/I)$ for Z again defined by a monomial ideal I; from this basis, one can recover Haiman's local parameters at $[Z]$.

By contrast with these results, Iarrobino's tangent space basis and local parameter construction are not restricted to monomial ideals; unfortunately, however, the latter construction does not always work, not even for monomial ideals (the counterexamples given in Section 5 are all monomial ideals). It is of interest to construct local parameters at all points of H^n. The positive results of this paper, which give conditions under which the map

$$\phi : \operatorname{Spec}(k[X]) = \mathbf{A}_k^{2n} \longrightarrow H^n$$

is a globally-defined open immersion, are a step in this direction, since in such cases, the indeterminates $X_{ij\mu} \in X$ give rise to local parameters at all points $[Z] \in \operatorname{im}(\phi)$.

In a forthcoming paper we give a description of certain open affine subschemes whose union covers H^n (these are the U_μ discussed in [**7**]); in particular, we express the coordinate ring $\mathcal{O}_{U_\mu}$ explicitly as a quotient of a polynomial ring. Generalizing the results proved here, we show that in certain cases U_μ is an affine cell, that is, isomorphic to $\mathbf{A}_k^{2n}$, with coordinate ring generated by functions giving the Haiman local parameters at the cell's "origin" $[Z]$, a point corresponding to a monomial ideal. In these cases we again obtain local parameters at all points of U_μ.

2. Summary of Iarrobino's approach to local parameters on $\mathrm{Hilb}^n_{\mathbf{P}^2_k}$. In this section of the paper, we summarize the construction, given in Iarrobino's memoir [**8**], of local parameters about a k-point $[Z]$ of $\mathrm{Hilb}^n_{\mathbf{P}^2_k} = H^n$. Recall at the outset that the ground field k is algebraically closed and of characteristic 0. We first observe that it suffices to construct local parameters whenever Z, the length-n closed subscheme of $\mathbf{P}^2_k$ corresponding to $[Z]$, is supported at a single point P. Indeed, if $\mathrm{Supp}\,(Z) = \{P_1, \dots, P_r\}$, then Z may be written as the coproduct of r subschemes $Z_1, Z_2, \dots, Z_r$ having lengths $n_1, n_2, \dots, n_r$, respectively, with $\sum_{j=1}^r n_j = n$. The natural rational map $\prod_{j=1}^r H^{n_j} \to H^n$ is defined and étale in a neighborhood of the point $([Z_1], [Z_2], \dots, [Z_r])$ lying over $[Z]$; if we can construct local parameters about each point $[Z_j] \in H^{n_j}$, we can in turn construct local parameters about $[Z] \in H^n$ [**8**, p. 74].

Consider therefore a closed subscheme $Z \subseteq \mathbf{P}^2_k$ of length n concentrated at a point P which we may, without loss of generality, take to be the origin of a standard affine patch $\mathrm{Spec}\,(k[x,y])$ of $\mathbf{P}^2_k$. Passing to the completion of the local ring $\mathcal{O}_P$ at its maximal ideal, we associate Z to an ideal I of colength n in the ring of power series $k[[x,y]]$. Families of such ideals are the main focus of Iarrobino's memoir [**8**], from which we now recall the concepts and results needed in the sequel.

2.1 Basic concepts: Type, pattern, standard generators. Let $\mathfrak{m} = (x,y)$ denote the maximal ideal of $R = k[[x,y]]$, $R_i = \mathfrak{m}^i/\mathfrak{m}^{i+1}$ the vector space of forms of degree i and $I_i = (\mathfrak{m}^i \cap I)/(\mathfrak{m}^{i+1} \cap I)$, the vector space of forms of degree i which are *initial forms*, i.e., of minimal degree, of elements of I. The *type* of I is the Hilbert function of R/I, that is, the sequence

$$T(I) = (t_0, t_1, \dots, t_j, \dots) \quad \text{where } t_j = \dim_k(R_j/I_j).$$

The *initial degree* of I is the least j such that $I_j \neq 0$ or, equivalently, such that $t_j < j+1$. For I of colength n, one has that $\mathfrak{m}^n \subseteq I$ [**8**, Lemma 1.1, p. 6]; consequently, $t_j = 0$ for $j \geq n$. Furthermore, one has that

(1)
$$T(I) = (1, 2, \dots, d, t_d, \dots, t_{n-1}, 0, 0, \dots), \quad \text{where } 0 \leq t_j \leq j+1,$$
$$\sum t_j = n \quad \text{and} \quad d \geq t_d \geq \cdots \geq t_{n-1}$$

[**8**, 1.2, p. 1].

We next describe a distinguished basis of the k-vector space R/I, with respect to which we can choose a distinguished set of generators of I. Let P be a set of monomials in the variables x, y, $\langle P \rangle$ the k-span of P in R, in which infinite sums are allowed if P is infinite, and $P_j = P \cap R_j$ the subset of monomials in P of degree j. The *type* of P is by definition the sequence $T(P) = (\#P_0, \#P_1, \dots)$. We say that I has *pattern* P if any one of the following equivalent conditions is satisfied [**8**, Lemma 1.4, p. 9]:

$$\begin{aligned} &\text{(i)} \quad \text{for all } j,\ \langle P \cap \mathfrak{m}^j \rangle \oplus (I \cap \mathfrak{m}^j) = \mathfrak{m}^j; \\ &\text{(ii)} \quad \text{for all } j,\ \langle P_j \rangle \oplus I_j = R_j; \\ &\text{(iii)} \quad \langle P \rangle \cap I = \varnothing \quad \text{and} \quad T(P) = T(I). \end{aligned} \tag{2}$$

In particular, if I has pattern P, then P is a basis of R/I; we call the monomials in P *pattern monomials*.

We now order the monomials using the graded lexicographical order with $x > y$, that is,

$$0 < y < x < y^2 < xy < x^2 < \cdots . \tag{3}$$

With respect to this order, the *normal pattern* of type $T = (t_0, t_1, \dots)$ is the set of polynomials P such that P_j consists of the last t_j monomials of degree j. For example, the pattern

$$P = \{1, x, y, x^2, xy, x^3\} \tag{4}$$

is the normal pattern of type $T = (1, 2, 2, 1, 0, 0, \dots)$. It can be helpful to visualize normal patterns within the "Pascal triangle" of monomials, as shown (the monomials in P are enclosed in a rectangular polygon; the parenthesized monomials will be explained shortly):

$$\begin{array}{ccccc} & & & & y^4 \\ & & & y^3 & xy^3 \\ & & (y^2) & xy^2 & x^2y^2 \\ & y & xy & (x^2y) & x^3y \\ 1 & x & x^2 & x^3 & (x^4) \end{array} \tag{5}$$

Conveniently, in this arrangement, the pattern monomials fall into columns that form the "bar graph" of the type T.

Let $I \subseteq R$ be an ideal of type T, initial degree d, and colength n. If $\operatorname{char}(k) = 0$, as we are assuming, or is "sufficiently large," Iarrobino proves that an ordered set of local parameters (y, x) for R can be chosen, from a finite list obtained by linear changes of the original coordinates, such that I has the normal pattern P of type T with respect to the corresponding ordered system of monomials (3) [**8**, Proposition 3.2, p. 32]. Assuming this is done, we may construct the unique set of *standard generators* $f_0, f_1, \dots, f_d$ of I, corresponding to P, as follows [**8**, pp. 10–12]: For $0 \le l \le d$, let $x^{k_l}y^l$ be the leftmost monomial on the lth row of the Pascal triangle which is not in P; we call these the *leading monomials*. For example, in (5), the leading monomials have been enclosed in parentheses; note that k_l is equal to the number of pattern monomials on the lth row. We then deduce from (2) that the lth leading monomial is congruent mod I to a unique k-linear combination of elements in $P \cap \mathfrak{m}^{k_l+l}$; that is, the ideal I contains $d+1$ uniquely determined polynomials of the form

$$f_l = x^{k_l}y^l + \text{ an element of } \langle P \cap \mathfrak{m}^{k_l+l} \rangle, \quad 0 \le l \le d. \tag{6}$$

By [**8**, Theorem 1.5, p. 11], the f_l generate I. We remark that the leading monomial $x^{k_l}y^l$ is minimal with respect to the monomial order (3) among the monomials which appear with nonzero coefficients in f_l; furthermore, the standard generators are indexed by the y-degrees of their leading monomials. Combining the definition with the conditions (1), one finds that $f_0 = x^{k_0}$, where $k_0 = j + 1$ is one greater than the index of the last nonzero entry t_j in the type of I, and that, as l increases from 0 to d, the total degree of the leading term decreases, not necessarily strictly, from k_0 down to d.

As an example, let P be the normal pattern displayed in (4) and (5). Then, for each choice of scalars p, q, r, s, we obtain an ideal $I_{(p,q,r,s)}$ having normal pattern P and standard generators

$$\begin{aligned} f_0 &= x^4, \\ f_1 &= x^2y + px^3, \\ f_2 &= y^2 + qxy + rx^2 + sx^3; \end{aligned} \tag{7}$$

moreover, it can be shown that the map $(p, q, r, s) \mapsto I_{(p,q,r,s)}$ is one-to-one. It follows that the family of all ideals of type $T = T(P) =$

$(1,2,2,1)$ in R is a (connected) union of four-dimensional affine spaces. Note that the coefficients cannot always be chosen arbitrarily, as in this example; in general, some of the coefficients can be chosen freely and the others are then determined. In any case, Iarrobino shows in this way that the locus of all ideals of a given type in R is a connected union of affine spaces of dimension equal to the number of coefficients which can be freely chosen. See [**8**, Chapter 2] for a thorough discussion of these results; similar results are proved by Briançon in [**2**].

2.2 Standard presentation of I, tangent space at $[Z]$. Let $I \in R$ be an ideal having normal pattern of type T with respect to the monomial order (3), and let $f_0, f_1, \dots, f_d$ be the standard generators (6). The latter satisfy d, uniquely determined, relations of the form

$$-yf_{i-1} + x^{w_i} f_i + \sum_{j=0}^{i-1} \alpha_{i,j} f_j = 0, \tag{8}$$

where $1 \le i \le d$, $w_i = k_{i-1} - k_i > 0$, and $\alpha_{i,j} \in k[x]$; in fact, Iarrobino proves that these relations freely generate the full syzygy module of relations among the f's [**8**, pp. 50–52]. To express this in another way, let M denote the $d \times (d+1)$ matrix of coefficients, that is,

$$M = \begin{bmatrix} \alpha_{1,0} - y & x^{w_1} & 0 & 0 & \cdots \\ \alpha_{2,0} & \alpha_{2,1} - y & x^{w_2} & 0 & \cdots \\ \vdots & & \ddots & & \\ \alpha_{d,0} & \cdots & \alpha_{d,d-2} & \alpha_{d,d-1} - y & x^{w_d} \end{bmatrix}, \tag{9}$$

and let N denote the $(d+1) \times 1$ matrix obtained by listing the standard generators in a column. Then R/I has the following free resolution of length 2, we view the elements of R^d and R^{d+1} as row vectors on which the matrices act by right multiplication:

$$0 \longrightarrow R^d \xrightarrow{M} R^{d+1} \xrightarrow{N} R \longrightarrow R/I \longrightarrow 0. \tag{10}$$

By a special case of the Hilbert-Burch theorem, see, e.g., [**4**, pp. 501–503] or [**11**, pp. 670–673], the "f_l are up to sign the $d \times d$ minors of M, and the above presentation is typical of that for a Cohen-Macaulay ideal of depth 2 in a local ring..." [**8**, p. 52]. Precisely, if we let r_j

denote the jth row of the matrix M (9), e_l the standard unit vector with 1 in the lth position, and $(e_l, r_1, \ldots, r_d)$ the $(d+1)\times(d+1)$ matrix having the indicated rows, then we have that

$$f_l = \det(e_l, r_1, r_2, \ldots, r_d). \tag{11}$$

For example, consider, for a given choice of the scalars, the ideal $I_{(p,q,r,s)}$ with standard generators f_0, f_1, f_2 defined by (7). One checks easily that f_0, f_1, f_2 satisfy the relations given by the rows of the matrix

$$M = \begin{bmatrix} -px - y & x^2 & 0 \\ -r + p(q-p) - sx & (p-q)x - y & x^2 \end{bmatrix}, \tag{12}$$

and that the f's are recovered using formula (11).

It is well known that the tangent space at the k-point $[Z] \in H^n$ corresponding to the ideal I is given by the module

$$TH^n_{[Z]} = \mathrm{Hom}_R(I, R/I). \tag{13}$$

By a result of Schaps [**11**, Corollary 3, p. 677], the latter module is generated as an R-module by the entries of a $d \times (d+1)$ matrix $(\theta_{i,j})$ defined as follows:

$$\theta_{i,j}(f_l) = \det(e_j, r_1, \ldots, r_{i-1}, e_l, r_{i+1}, \ldots, r_d) \bmod I \tag{14}$$

(it suffices to define each map $\theta_{i,j}$ on the generators $f_0, f_1, \ldots, f_d$ of I in a way that respects the relations $\{r_i\}$ among the f's). Iarrobino extends this result by proving that the set

$$\{x^\mu \cdot \theta_{i,j} \mid 0 \le \mu < w_{\max(i,j)},\ 1 \le i \le d,\ 0 \le j \le d\} \tag{15}$$

is a k-basis of $TH^n_{[Z]}$ (the $\{w_i\}$ are those appearing in the relations (8)); he shows that this set, of cardinality

$$\dim_k(TH^n_{[Z]}) = \dim(H^n) = 2n,$$

is linearly independent over k [**8**, Theorem 4.15, p. 69].

2.3 Construction of local parameters at $[Z] \in \mathrm{Hilb}^n_{\mathbf{P}^2_k}$. Let I, Z, etc., be as in the preceding paragraphs, and let U denote an open neighborhood of the origin $0 \in \mathbf{A}_k^{2n}$. Our goal is to construct local parameters at $[Z] \in \mathrm{Hilb}^n_{\mathbf{P}_k^2} = H^n$; that is, we seek to construct a map $\phi : U \to H^n$ with $0 \mapsto [Z]$ such that ϕ is étale in a neighborhood of 0. By the universal property of H^n, the map ϕ will correspond to a family of subschemes $W \subseteq U \times \mathbf{P}_k^2$ that is finite and flat of rank n over U. Here is Iarrobino's procedure for constructing such a family [**8**, p. 72] (modulo slight notational changes):

> We choose the local parameters x, y on $\mathbf{P}_k^2$ at P [the point at which Z is supported] so that $\mathrm{Spec}\,(k[x,y])$ describes an affine plane neighborhood $\mathbf{A}_k^2$ of P in $\mathbf{P}_k^2$. We now regard the matrix M [i.e., matrix (9)] for I as having its entries in $k[x,y]$, the polynomial ring. Given $\beta = \{\beta_{ij\mu}\}$ with $\beta_{ij\mu} \in k$ and i, j, μ satisfying [the inequalities in (15)], consider the matrix $M(\beta)$ whose entries are
>
> $$M(\beta)_{i,j} = M_{i,j} + \sum_{\mu} \beta_{ij\mu} x^{\mu}$$
>
> and with rows $r_i(\beta)$. $M(\beta)$ also has entries in $k[x,y]$. We let $I(\beta) \subseteq k[x,y]$ be the ideal
>
> $$\begin{aligned} I(\beta) &= (f_0(\beta), \dots, f_d(\beta)), \\ f_i(\beta) &= \det(e_i, r_1(\beta), \dots, r_d(\beta)) \end{aligned}$$
>
> generated by the $d \times d$ minors of $M(\beta)$. The ideal $I(\beta)$ is Cohen-Macaulay by Schaps [[11]], since it is determinantal. We let $k[X]$ denote the polynomial ring on variables $X_{ij\mu}$ corresponding to $\beta_{ij\mu}$, with i, j, μ [satisfying the inequalities in (15)]. Then we have a morphism F
>
> $$\mathrm{Spec}\,(k[x,y,X]/I(X)) \xrightarrow{F} \mathrm{Spec}\,(k[X]) \approx \mathbf{A}_k^{2n}$$
>
> whose fiber over the closed point β is $\mathrm{Spec}\,(k[x,y]/I(\beta))$. By Schaps [[11]], Corollary 2, and its proof, F is flat over any neighborhood U of 0 in $\mathbf{A}_k^{2n}$ for which the closed fibers of F have codimension two. This fact depends on the following lemma [**11**, Lemma, p. 675]:
>
> **Lemma 2.1.** *Let $f : Z \to Y$ be a morphism of algebraic schemes, f of finite type, Z Cohen-Macaulay, Y regular, and the closed fibers equidimensional. Then f is flat.*

Iarrobino proceeds to demonstrate the existence of a neighborhood U of $0 \in \operatorname{Spec}(k[X])$ over which the closed fibers of F have codimension 2, that is, F is quasi-finite over U, and argues that the resulting map $\phi : U \to H^n$ will be étale at 0 since, by construction, ϕ induces an isomorphism of tangent spaces there [**8**, Theorem 4.16, p. 72].

Unfortunately, there is a gap in Iarrobino's argument: the map F, even if quasi-finite and flat near 0, need not be finite there, meaning that the map ϕ need not be defined. In Section 5 we give three examples in which the family of subschemes defined by F contains members of the wrong length arbitrarily close to $F^{-1}(0) = Z$. On the other hand, Iarrobino's procedure works very well—that is, the map $\phi : \mathbf{A}_k^{2n} \to H^n$ is globally defined and an open immersion—in several important cases, including the ideals of "generic type" $T = (1, 1, \ldots, 1, 0, 0, \ldots)$ and the "fat point" ideals $I = \mathfrak{m}^r$, $r = 1, 2, \ldots$; this is the subject of Section 4. We continue our preparations for these results in the next section.

3. Consequences of (quasi)finiteness for F. At the end of the previous section, we asserted that Iarrobino's construction of the map ϕ sometimes fails because the map F can fail to be finite over any neighborhood of $0 \in \mathbf{A}_k^{2n}$. Our first goal in this section is to confirm that Iarrobino's construction succeeds whenever F is finite over some neighborhood U of zero. Following this, we present a free resolution of the A-module $\Gamma(F^{-1}(U)) = B$ which is needed in Section 4, see (18) for the notation; this resolution exists whenever F is quasi-finite over U.

Let $I \subseteq k[[x, y]]$ be an ideal of colength n having type T, normal pattern P with respect to the monomial order (3), associated standard generators $f_0, \ldots, f_d$, and matrix of relations M, as in Sections 2.1 and 2.2. Following Iarrobino's procedure quoted in Section 2.3, we introduce the set of variables $X = \{X_{ij\mu}\}$, form the matrix $M(X) = (r_1(X), \ldots, r_d(X))$, with entries given by

$$M(X)_{i,j} = M_{i,j} + \sum_{\mu=0}^{w_{\max(i,j)}-1} X_{ij\mu} x^{\mu}, \tag{16}$$

and define, for $0 \leq l \leq d$,

$$f_l(X) = \det(e_l, r_1(X), \ldots, r_d(X)). \tag{17}$$

We set

$$I(X) = (f_0(X), \dots, f_d(X)) \subseteq k[x, y, X],$$

and consider the map

$$\mathrm{Spec}\,(k[x, y, X]/I(X)) \xrightarrow{F} \mathrm{Spec}\,(k[X]).$$

If there exists an open set U of the target over which F is finite and flat of degree n, then F induces a map

$$\phi : U \to \mathrm{Hilb}^n_{\mathbf{P}^2_k} = H^n,$$

by the universal property of H^n. We of course want U to contain the origin $0 \in \mathrm{Spec}\,(k[X]) = \mathbf{A}^{2n}_k$, which will then map under ϕ to the k-point $[Z] \in H^n$ corresponding to the ideal I; we also want ϕ to be étale at zero. To achieve these ends, it suffices that U be a neighborhood of 0 over which the map F is finite; we state this as

Proposition 3.1. *Let $U = \mathrm{Spec}\,(C)$ be an open neighborhood of the origin $0 \in \mathrm{Spec}\,(k[X])$ over which the map F is finite. Then F is also flat over U of rank n, and the induced map $\phi : U \to H^n$ is étale at zero.*

Remark 3.2. The proof which follows is merely a fleshing out of the arguments given by Iarrobino, which we sketched in Section 2.3.

Proof. We write

$$\begin{aligned} Z(U) &= F^{-1}(U), \\ A &= C[x, y], \\ B &= \Gamma(Z(U)) = A/(f_0(X), \dots, f_d(X)). \end{aligned} \tag{18}$$

Since F is by hypothesis finite over U, and therefore quasi-finite, we have that every component of $Z(U)$ has codimension ≥ 2 (the fibers of F over U are all zero-dimensional). On the other hand, because the ideal $I(X)$ is defined by the $d \times d$ subdeterminants of a $d \times (d+1)$ matrix, every component of $Z(U)$ has codimension $\leq (d-d+1)((d+1)-d+1) = 2$, see, e.g., [**4**, Example 10.9, p. 244]; whence, every component has codimension equal to $(d-d+1)((d+1)-d+1)$ and $Z(U)$ is therefore a determinantal scheme. It now follows from the theorem of Eagon and Hochster, see, e.g., [**11**, p. 670] or [**4**, p. 463], that

$$Z(U) = \operatorname{Spec}(B) \quad \text{is } \textit{Cohen-Macaulay.} \tag{19}$$

Therefore, by Lemma 2.1, we have that the *map*

$$F|_{Z(U)} : Z(U) \longrightarrow U \quad \textit{is flat.} \tag{20}$$

Since $F|_{Z(U)}$ is finite and flat, we have that the ring B is locally free over C, by [**10**, Proposition 7, p. 43]. To compute the rank of B over C, tensor with the residue field k of the origin $0 \in U$:

$$B \otimes_C k \approx k[x,y]/I(0) = \Gamma(\mathcal{O}_Z).$$

Since Z, the subscheme corresponding to the ideal I we started with, has length n, and $U \subseteq \mathbf{A}_k^{2n}$ is connected, we see that

$$B \textit{ is locally free of (constant) rank } n \textit{ over } C. \tag{21}$$

The universal property of the Hilbert scheme H^n now yields a map $\phi : U \to H^n$; by definition, the image of the k-point $\beta = (\beta_{ij\mu}) \in U$ is

$$\phi(\beta) = [Z(\beta)],$$
$$Z(\beta) = \operatorname{Spec}(B \otimes_C k_\beta) = \operatorname{Spec}(k[x,y]/I(\beta)),$$

where k_β denotes k viewed as the residue field of β, and $I(\beta)$ the ideal obtained from $I(X)$ by replacing each $X_{ij\mu}$ with $\beta_{ij\mu}$. It remains to prove that ϕ is étale at $0 \in U$. Since the source and target of ϕ are nonsingular schemes of the same dimension, it suffices to show that the induced map on tangent spaces

$$TU_0 \xrightarrow{\phi_0'} TH^n_{[Z]}$$

is surjective (and therefore an isomorphism) [**1**, Proposition 5.2, p. 148]. To do this, we will show that each of the elements in Iarrobino's k-basis (15) of $TH^n_{[Z]}$ is in the image of ϕ_0'; indeed, we claim that

$$x^\mu \cdot \theta_{i,j} = \phi_0'\left(-\frac{\partial}{\partial X_{ij\mu}}\right) \tag{22}$$

for $0 \leq \mu < w_{\max(i,j)}$, $1 \leq i \leq d$, $0 \leq j \leq d$.

Proof of claim. Let $k[\varepsilon]$ be the dual numbers, i.e., $\varepsilon^2 = 0$. We will express the lefthand side and righthand side of (22) as maps $\mathrm{Spec}\,(k[\varepsilon]) \to H^n$, that is, as families of subschemes over $\mathrm{Spec}\,(k[\varepsilon])$, and observe that these families are the same.

We first consider the lefthand side. The proof of (13) shows that the tangent vector $v \in \mathrm{Hom}_R(I, R/I)$, when viewed as a map $\mathrm{Spec}\,(k[\varepsilon]) \to H^n$, corresponds to the closed subscheme of $\mathrm{Spec}\,(k[\varepsilon][x,y])$ cut out by the ideal
$$(f_0 + \varepsilon \cdot v(f_0), \dots, f_d + \varepsilon \cdot v(f_d)),$$
see, e.g., [**11**, p. 677]. Therefore, the lefthand side of (22) corresponds to the ideal

$$(f_0 + \varepsilon \cdot x^\mu \cdot \theta_{i,j}(f_0), \dots, f_d + \varepsilon \cdot x^\mu \cdot \theta_{i,j}(f_d)). \tag{23}$$

We now consider the righthand side. First note that the tangent vector $v = -(\partial/\partial X_{ij\mu})$ at $0 \in U \subseteq \mathrm{Spec}\,(k[X])$ corresponds to the map $\mathrm{Spec}\,(k[\varepsilon]) \to \mathrm{Spec}\,(k[X])$ which is given on rings by

$$X_{ij\mu} \longmapsto -\varepsilon \quad \text{and} \quad X_{i'j'\mu'} \longmapsto 0 \quad \text{otherwise.} \tag{24}$$

The composition
$$\mathrm{Spec}\,(k[\varepsilon]) \xrightarrow{v} U \xrightarrow{\phi} H^n,$$
by definition of ϕ, corresponds to the closed subscheme of $\mathrm{Spec}\,(k[\varepsilon][x,y])$ cut out by the pullback of the ideal $(f_0(X), \dots, f_d(X))$; that is, the ideal $(f_0(\varepsilon), \dots, f_d(\varepsilon))$ obtained by making the substitutions (24). Recalling the definition (17) of the $f_l(X)$, and that the row r_p is obtained from the row $r_p(X)$ by setting $X_{pq\mu}$ to zero for all q, μ, we see that
$$f_l(\varepsilon) = \det(e_l, r_1, \dots, r_{i-1}, r_i + -\varepsilon \cdot x^\mu \cdot e_j, r_{i+1}, \dots, r_d).$$
Using linearity of the determinant in the ith row, the determinantal description (11) of the standard generators f_l, and the definition (14), we obtain
$$\begin{aligned} f_l(\varepsilon) &= f_l - \varepsilon \cdot x^\mu \cdot \det(e_l, r_1, \dots, r_{i-1}, e_j, r_{i+1}, \dots, r_d) \\ &= f_l + \varepsilon \cdot x^\mu \cdot \det(e_j, r_1, \dots, r_{i-1}, e_l, r_{i+1}, \dots, r_d) \\ &= f_l + \varepsilon \cdot x^\mu \cdot \theta_{i,j}(f_l). \end{aligned}$$

In other words, the generators of the ideal corresponding to the righthand side are the same as the generators of the ideal (23) corresponding to the lefthand side; whence, the claim.

The proof of the proposition is now complete. □

Remark 3.3. In Theorem 4.2, we give a condition under which the map $\phi : U \to H^n$ is an open immersion.

Our second goal in this section is to present a free resolution of the A-module B which specializes to the free resolution (10) of $k[[x,y]]/I$. This resolution exists whenever the map F is quasi-finite over $U = \mathrm{Spec}\,(C)$ (we continue to use the notation (18)); the quasi-finiteness ensures, as in the proof of Proposition 3.1, that B is Cohen-Macaulay (19). As noted in Section 2.3, Iarrobino showed that the map F is quasi-finite over some neighborhood U of the origin in $\mathrm{Spec}\,(k[X])$ for every I; therefore, the resolution is available in every case, not just when F is finite over U. In addition to being interesting for its own sake, the resolution plays an important role in the next section.

We begin by defining the maps

$$(25)\qquad \begin{aligned} &\alpha : A^d \longrightarrow A^{d+1}, \quad (a_1, \dots, a_d) \longmapsto (a_1, \dots, a_d) \cdot M(X), \\ &\delta : A^{d+1} \longrightarrow A, \quad\ (a_1, \dots, a_{d+1}) \longmapsto \sum_{i=0}^{d} a_i \cdot f_i(X); \end{aligned}$$

recall that $M(X) = (r_1(X), \dots, r_d(X))$ is the $d \times (d+1)$ matrix (16) whose $d \times d$ minors yield the generators $f_l(X)$ up to sign. These fit into the sequence

$$(26)\qquad A^d \xrightarrow{\alpha} A^{d+1} \xrightarrow{\delta} A \longrightarrow B = A/(f_0(X), \dots, f_d(X)) \longrightarrow 0.$$

We will show that α is injective and that

$$\mathrm{im}\,(\alpha) = \ker\,(\delta), \text{ the first syzygy module of the } \{f_l(X)\};$$

this will yield the desired free resolution of B.

To prove that α is injective, first recall from the construction of $M(X)$ that its ith row has the form

$$(27)\qquad \begin{aligned} r_i(X) = (&g_{i,0}(x), g_{i,1}(x), \dots, -y + g_{i,i-1}(x), g_{i,i}(x), \\ &g_{i,i+1}(x), \dots, g_{i,d}(x)), \end{aligned}$$

where $g_{i,j}(x) \in C[x]$; in particular, the only appearance of the variable y in $r_i(X)$ occurs in the $(i-1)$st component, as shown (the term x^{w_i} is included in $g_{i,i}(x)$). Choose a nonzero element $(\gamma_1, \dots, \gamma_d) \in A^d$ and consider its image under α, that is,

$$\gamma_1 \cdot r_1(X) + \gamma_2 \cdot r_2(X) + \cdots + \gamma_d \cdot r_d(X) \in A^{d+1}. \tag{28}$$

Viewing the elements of $A = C[x, y]$ as polynomials in y over $C[x]$, choose $\gamma_j \neq 0$ of maximum y-degree among the γ_i's and consider the $(j-1)$th component of (28):

$$(-y + g_{j,j-1}(x)) \cdot \gamma_j + \sum_{i \neq j} (g_{i,j-1}(x) \cdot \gamma_i).$$

A moment's reflection shows that the term of maximum y-degree in $-y \cdot \gamma_j$ cannot cancel out of this sum, which implies that $\alpha(\gamma_1, \dots, \gamma_d) \neq 0$, that is, α is injective.

Remark 3.4. The argument just given, borrowed from [**8**, Proof of Proposition 4.1, p. 51], shows that any nontrivial A-linear combination of the rows $r_i(X)$ must involve y nontrivially in at least one component, in fact, with index $0 \leq j \leq d-1$.

To prove that $\mathrm{im}\,(\alpha) = \ker(\delta)$ in (26), we first observe that $\mathrm{im}\,(\alpha) \subseteq \ker(\delta)$ since, for each i, $1 \leq i \leq d$, we have that

$$\begin{aligned} \delta(r_i(X)) &= r_i(X) \cdot (f_0(X), \dots, f_d(X)) \\ &= \det(r_i(X), r_1(X), r_2(X), \dots, r_d(X)) = 0 \end{aligned}$$

(use linearity of the determinant in the first row and recall the definition of the $f_l(X)$). Next we consider the exact sequence

$$0 \longrightarrow \ker(\delta) \longrightarrow A^{d+1} \xrightarrow{\delta} A \longrightarrow B \longrightarrow 0$$

and its localization

$$0 \longrightarrow \ker(\delta)_{\mathfrak{M}} \longrightarrow A^{d+1}_{\mathfrak{M}} \xrightarrow{\delta} A_{\mathfrak{M}} \longrightarrow B_{\mathfrak{M}} \longrightarrow 0$$

at any maximal ideal $\mathfrak{M}$ of A, and quote portions of the proof of [**11**, Theorem 1, pp. 671–673]:

Since $B_{\mathfrak{M}}$ is Cohen-Macaulay [(19)], either $B_{\mathfrak{M}} = 0$ or the homological dimension of $B_{\mathfrak{M}}$ is equal to its codimension, that is, two. Thus $\ker(\delta_{\mathfrak{M}})$ is projective, and since $A_{\mathfrak{M}}$ is a local ring, it must in fact be free. Passing to the quotient field K of A shows that rank $\ker(\delta)_{\mathfrak{M}} = d$. Thus $\ker(\delta)$ is locally free of rank d. ... [Since the question of whether the inclusion $\operatorname{im}(\alpha) \hookrightarrow \ker(\delta)$ is an isomorphism is local, we may replace A by a localization over which $\ker(\delta)$ is free. We then] have a resolution

$$0 \longrightarrow A^d \longrightarrow A^{d+1} \xrightarrow{\delta} A \longrightarrow B \longrightarrow 0;$$

[b]y Burch's theorem, we can choose a basis $r_1', \dots, r_d' \in A^{d+1}$ of $\ker(\delta)$ such that

$$f_l(X) = \det(e_l, r_1', \dots, r_d'),$$

for all $l = 1, \dots, d$, as follows ... [the passage goes on to prove the existence of the r_i' using an argument having [**9**, p. 148] as citation].

Let $R' = (r_1', \dots, r_d')$ denote the $d \times (d+1)$ matrix with rows r_i', and recall that $M(X) = (r_1(X), \dots, r_d(X))$. We previously showed that $\operatorname{im}(\alpha) \subseteq \ker(\delta)$; it follows that there exists a unique $d \times d$ matrix (a_{ij}) with entries in A such that

$$M(X) = (a_{ij}) \cdot R',$$

i.e., each $r_i(X) \in \operatorname{im}(\alpha)$ can be uniquely expressed as an A-linear combination of the r_i'. Now observe that the lth $d \times d$ minor of $M(X)$, which is $f_l(X)$ up to sign, is equal to $\det(a_{ij}) \cdot (l\text{th } d \times d \text{ minor of } R')$. That is,

$$f_l(X) = \det(a_{ij}) \cdot f_l(X).$$

Since A is a domain, we have that $\det(a_{ij}) = 1$, which implies that (a_{ij}) is invertible (at least one of the $f_l(X)$ must be nonzero, since the matrix $M(X)$ has rank d). It follows that the r_i' can be expressed as A-linear combinations of the $r_i(X)$; whence, $\operatorname{im}(\alpha) = \ker(\delta)$, as desired.

We have now established the desired free resolution of B. We summarize the preceding discussion in

Theorem 3.5. *Let $U = \mathrm{Spec}\,(C)$ be an open neighborhood of the origin $0 \in \mathrm{Spec}\,(k[X])$ over which the map F is quasi-finite, and let*

$$A = C[x,y], \qquad B = A/(f_0(X),\dots,f_d(X)),$$

as in Proposition 3.1. *We have that the sequence*

$$0 \longrightarrow A^d \xrightarrow{\alpha} A^{d+1} \xrightarrow{\delta} A \longrightarrow B \longrightarrow 0$$

is a free resolution of the A-module B, where the maps α and δ are defined in (25).

Corollary 3.6. *Under the hypotheses of the theorem, we have that for any C-module N, the sequence obtained from the free resolution of B by applying the functor $(\cdot) \otimes_C N$ is exact.*

Proof. Since $A = C[x,y]$, we have that the free resolution of the theorem is a free resolution of B as a C-module. Therefore, the modules $\mathrm{Tor}_q^C(N,B)$ are the homology modules of the tensored sequence. But since B is flat over C (20), we have that

$$\mathrm{Tor}_q^C(N,B) \approx \mathrm{Tor}_q^C(B,N) = 0;$$

whence, the corollary. $\square$

4. Cases in which Iarrobino's method works well. Once again, let $I \subseteq k[[x,y]]$ be an ideal of colength n having type T, normal pattern P with respect to the monomial order (3), associated standard generators $f_0,\dots,f_d$, and matrix of relations M. As in Section 3, we introduce the set of variables $X = \{X_{ij\mu}\}$, form the matrix $M(X) = (r_1(X),\dots,r_d(X))$, and define

$$f_l(X) = \det\,(e_l, r_1(X),\dots,r_d(X)), \quad 0 \le l \le d,$$
$$I(X) = (f_0(X),\dots,f_d(X)) \subseteq k[x,y,X], \quad \text{and the map}$$
$$\mathrm{Spec}\,(k[x,y,X]/I(X)) \xrightarrow{F} \mathrm{Spec}\,(k[X]).$$

Note that $f_l(X)$ has the same leading monomial $x^{k_l}y^l$ as f_l (6), since $f_l(X) \to f_l$ as all $X_{ij\mu} \to 0$.

The main goal of this section is to identify certain classes of ideals I for which the map F is globally finite and the induced map (globally defined by Proposition 3.1)

$$\phi : \mathrm{Spec}\,(k[X]) = \mathbf{A}_k^{2n} \longrightarrow \mathrm{Hilb}^n_{\mathbf{P}^2_k} = H^n$$

is an open immersion. In overview, we proceed as follows. First, in Theorem 4.2, we give conditions that ensure that the map ϕ is defined and an open immersion on an open neighborhood $U = \mathrm{Spec}\,(C)$ of $0 \in \mathrm{Spec}\,(k[X])$, as promised in Remark 3.3. We then show that these conditions hold for $U = \mathrm{Spec}\,(k[X])$ if all the $f_l(X)$ have "good form," Definition 4.3, Corollary 4.6. Finally we present a condition, depending only on the type of I, sufficient to ensure that the $f_l(X)$ all have good form, Proposition 4.8; this condition subsumes the ideals of generic type and the fat point ideals, Examples 4.12 and 4.13. We will use the notation (18) and in addition will write

$$I(U) = A \cdot (f_0(X), \dots , f_d(X)) \subseteq A = C[x, y]$$

for the extension of $I(X)$ to A.

Definition 4.1. We say that $g \in I(U)$ has the *pullback uniqueness property* provided that the following condition holds. Whenever we are given a k-algebra D and two maps $s, t : C \to D$ such that

$$I(s) = I(t) \subseteq D[x, y],$$

where $I(s)$ and $I(t)$ are the extensions of $I(U)$ via the induced maps $\tilde{s}, \tilde{t} : C[x, y] \to D[x, y]$, then

$$\tilde{s}(g) = \tilde{t}(g).$$

Our main result is

Theorem 4.2. *Suppose that $C[x, y]/I(U) = B$ is finite over C and that each of the generators $f_0(X), \dots , f_d(X) \in I(U)$ has the pullback uniqueness property. Then the map $\phi : U \to H^n$ induced by the universal property of H^n is an open immersion.*

Proof. Since B is finite over C, Proposition 3.1 yields that B is locally free of rank n over C (21), the map $\phi : U \to H^n$ is defined, and ϕ is étale at the origin. We are out to prove that ϕ is an open immersion or, equivalently, that ϕ is étale (everywhere) and radicial [**1**, Theorem 5.5, p. 121].

Recall that ϕ is *radicial*, provided that it is universally injective or, equivalently, that for any field K, the induced map $\phi(K) : U(K) \to H^n(K)$ of K-points is injective ([**6**, p. 246] or [**1**, Proposition 5.2, p. 119]); we proceed to show that ϕ satisfies the latter criterion. Let s and t be two K-points of U; we write $s, t : C \to K$ for the comorphisms of the inclusion maps $\mathrm{Spec}\,(K) \to U = \mathrm{Spec}\,(C)$. If these points have the same image under the map $\phi(K)$, then the ideals $I(s)$ and $I(t)$ in $K[x, y]$, obtained by extending $I(U)$ via the induced maps $\tilde{s}, \tilde{t} : C[x, y] \to K[x, y]$, respectively, are equal. Since we are assuming that the generators $f_0(X), \dots, f_d(X)$ each have the pullback uniqueness property, we have that

$$\tilde{s}(f_l(X)) = \tilde{t}(f_l(X)), \quad 0 \le l \le d. \tag{29}$$

We now exploit the free resolution of B developed in the last section, and summarized in Theorem 3.5: this is the exact sequence

$$0 \longrightarrow A^d \xrightarrow{\alpha} A^{d+1} \xrightarrow{\delta} A \longrightarrow B \longrightarrow 0,$$

where $A = C[x, y]$ and the maps α and δ are given in (25). We write K_s for K viewed as a C-algebra via the map s; $M(s)$ for the matrix obtained from $M(X)$ by making the substitutions $X_{ij\mu} \mapsto s(X_{ij\mu})$; and $r_i(s)$ for the ith row of the matrix $M(s)$; and likewise for the point t (we view $X_{ij\mu} \in k[X]$ as an element of C via the comorphism $k[X] \to C$ of the inclusion map $U \hookrightarrow \mathrm{Spec}\,(k[X])$). By Corollary 3.6, if we apply the functor $(\cdot) \otimes_C K_s$ to the preceding resolution, we obtain an exact sequence

$$0 \longrightarrow K[x, y]^d \xrightarrow{\alpha(s)} K[x, y]^{d+1} \xrightarrow{\delta(s)} K[x, y] \longrightarrow K[x, y]/I(s) \longrightarrow 0,$$

where $\alpha(s)$ is defined by the matrix $M(s)$ and $\delta(s)$ by the generators $\tilde{s}(f_0(X)), \dots, \tilde{s}(f_d(X))$ of $I(s)$. In particular, the rows $r_i(s)$ of $M(s)$ are a $K[x, y]$-basis of $\ker\,(\delta(s))$. Repeating this argument for t, we find that the rows $r_i(t)$ of $M(t)$ are a $K[x,y]$-basis of $\ker(\delta(t))$. But by (29), we have that $\delta(s) = \delta(t)$; whence, each row $r_l(t)$ lies in the $K[x,y]$-span of the $r_i(s)$ and vice versa. For $1 \leq i \leq d$, consider the element

$$r_i^* = r_i(s) - r_i(t) \in \ker(\delta(s)) = \ker(\delta(t)).$$

Recalling the form (27) of the rows $r_i(X)$, we see that the vector r_i^* does not involve y. By the same argument that led to Remark 3.4, we conclude that r_i^* must be zero, since otherwise it would be a nontrivial $K[x,y]$-linear combination of the $r_i(s)$ and therefore would involve y nontrivially. In other words, we have that $r_i(s) = r_i(t)$ for $1 \leq i \leq d$, that is, $M(s) = M(t)$. Recalling (16), we see that this implies that $s(X_{ij\mu}) = t(X_{ij\mu})$ for all i, j, μ. It follows at once that $\phi(K)$ is injective, which completes the proof of radiciality.

The proof that ϕ is étale uses the same idea as the proof of radiciality. Since the source and target of ϕ are nonsingular schemes of dimension $2n$, it suffices to show that the tangent space mapping

$$TU_s \xrightarrow{\phi_s'} TH^n_{\phi(s)}$$

is injective, and therefore an isomorphism, at each k-point $s = (s_{ij\mu}) \in U$. Choose tangent vectors $v, w \in TU_s$; these correspond to maps $\operatorname{Spec}(k[\varepsilon]) \to U \subseteq \operatorname{Spec}(k[X])$, whose comorphisms, which we shall denote by v and w, are given by

$$X_{ij\mu} \longmapsto s_{ij\mu} + \varepsilon \cdot v_{ij\mu}, \quad X_{ij\mu} \longmapsto s_{ij\mu} + \varepsilon \cdot w_{ij\mu}, \tag{30}$$

respectively. If $\phi'(v) = \phi'(w)$, then the ideals $I(v)$ and $I(w)$ in $K[\varepsilon][x,y]$, obtained by extending $I(U)$ via the induced maps $\tilde{v}, \tilde{w} : C[x,y] \to k[\varepsilon][x,y]$, respectively, are equal. The argument now proceeds exactly as in the foregoing proof of radiciality. From the hypothesis that the $f_l(X)$ all have the pullback uniqueness property, we have that $\tilde{v}(f_l(X)) = \tilde{w}(f_l(x))$ for all l. It follows that the rows of each of the matrices $M(v)$ and $M(w)$, obtained from $M(X)$ by applying the substitutions (30), respectively, constitute a $k[\varepsilon][x,y]$-basis of the first syzygy module of the $\tilde{v}(f_l(X))$. From this we deduce that $M(v) = M(w)$, which yields $v = w$; that is, ϕ_s' is injective, as desired. This completes the proof of the theorem. $\square$

For the theorem to be useful, we need to have cases for which its hypotheses hold. We proceed to develop one particularly nice family of cases for which the hypotheses of the theorem hold globally, that is, with $U = \operatorname{Spec}(k[X])$.

Definition 4.3. Let $I(X) = (f_0(X), \dots, f_d(X))$ be as above. We say that the generator $f_l(X)$ has *good form* provided that

$$f_l(X) = x^{k_l} y^l + \begin{pmatrix} \text{a } k[X]\text{-linear combination of monomials} \\ \text{in the pattern } P \end{pmatrix}.$$

Lemma 4.4. *Suppose that all of the generators $f_l(X)$ of $I(X)$ have good form. Then the pattern monomials generate $k[x, y, X]/I(X)$ as a $k[X]$-module. Consequently, $k[x, y, X]/I(X)$ is finite over $k[X]$.*

Proof. It suffices to show that every monomial in x and y is congruent modulo $I(X)$ to a $k[X]$-linear combination of pattern monomials. This is trivially true for the pattern monomials themselves, and the hypothesis that each of the generators $f_l(X)$ has good form implies that it is true as well for the leading monomials $x^{k_0}, x^{k_1}y, \dots, x^{k_{d-1}}y^{d-1}, y^d$ of the generators. Let (P) denote an arbitrary $k[X]$-linear combination of pattern monomials. We claim that, for $r = 1, 2, \dots,$

$$x^r \cdot (P) \equiv (P) \bmod I(X) \quad \text{and} \quad y^r \cdot (P) \equiv (P) \bmod I(X).$$

To see this, choose a pattern monomial m and note that $x \cdot m$ is either again a pattern monomial or one of the leading monomials, for example, consider (5); the first statement therefore follows by induction on r. The second statement now follows in the same way from the observation that $y \cdot m$ is either a pattern monomial or of the form $x^r \cdot$ (leading monomial). Since any x, y-monomial m' other than a pattern monomial or a leading monomial can be written in the form $x^r \cdot y^s \cdot$ (leading monomial), we have that $m' \equiv (P) \bmod I(X)$, as desired. $\square$

Proposition 4.5. *Suppose that all of the generators $f_l(X)$ of $I(X)$ have good form. Then $k[x, y, X]/I(X)$ is a free $k[X]$-module of* *dimension n, with the set of pattern monomials constituting a basis. Moreover, the $f_l(X)$ all have the pullback uniqueness property.*

Proof. By Lemma 4.4, we have that $k[x, y, X]/I(X) = B$ is generated as a $k[X] = C$-module by the pattern monomials and is therefore finite over C; consequently, by (21) in the Proof of Proposition 3.1, we have that B is locally free of rank n over C. Since the n pattern monomials generate B over C, they generate the localization $S^{-1}B$ over $S^{-1}C$, where we may choose the latter, by the local freeness, to be the coordinate ring of a nonempty open subscheme of Spec (C) such that $S^{-1}B$ is free of rank n over $S^{-1}C$. We deduce that the pattern monomials, viewed as elements of $S^{-1}B$, give an $S^{-1}C$-basis of this module; in particular, they are $S^{-1}C$-linearly independent. Since C is an integral domain, it follows that the pattern monomials, viewed as elements of B, are C-linearly independent and therefore constitute a C-basis of B. (Alternatively, from the local freeness of B over C, we conclude that B is free over C, of rank n, by the well-known theorem of Quillen and Suslin, see, e.g., [**4**, p. 481]; then, since the pattern monomials form a C-generating set of B of cardinality n, they must constitute a basis.)

It remains to show that the $f_l(X)$ have the pullback uniqueness property. Let D be a k-algebra and $s, t : C \to D$ two maps such that $I(s) = I(t) \subseteq D[x, y]$, where $I(s)$ and $I(t)$ are the extensions of $I(X)$ via the induced maps $\tilde{s}, \tilde{t} : C[x, y] \to D[x, y]$, respectively. We write D_s, respectively D_t, for D viewed as a C-algebra via the map s, respectively, t. Since the pattern monomials constitute a C-basis of B, they also constitute a D-basis of

$$\begin{aligned} B \otimes_C D_s = C[x, y]/I(X) \otimes_C D_s &= D[x, y]/I(s) \\ &= D[x, y]/I(t) = B \otimes_C D_t. \end{aligned}$$

Furthermore, since the $f_l(X)$ all have good form, by hypothesis we see that $\tilde{s}(f_l(X))$ and $\tilde{t}(f_l(X))$ have "good form" as well; that is,

$$(\tilde{s} \text{ or}) \ \tilde{t}(f_l(X)) = x^{k_l} y^l + \text{ a } D\text{-linear combination of monomials in } P.$$

Therefore, since each leading monomial $x^{k_l} y^l$ can be uniquely expressed as a D-linear combination of pattern monomials mod $I(s) = I(t)$, it follows that

$$\tilde{s}(f_l(X)) = \tilde{t}(f_l(X)) \quad \text{for } 0 \le l \le d;$$

that is, the $f_l(X)$ all have the pullback uniqueness property. $\square$

Combining Proposition 4.5 and Theorem 4.2, we obtain

Corollary 4.6. *Suppose that the $f_l(X)$ all have good form. Then the $k[X]$-module $k[x, y, X]/I(X)$ is free of dimension n, with the set of pattern monomials P constituting a basis. Moreover, the map*

$$\phi : \mathrm{Spec}\,(k[X]) = \mathbf{A}_k^{2n} \longrightarrow H^n$$

induced by the universal property of H^n is an open immersion. $\square$

Remark 4.7. In the next section we give an example (5.5) of an ideal I for which the conclusion of the preceding corollary holds, even though not all of the generators $f_l(X)$ have good form. In other words, the hypothesis of Corollary 4.6 is a sufficient, but not necessary, condition for its conclusion.

The remainder of this section is devoted to the identification of some specific classes of ideals I which satisfy the hypothesis of Corollary 4.6; namely, that the generators $f_0(X), \ldots, f_d(X)$ of $I(X)$ all have good form. The examples we present are subsumed by the following

Proposition 4.8. *Let $I \subseteq k[[x, y]]$ be an ideal of colength n having type T, normal pattern P with respect to the monomial order* (3), *associated standard generators $f_0, \ldots, f_d$, and matrix of relations M. If $w_1 = w_2 = \cdots = w_d$* (8), *then the generators $f_0(X), \ldots, f_d(X)$ of the ideal $I(X)$ all have good form.*

For the proof, we need to bound the degrees of the polynomials $\alpha_{i,j} \in k[x]$ which appear in the matrix of relations M (9); a crude bound sufficient for our purposes is given by

Lemma 4.9. *Let I be as in the first sentence of the proposition* (*note that we are not assuming here that the w_i are all equal to one another*). *Then, for $1 \leq i \leq d$, $0 \leq j \leq i - 1$, we have that*

$$\deg\,(\alpha_{i,j}) < \max(w_1, w_2, \ldots, w_d).$$

Proof. First recall the form (6) of the standard generator f_l: the leading monomial $x^{k_l}y^l$ is the unique monomial of maximal y-degree among the monomials appearing in f_l, and the other monomials appearing in f_l are pattern monomials which are greater than the leading monomial in the monomial order (3). It may be helpful to keep in mind the "Pascal triangle" (5) of monomials.

Consider the relation (8) among the standard generators defined by the ith row of M:

$$-yf_{i-1} + x^{w_i}f_i + \sum_{j=0}^{i-1} \alpha_{i,j}f_j = 0.$$

Observe first of all that the relation contains two terms of y-degree i, coming from $-y \cdot f_{i-1}$ and $x^{w_i} \cdot f_i$; these cancel out because

$$x^{w_i} \cdot x^{k_i} \cdot y^i = x^{w_i+k_i} \cdot y^i = y \cdot x^{k_{i-1}} \cdot y^{i-1}.$$

The proof now proceeds by descending induction on j; the key point is that the polynomial $\alpha_{i,j}$ is uniquely determined by the requirement that all terms of y-degree j must cancel.

Suppose that the lemma has been established for $\alpha_{i,j+1}, \dots, a_{i,i-1}$, with $j \geq 0$. Consider the terms of y-degree j in the relation; these fall into several groups:

(a) the terms of y-degree $j-1$ in f_{i-1}, multiplied by $-y$ (note that this set is empty if $j = 0$);

(b) the terms of y-degree j in f_i, multiplied by x^{w_i};

(c) the terms of y-degree j in $f_{j+1}, \dots, f_{i-1}$, multiplied by $\alpha_{i,j+1}, \dots, \alpha_{i,i-1}$, respectively (note that this set is empty if $j = i-1$); and

(d) the leading monomial $x^{k_j}y^j$ of f_j, multiplied by $\alpha_{i,j}$.

The maximum possible x-degree of a term of type (a), if $j > 0$, is $k_{j-1} - 1$, and the maximum possible x-degree of a term of type (b) is $(k_j - 1) + w_i$. From the induction hypothesis, the terms of type (c) have maximum possible x-degree strictly less than $k_j - 1 + \max(w_1, w_2, \dots, w_d)$. Therefore, to produce a term of type (d) to cancel the maximum x-degree term(s) of types (a), (b) and (c), $\alpha_{i,j}$ must have a term of maximal degree bounded above by the maximum of

$$\begin{gathered}(k_{j-1}-1)-k_j = w_j - 1,\\ (k_j-1)+w_i-k_j = w_i-1,\\ (k_j-1)+\max(w_1,w_2,\dots,w_d)-k_j = \max(w_1,w_2,\dots,w_d)-1,\end{gathered}$$

if $j>0$ or simply by the maximum of the last two values if $j=0$. It follows at once that $\deg(\alpha_{i,j}) < \max(w_1,\dots,w_d)$, which completes the induction step. The base case, $j=i-1$, follows in the same way; in that case, there are no terms of type (c) and no terms of type (a) if $i=1 \Rightarrow j=i-1=0$. $\square$

Remark 4.10. The conclusion of Lemma 4.9 is easily verified in example (12).

Proof of Proposition 4.8. We are assuming that $w_1 = w_2 = \cdots = w_d = w$. The lemma yields that $\deg(\alpha_{i,j}) < w$ for all $1 \le i \le d$, $0 \le j \le d$. Therefore, when we form the matrix $M(X)$ for I, recall (16), we obtain a $d \times (d+1)$ matrix of the form

$$\begin{bmatrix} -y+g_{1,0} & x^w+g_{1,1} & g_{1,2} & \cdots & g_{1,d}\\ g_{2,0} & -y+g_{2,1} & x^w+g_{2,2} & g_{2,3}\cdots & g_{2,d}\\ \vdots & & \ddots & & \vdots\\ g_{d-1,0} & \cdots g_{d-1,d-3} & -y+g_{d-1,d-2} & x^w+g_{d-1,d-1} & g_{d-1,d}\\ g_{d,0} & \cdots & g_{d,d-2} & -y+g_{d,d-1} & x^w+g_{d,d} \end{bmatrix},$$

in which each $g_{i,j} \in k[X][x]$ has x-degree $< w$. We are out to show that each

$$f_l(X) = \det(e_l, r_1(X), \dots, r_d(X))$$

has good form; that is, that $f_l(X)$ is the sum of its leading monomial $x^{k_l}y^l$ and a $k[X]$-linear combination of pattern monomials. Since $w_i = w$ for all i, we have that $k_l = (d-l)\cdot w$, and the set of pattern monomials is

$$P = \{x^r y^s \mid 0 \le s < d,\ 0 \le r < (d-s)\cdot w\}. \tag{31}$$

Up to sign, $f_l(X)$ is the determinant of the $d \times d$ matrix obtained from $M(X)$ by deleting the lth column; for example, when $d=5$ and

$l = 2$, we have the matrix

$$\begin{bmatrix} -y + g_{1,0} & x^w + g_{1,1} & g_{1,3} & g_{1,4} & g_{1,5} \\ g_{2,0} & -y + g_{2,1} & g_{2,3} & g_{2,4} & g_{2,5} \\ g_{3,0} & g_{3,1} & x^w + g_{3,3} & g_{3,4} & g_{3,5} \\ g_{4,0} & g_{4,1} & -y + g_{4,3} & x^w + g_{4,4} & g_{4,5} \\ g_{5,0} & g_{5,1} & g_{5,3} & -y + g_{5,4} & x^w + g_{5,5} \end{bmatrix}.$$

We evaluate the determinant as a sum of signed products of d entries of the matrix, drawn in such a way that each row and column is represented exactly once in each product. By expanding the products involving one or more binomial entries, $-y + g_{i,i-1}$ or $x^w + g_{i,i}$, we can express the determinant as a sum of signed products with d factors, each factor being either y, x^w or one of the $g_{i,j}$. Using (31), we see that any product having one or more of the g's (which have x-degree $< w$) as factors expands to a $k[X]$-linear combination of pattern monomials. Moreover, if any product involves a factor of x^w from row i, $1 \leq i \leq l-1$, i.e., from above the main diagonal, then one sees that any attempt to avoid drawing one of the g's for that product is doomed to failure: one will be forced to draw one of the g's from row l to form the product. Similarly, if any product involves a factor of y from row i, $l+2 \leq i \leq d$, i.e., from below the main diagonal, then that product must also involve one of the g's.

In other words, the only product in the expansion of the determinant that does not involve one or more of the $g_{i,j}$ is the product

$$x^{(d-l)\cdot w} \cdot y^l = x^{k_l} y^l = \text{ leading monomial of } f_l(X),$$

which comes from the main diagonal. It follows that $f_l(X)$ has good form, and the proposition is proved. □

In light of Proposition 4.8, Corollary 4.6 specializes to

Corollary 4.11. *If I satisfies the hypotheses of Proposition 4.8, then the $k[X]$-module $k[x, y, X]/I(X)$ is free of dimension $n =$ colength of I, with the set of pattern monomials P constituting a basis. Moreover, the map $\phi : \mathbf{A}_k^{2n} \to H^n$ induced by the universal property of H^n is an open immersion.* □

To bring this section to a close, we highlight as examples two important families of ideals for which the hypotheses of Proposition 4.8 hold:

Example 4.12. Ideals of generic type. Recall that the generic type of length n is

$$T = (1, 1, \dots, 1, 0, \dots), \quad \text{with } n \text{ 1's.}$$

The normal pattern of type T with respect to the monomial order (3) is then

$$P = \{1, x, x^2, \dots, x^{n-1}\}.$$

If I has normal pattern P, then $d = 1$ and the standard generators are

$$f_0 = x^n, \qquad f_1 = y + \sum_{i=1}^{n-1} p_i x^i, \quad p_i \in k.$$

There is only one exponent w_i, namely, $w_1 = n$, so Proposition 4.8 applies. The matrix $M(X)$ consists of one row, whose entries are, up to sign, the generators $f_l(X)$ of the ideal $I(X)$:

$$M(X) = \Big[-y - \sum_{i=1}^{n-1} p_i x^i + \sum_{\mu=0}^{n-1} (X_{10\mu}) \cdot x^\mu, x^n + \sum_{\mu=0}^{n-1} X_{11\mu} \cdot x^\mu \Big].$$

In this case it is manifest that the generators of $I(X)$ have good form.

Example 4.13. Fat point ideals $\mathfrak{m}^r$. Let

$$I = \mathfrak{m}^r = (x^r, x^{r-1}y, \dots, y^r),$$

an ideal of colength $n = r \cdot (r+1)/2$. This ideal has normal pattern

$$P = \{x^p y^q \mid 0 \le p + q < r\};$$

the standard generators are

$$f_0 = x^r, \quad f_1 = x^{r-1}y, \dots, \quad f_d = y^r = y^d,$$

and each exponent $w_i = 1$, so Proposition 4.8 applies. In this case the ith row of the matrix $M(X)$ has the form

$$(X_{i0}, \dots, X_{i(i-2)}, -y + X_{i(i-1)}, x + X_{ii}, X_{i(i+1)}, \dots, X_{id}),$$

where we write X_{ij} as shorthand for X_{ij0}.

In the next section of the paper, we present three examples (5.1), (5.2) and (5.3) of ideals I for which the map ϕ fails to be defined in a neighborhood of the origin $0 \in \mathbf{A}_k^{2n}$. It should not be surprising that these examples are produced by causing large jumps in the sequence of exponents w_i.

5. Further examples of Iarrobino's procedure. In this final section of the paper, we apply Iarrobino's construction of local parameters to several ideals, with mixed results. We begin with three examples in which Iarrobino's procedure fails: the first of these is discussed in detail; the second and third, being similar, are dealt with more briefly. Iarrobino's procedure succeeds in the remaining two examples. The penultimate shows that Iarrobino's construction can yield a globally-defined open immersion $\phi : \mathrm{Spec}\,(k[X]) \to H^n$, even though the generators $f_l(X)$ do not all have good form. The final example is one in which the map ϕ is not defined on all of $\mathrm{Spec}\,(k[X])$, but is an open immersion on a neighborhood U of the origin in $\mathrm{Spec}\,(k[X])$.

Example 5.1. This is the first of three examples in which Iarrobino's construction of local parameters fails. We will show that the map

$$F : \mathrm{Spec}\,(k[x, y, X]/I(X)) \longrightarrow \mathrm{Spec}\,(k[X])$$

is not finite over any open subscheme $U \subseteq \mathrm{Spec}\,(k[X])$ that contains the origin; indeed, the family of subschemes defined by F contains members of the wrong colength arbitrarily close to $F^{-1}(0)$.

$$\begin{aligned} \text{ideal:}\quad & I = (x^5, x^4y, y^2) \subseteq R = k[[x, y]], \\ \text{type:}\quad & T = (1, 2, 2, 2, 2, 0, \dots), \\ \text{colength:}\quad & 9, \\ w\text{-sequence:}\quad & w_1 = 1, w_2 = 4. \end{aligned}$$

This ideal has the normal pattern shown, as in (5), with respect to which the standard generators are $f_0 = x^5$, $f_1 = x^4y$ and $f_2 = y^2$ (being leading terms, these monomials are shown in parentheses in the diagram).

$$\begin{array}{cccccc}
 & & & & y^4 & xy^4 \\
 & & & y^3 & xy^3 & x^2y^3 \\
 & & (y^2) & xy^2 & x^2y^2 & x^3y^2 \\
 & y & xy & x^2y & x^3y & (x^4y) \\
1 & x & x^2 & x^3 & x^4 & (x^5)
\end{array}.$$

The matrix M (9) of relations among the f's is the following:

$$M = \begin{bmatrix} -y & x & 0 \\ 0 & -y & x^4 \end{bmatrix}.$$

Following Iarrobino's procedure described in Section 2.3, we introduce the variables $X = \{X_{ij\mu}\}$ and build the matrix $M(X)$ with rows

$$\begin{aligned}
r_1(X) &= (-y + X_{100}, x + X_{110}, \\
&\qquad X_{120} + X_{121} \cdot x + X_{122} \cdot x^2 + X_{123} \cdot x^3), \\
r_2(X) &= (X_{200} + X_{201} \cdot x + X_{202} \cdot x^2 + X_{203} \cdot x^3, \\
&\qquad - y + X_{210} + X_{211} \cdot x + X_{212} \cdot x^2 + X_{213} \cdot x^3, \\
&\qquad x^4 + X_{220} + X_{221} \cdot x + X_{222} \cdot x^2 + X_{223} \cdot x^3);
\end{aligned} \tag{32}$$

we then form the ideal

$$I(X) = (f_0(X), f_1(X), f_2(X)) \subseteq k[x, y, X]$$

with generators

$$f_l(X) = \det(e_l, r_1(X), r_2(X)),$$

and consider the map

$$F : \mathrm{Spec}\,(k[x, y, X]/I(X)) \longrightarrow \mathrm{Spec}\,(k[X]).$$

As shown in Section 3, Iarrobino's construction of local parameters will succeed if F is finite over a neighborhood U of the origin in

$\mathrm{Spec}\,(k[X]) = \mathbf{A}_k^{18}$, $18 = 2n = 2 \cdot 9$. However, we claim that there is no such neighborhood in this case.

Proof of claim. Consider the one-parameter subfamily of subschemes obtained by pulling back the map F over the map

$$\lambda : \mathrm{Spec}\,(k[t]) \longrightarrow \mathrm{Spec}\,(k[X])$$

with comorphism $\lambda^* : k[X] \to k[t]$ given by

$$X_{213} \longmapsto t, \quad X_{123} \longmapsto t \quad \text{and} \quad X_{ij\mu} \longmapsto 0 \text{ otherwise.}$$

The subfamily is cut out by the ideal

$$I(t) = (f_0(t), f_1(t), f_2(t)) \subseteq k[x, y, t],$$

whose generators are obtained as before from the matrix

$$M(t) = \begin{bmatrix} -y & x & tx^3 \\ 0 & -y + tx^3 & x^4 \end{bmatrix};$$

whence, we have that

$$f_0(t) = x^5 + tx^3y - t^2x^6, \quad f_1(t) = x^4y, \quad f_2(t) = y^2 - tx^3y. \tag{33}$$

Note that when $t = 0$, we recover the original ideal I, as we should, since λ maps the origin of its source to the origin of its target.

We now compute a Groebner basis for $I(t)$ using the lexicographical order with $y > x > t$ (the *Mathematica* package was used):

$$\mathbf{GroebnerBasis}[\{\mathbf{x^5 + tx^3y - t^2x^6, x^4y, y^2 - tx^3y}\}, \{\mathbf{y, x, t}\}]$$

$$\{-x^6 + t^2x^7, x^5 - t^2x^6 + tx^3y, x^4y, x^5 - t^2x^6 + y^2\}.$$

The result yields a proof that the ring $k[x, y, t]/I(t)$ is not finite over $k[t]$: otherwise, the variable x would be integral over $k[t]$; whence, $I(t)$ would have to contain a monic polynomial

$$g = x^r + \gamma_{r-1} \cdot x^{r-1} + \cdots + \gamma_0$$

in x with coefficients $\gamma_j \in k[t]$. The (lex) leading term of this polynomial, namely x^r, would then have to be divisible by the (lex) leading term of one of the members of the Groebner basis, but this is manifestly impossible. It follows that the ring $k[x,y,X]/I(X)$ is not finite over $k[X]$.

Consider next the restriction of our one-parameter subfamily to the open subset of Spec $(k[t])$ defined by $t \neq 0$. Since we can divide by t on this open subset, we can rewrite the pullback of $I(t)$ over this open subset as

$$I(t,t^{-1}) = \left(-\frac{1}{t^2} \cdot x^6 + x^7, \frac{1}{t} \cdot x^5 - tx^6 + x^3y, x^5 - t^2x^6 + y^2 \right).$$

We write

$$B = k[x,y,t,t^{-1}]/I(t,t^{-1}),$$
$$C = k[t,t^{-1}].$$

Since x^7, x^3y and y^2 are congruent to C-linear combinations of x^5 and x^6, we have that B is generated as a C-module by the following set of ten monomials:

$$\{1, x, y, x^2, xy, x^3, x^2y, x^4, x^5, x^6\} \tag{34}$$

(adapt the proof of Lemma 4.4). It follows at once that B is finite over C; it further follows, as in the proof of Proposition 3.1, that B is flat, and hence locally free, over C. Since Spec (C) is irreducible, the colength of the ideal $I(t_0) \subseteq k[x,y]$, obtained by replacing t by any nonzero value $t_0 \in k$, is constant, and so may be computed at $t = 1$. Substituting this value in (33), we find that $I(1) \subseteq k[x,y]$ has generators

$$f_0(1) = x^5 + x^3y - x^6, f_1(1) = x^4y, f_2(1) = y^2 - x^3y.$$

Computing a Groebner basis for this ideal with respect to the lexicographic order with $y > x$, we find:

$$\mathbf{GroebnerBasis}[\{\mathbf{x^5 + x^3y - x^6, x^4y, y^2 - x^3y}\}, \{\mathbf{y, x}\}]$$

$$\{-x^6 + x^7, x^5 - x^6 + x^3y, x^5 - x^6 + y^2\}.$$

We can now see that the set of monomials (34) is k-linearly independent $\operatorname{mod} I(1)$; if not, then a nontrivial k-linear combination of the monomials in (34) would lie in $I(1)$, and hence the (lex) leading term thereof would be divisible by one of the leading monomials of the Groebner basis of $I(1)$. However, no monomial in (34) is divisible by any of the leading monomials in the Groebner basis. Since we have already seen that (34) generates B over C, and therefore generates $k[x,y]/I(1)$ over k, we conclude that $I(1)$ has colength 10, whence $I(t_0)$ has colength 10 for every nonzero value t_0 of t. Therefore, the family of subschemes defined by F cannot induce a well-defined map $\phi : U \to H^9$ for any open neighborhood U of the origin, since the family contains members of length 10 arbitrarily close to $F^{-1}(0)$; equivalently, the map F cannot be finite over any such U. The proof of the claim is now complete. $\square$

Example 5.2. This is the second of three examples in which Iarrobino's construction of local parameters fails. The problem is the same as in the previous example.

$$\begin{aligned} \text{ideal:}\quad & I = (x^6, x^2y, xy^2, y^3) \subseteq R = k[[x,y]], \\ \text{type:}\quad & T = (1,2,3,1,1,1,0,\dots), \\ \text{colength:}\quad & 9, \\ w\text{-sequence:}\quad & w_1 = 4,\ w_2 = 1,\ w_3 = 1. \end{aligned}$$

In this case, we consider the one-parameter subfamily of subschemes obtained by pulling back the map F over the map

$$\lambda : \operatorname{Spec}(k[t]) \longrightarrow \operatorname{Spec}(k[X])$$

with comorphism $\lambda^* : k[X] \to k[t]$ given by

$$X_{103} \longmapsto t, \quad X_{230} \longmapsto t \quad \text{and} \quad X_{ij\mu} \longmapsto 0 \text{ otherwise.}$$

The subfamily is cut out by the ideal

$$I(t) = (f_0(t), f_1(t), f_2(t), f_3(t)) \subseteq k[x,y,t]$$

whose generators are the (signed) maximal minors of the matrix

$$M(t) = \begin{bmatrix} -y + tx^3 & x^4 & 0 & 0 \\ 0 & -y & x & t \\ 0 & 0 & -y & x \end{bmatrix};$$

whence,

$$f_0(t) = x^6 + tx^4y, \qquad f_1(t) = -tx^5 + x^2y - t^2x^3y + ty^2,$$
$$f_2(t) = -tx^4y + xy^2, \quad f_3(t) = -tx^3y^2 + y^3.$$

Arguing as in the previous example, one can show that the subfamily of subschemes is finite and flat over the complement of the origin in Spec $(k[t])$; therefore, the colengths of the ideals $I(t_0) \subseteq k[x,y]$, defined by replacing t by any nonzero $t_0 \in k$, are all the same. Setting t_0 to 1 and computing the Groebner basis of the $f_l(1)$ with respect to the lexicographical ordering $y > x$, we find that

$$I(1) = (x^6 + x^7, -x^6 + x^3y, -x^5 - x^6 + x^2y + y^2),$$

which has colength $10 \neq n = 9$: one checks easily that the ten monomials in (34) form a basis on the quotient $k[x,y]/I(1)$. Once again, the family of subschemes defined by the map F has members of the wrong colength arbitrarily close to the "original" subscheme $F^{-1}(0)$.

Example 5.3. This is the third of three examples in which Iarrobino's construction of local parameters fails. The problem is the same as in the previous examples.

$$\begin{aligned} \text{ideal: } & I = (x^{12}, x^{11}y, x^5y^2, y^3) \subseteq R = k[[x,y]], \\ \text{type: } & T = (1,2,3,3,3,3,3,2,2,2,2,2,2,0,\dots), \\ \text{colength: } & 28, \\ w\text{-sequence: } & w_1 = 1,\ w_2 = 6,\ w_3 = 5. \end{aligned}$$

We leave it to the reader to check that the one-parameter family of subschemes defined by the substitutions

$$X_{125} \longmapsto t, \quad X_{234} \longmapsto t, \quad X_{314} \longmapsto t \quad \text{and} \quad X_{ij\mu} \longmapsto 0 \text{ otherwise},$$

has colength 29 everywhere except at $t = 0$, where the colength is $n = 28$.

Remark 5.4. The foregoing examples show that [**8**, Theorem 4.16, p. 72] is false in full generality.

Example 5.5. This example, promised in Remark 4.7, is one in which the quotient $k[x,y,X]/I(X)$ is free over $k[X]$ with the pattern monomials constituting a basis, and the induced map $\phi : \mathrm{Spec}\,(k[X]) \to H^n$ is an open immersion, but for which the generators $f_l(X)$ do not all have good form.

$$\begin{aligned}
\text{ideal:}\ \ & I = (x^4, xy, y^2) \subseteq R = k[[x,y]],\\
\text{type:}\ \ & T = (1,2,1,1,0,\dots),\\
\text{colength:}\ \ & 5,\\
w\text{-sequence:}\ \ & w_1 = 3,\ \ w_2 = 1.
\end{aligned}$$

The ideal I has normal pattern

$$P = \{1, x, y, x^2, x^3\},$$

with respect to which the given generators are standard. The generators of the ideal $I(X)$ are, up to sign, the 2×2 minors of the matrix $M(X)$ having rows

$$\begin{aligned}
r_1(X) = (&-y + X_{100} + X_{101}\cdot x + X_{102}\cdot x^2,\\
&x^3 + X_{110} + X_{111}\cdot x + X_{112}\cdot x^2, X_{120}),\\
r_2(X) = (&X_{200}, -y + X_{210}, x + X_{220});
\end{aligned}$$

whence, we have

$$\begin{aligned}
f_0(X) = {} & x^4 - X_{120}X_{210} + X_{110}X_{220} + x^3(X_{112} + X_{220})\\
& + x^2(X_{111} + X_{112}X_{220}) + x(X_{110} + X_{111}X_{220}) + yX_{120},\\
f_1(X) = {} & xy + X_{120}X_{200} - X_{100}X_{220} - x^3X_{102}\\
& + x^2(-X_{101} - X_{102}X_{220}) + x(-X_{100} - X_{101}X_{220}) + yX_{220},\\
f_2(X) = {} & y^2 - x^2yX_{102} - xyX_{101} - X_{110}X_{200} + X_{100}X_{210}\\
& - x^3X_{200} + x^2(-X_{112}X_{200} + X_{102}X_{210})\\
& + x(-X_{111}X_{200} + X_{101}X_{210}) - y(X_{100} + X_{210}).
\end{aligned}$$

We see that $f_0(X)$ and $f_1(X)$ have good form, but that $f_2(X)$ contains, in addition to its "leading term" y^2, two terms involving nonpattern monomials in x and y, namely x^2y and xy. We obtain immediately that x^4 and xy are each congruent to $k[X]$-linear combinations of pattern monomials modulo $I(X)$, and a brief computation shows that the same is true of y^2. The proof of Lemma 4.4 now yields that the ring $k[x,y,X]/I(X)$ is finite over $k[X]$ with the pattern monomials constituting a generating set, and therefore, as in the proof of Proposition 4.5, a free $k[X]$-basis. By Proposition 3.1 we obtain a globally defined map $\phi : \operatorname{Spec}(k[X]) \to H^5$ which is étale in a neighborhood of the origin. It remains to prove that this map is an open immersion, for which it suffices, by Theorem 4.2, to show that the generators $f_l(X)$ all have the pullback uniqueness property.

Since the pattern monomials give a $k[X]$-basis of $k[x,y,X]/I(X)$, we can uniquely express each of the leading monomials as a $k[X]$-linear combination of the pattern monomials modulo $I(X)$. Doing this yields the following distinguished elements of $I(X)$:

$$\begin{aligned} g_0(X) &= x^4 - \langle P_0 \rangle = f_0(X), \\ g_1(X) &= xy - \langle P_1 \rangle = f_1(X), \\ g_2(X) &= y^2 - \langle P_2 \rangle = f_2(X) + X_{102}^2 \cdot f_0(X) \\ &\qquad + (X_{101} - X_{102}X_{220} + xX_{102}) \cdot f_1(X), \end{aligned} \tag{35}$$

where $\langle P_l \rangle$ denotes the appropriate $k[X]$-linear combination of pattern monomials. (Since $f_0(X)$ and $f_1(X)$ have good form, they are equal to $g_0(X)$ and $g_1(X)$, respectively; one may check that the given expression for $g_2(X)$ has the form y^2+ a $k[X]$-linear combination of pattern monomials.)

Assume now that we are given a k-algebra D and maps $s, t : k[X] \to D$ such that

$$I(s) = I(t) \subseteq D[x,y],$$

where $I(s)$ and $I(t)$ are obtained by extending $I(X)$ via the induced maps $\tilde{s}, \tilde{t} : k[X][x,y] \to D[x,y]$, respectively. Arguing as in the proof of Proposition 4.5, we see that the $g_l(X)$ have the pullback uniqueness property; whence,

$$\tilde{s}(g_l(X)) = \tilde{t}(g_l(X)), \quad 0 \le l \le 2.$$

By inspection of $g_1(X)$ $(= f_1(X))$, we see that the preceding equation, for $l = 1$, implies that

$$\begin{gathered} s(X_{102}) = t(X_{102}), \qquad s(X_{220}) = t(X_{220}), \\ s(X_{101}) = t(X_{101}). \end{gathered} \tag{36}$$

Now solve (35) for $f_2(X)$; from the result and the equalities (36), it is clear that $\tilde{s}(f_2(X)) = \tilde{t}(f_2(X))$, that is, $f_2(X)$ has the pullback uniqueness property. Since $f_0(X), f_1(X)$ and $f_2(X)$ all have the pullback uniqueness property, Theorem 4.2 now yields that the induced map $\phi : \mathrm{Spec}\,(k[X]) = \mathbf{A}_k^{10} \to H^5$ is an open immersion, as desired.

Example 5.6. The preceding examples represent the two extremes: on the one hand, Iarrobino's procedure sometimes fails to produce a well-defined map $\phi : U \to H^n$; on the other hand, the procedure sometimes leads to an open immersion $\phi : \mathbf{A}_k^{2n} \to H^n$. An "intermediate" example, for which the map $\phi : U \to H^n$ is locally an open immersion but not globally defined, is given by

$$\begin{aligned} \text{ideal: } & I = (x^4, x^3y, y^2) \subseteq R = k[[x,y]], \\ \text{type: } & T = (1,2,2,2,0,\dots), \\ \text{colength: } & 7, \\ w\text{-sequence: } & w_1 = 1,\ w_2 = 3. \end{aligned}$$

The ideal I has the normal pattern

$$P = \{1, x, y, x^2, xy, x^3, x^2y\},$$

with respect to which the given generators are standard. The matrix $M(X)$ is obtained from (32) by changing the value of w_2 from 4 to 3; that is, by changing x^4 to x^3 in the last component of $r_2(X)$ and then deleting all other terms of degree 3 in x. The generators of the ideal $I(X)$, that is, the signed 2×2 minors of $M(X)$, are as follows:

$$\begin{aligned} f_0(X) = {} & x^4(1 - X_{122}X_{212}) - X_{120}X_{210} + X_{110}X_{220} + x^2yX_{122} \\ & + x^3(X_{110} - X_{122}X_{211} - X_{121}X_{212} + X_{222}) + yX_{120} \\ & + x(-X_{121}X_{210} - X_{120}X_{211} + X_{220} + X_{110}X_{221}) + xyX_{121} \\ & + x^2(-X_{122}X_{210} - X_{121}X_{211} - X_{120}X_{212} + X_{221} + X_{110}X_{222}), \end{aligned}$$

$$\begin{aligned} f_1(X) = {} & x^3y + X_{120}X_{200} - X_{100}X_{220} + x^4X_{122}X_{202} + yX_{220} \\ & + x^3(-X_{100} + X_{122}X_{201} + X_{121}X_{202}) \\ & + x(X_{121}X_{200} + X_{120}X_{201} - X_{100}X_{221}) + xyX_{221} \\ & + x^2(X_{122}X_{200} + X_{121}X_{201} + X_{120}X_{202} - X_{100}X_{222}) \end{aligned}$$

$$
\begin{aligned}
&+ x^2 y X_{222}, \\
f_2(X) = {} & y^2 + X_{100}X_{210} - X_{110}X_{200} - x^3 X_{202} - y(X_{100} + X_{210}) \\
&+ x(-X_{200} - X_{110}X_{201} + X_{100}X_{211}) - xyX_{211} \\
&+ x^2(-X_{201} - X_{110}X_{202} + X_{100}X_{212}) - x^2 y X_{212}.
\end{aligned}
$$

We see by inspection that $f_2(X)$ is the only generator having good form, which implies that y^2 is congruent modulo $I(X)$ to a $k[X]$-linear combination of pattern monomials. However, if C is the localization of $k[X]$ defined by inverting $(1 - X_{122}X_{212})$, then x^4 will be congruent to a C-linear combination of pattern monomials modulo $C[x,y] \cdot I(X) = I(U)$, where $U = \mathrm{Spec}\,(C)$; whence, the same is true of $x^3 y$. Arguing as in the proof of Lemma 4.4, we obtain that the ring $C[x,y]/I(U) = B$ is generated over C by the pattern monomials and is therefore finite over C, so Proposition 3.1 applies and yields a well-defined map $\phi : U \to H^7$ which is étale at the origin; in particular, B is locally free of rank n over C by (21). It is easy to see that $k[x,y,X]/I(X)$ is not globally finite, or even quasi-finite, over $k[X]$; indeed, if we specialize to the k-point defined, for any $0 \neq a \in k$, by

$$X_{122} \longmapsto a, \quad X_{212} \longmapsto 1/a \quad \text{and} \quad X_{ijk} \longmapsto 0 \text{ otherwise},$$

we find that the subscheme lying above this point is

$$
\begin{aligned}
\mathrm{Spec}\,(k[x,y])/I(a) &= \mathrm{Spec}\,(k[x,y]/(ax^2y, x^3y, -(x^2y/a) + y^2)) \\
&= \mathrm{Spec}\,(k[x,y]/(x^2y, y^2)),
\end{aligned} \tag{37}
$$

which has the one-dimensional locus $y = 0$ as support. Therefore, the map ϕ cannot be extended from U to all of $\mathrm{Spec}\,(k[X])$.

To complete this example, we must show that the map $\phi : U \to H^7$ is an open immersion. To do this, it suffices by Theorem 4.2 to show that each of the $f_l(X) \in I(U)$ has the pullback uniqueness property. As a prelude, we note that this is not the case globally, that is, over $\mathrm{Spec}\,(k[X])$. Indeed, if a, b in (37) are distinct nonzero elements of k, then the two extensions of $I(X)$ satisfy

$$I(a) = I(b) = (x^2y, y^2),$$

but the two specializations of $f_0(X)$, namely, ax^2y and bx^2y, are not equal; the same is true of the two specializations of $f_2(X)$.

The proof that the $f_l(X) \in I(U)$ have the pullback uniqueness property is similar to that used in the previous example. We begin with the observation that $C[x,y]/I(U) = B$ is free of rank $n = 7$ over C with the pattern monomials constituting a basis. (Since B is locally free of rank n over C and is generated as a C-module by the pattern monomials, we may repeat the argument used at the start of the proof of Proposition 4.5.) By expressing each of the leading monomials as a unique linear combination of the pattern monomials modulo $I(U)$, we obtain the following distinguished elements of $I(U)$:

$$\begin{aligned} g_0(X) &= x^4 - \langle P_0 \rangle = \frac{1}{(1 - X_{122}X_{212})} \cdot f_0(X), \\ g_1(X) &= x^3y - \langle P_1 \rangle \\ &= f_1(X) - \frac{X_{122}X_{202}}{(1 - X_{122}X_{212})} \cdot f_0(X), \\ g_2(X) &= y^2 - \langle P_2 \rangle = f_2(X), \end{aligned} \tag{38}$$

where $\langle P_l \rangle$ denotes the appropriate C-linear combination of pattern monomials.

Now suppose given a k-algebra D and maps $s, t : C \to D$ such that

$$I(s) = I(t) \subseteq D[x,y],$$

where $I(s)$ and $I(t)$ are obtained by extending $I(U)$ via the induced maps $\tilde{s}, \tilde{t} : C[x,y] \to D[x,y]$, respectively. As in the preceding example, we have that the $g_l(X)$ have the pullback uniqueness property, that is,

$$\tilde{s}(g_l(X)) = \tilde{t}(g_l(X)), \quad 0 \le l \le 2.$$

By inspection of $g_2(X) = f_2(X)$, we see that the last equation, for $l = 2$, implies that

$$s(X_{202}) = t(X_{202}), \qquad s(X_{212}) = t(X_{212}); \tag{39}$$

similarly, inspection of $g_0(X)$ yields

$$s\left(\frac{X_{122}}{1 - X_{122}X_{212}}\right) = t\left(\frac{X_{122}}{1 - X_{122}X_{212}}\right),$$

which, in light of (39), quickly simplifies to

$$s(X_{122}) = t(X_{122}). \tag{40}$$

Now solve the equations (38) for the $f_l(X)$; the results, together with the relations (39) and (40), imply that $\tilde{s}(f_l(X)) = \tilde{t}(f_l(X))$ for $0 \leq l \leq 2$; whence, the $f_l(X)$ have the pullback uniqueness property. Theorem 4.2 now yields the desired conclusion; namely, that the map $\phi : U \to H^7$ is an open immersion.

Lacking a counterexample, we end this paper with the following

Conjecture 5.7. *Whenever the map F is finite over an open neighborhood U of the origin in* Spec $(k[X])$*, the induced map $\phi : U \to H^n$ is an open immersion.*

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