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# EXTENSIONS, DILATIONS AND FUNCTIONAL MODELS OF SINGULAR STURM-LIOUVILLE OPERATORS

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ABSTRACT. A space of boundary values is constructed for minimal symmetric singular Sturm-Liouville operator acting in the Hilbert space $L_w^2[a,b)$, $-\infty < a < b \le \infty$, with deficiency indices $(2,2)$ (in Weyl's limit-circle case). A description of all maximal dissipative, maximal accretive, self-adjoint, and other extensions of such a symmetric operator is given in terms of boundary conditions at end points $a$ and $b$. We investigate maximal dissipative operators with general (coupled or separated) boundary conditions. We construct a self-adjoint dilation of the maximal dissipative operator and its incoming and outgoing spectral representations, which makes it possible to determine the scattering matrix of the dilation. We also construct a functional model of the maximal dissipative operator and determine its characteristic function. We prove the theorem on completeness of the system of eigenfunctions and associated functions of the maximal dissipative operators.



**1. Introduction.** The theory of extensions of symmetric operators is one of the basic directions in operator theory. The first fundamental results in this theory were obtained by von Neumann [**17**], although the apparent origins can be found in the famous works of Weyl, see [**22**]. The theorems on representation of linear relations turned out to be useful for the description of various classes of extensions of symmetric operators. The first result of this type is due to Rofe-Beketov [**18**]. Kochubei [**11**] and Bruk [**3**] independently introduced the term 'space of boundary values' and in terms of this notion all maximal dissipative, maximal accretive, self-adjoint, and other extensions of symmetric operators, see [**9**] (also in the survey article [**8**]). However, regardless of the general scheme, the problem of the description of the maximal dissipative (accretive), self-adjoint and other extensions of a given symmetric operator via the boundary conditions is of considerable interest. This problem is particularly interesting in the case of singular

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differential operators, because at the singular ends of the interval under consideration the usual boundary conditions in general are meaningless.

The theory of dilations with application of functional models represents a new trend in the spectral analysis of dissipative (contractive) operators [**13, 15**]. A central part in this theory is played by the characteristic function, which carries the complete information on the spectral properties of the dissipative operator. Thus, in the spectral representation of the dilation, the dissipative operator becomes the model. The problem of the completeness of the system of eigenvectors and associated vectors is solved in terms of the factorization of the characteristic function. The computation of the characteristic functions of dissipative operators is preceded by the construction and investigation of a self-adjoint dilation and of the corresponding scattering problem, in which the characteristic function is realized as the scattering matrix.

In this paper, we consider the minimal symmetric singular Sturm-Liouville operator $L_0$ acting in the Hilbert space $L_w^2[a,b)$, $-\infty < a < b \leq \infty$, with deficiency indices $(2,2)$ (in Weyl's limit-circle case). We construct a space of boundary values of the minimal operator $L_0$ and describe all the maximal dissipative (accretive), self-adjoint, and other extensions of such a symmetric operator in terms of boundary conditions at end points $a$ and $b$. We investigate maximal dissipative operators with general (coupled or separated) boundary conditions. In particular, if we consider separated boundary conditions, at the points $a$ and $b$ the nonself-adjoint (dissipative) boundary conditions are prescribed simultaneously. We construct a self-adjoint dilation of the maximal dissipative operator and its incoming and outgoing spectral representations, which makes it possible to determine the scattering matrix of the dilation according to the scheme of Lax and Phillips [**14**]. We also construct a functional model of the maximal dissipative operator and define its characteristic function in terms of the Titchmarsh-Weyl function of the corresponding self-adjoint operator. We prove the theorem on completeness of the system of eigenfunctions and associated functions of the maximal dissipative Sturm-Liouville operators.

**2. Self-adjoint and nonself-adjoint extensions of a minimal symmetric operator.** We consider the Sturm-Liouville differential expression with singular point $b$:

$$\ell y := \frac{1}{w(x)}[(-p(x)y'(x))' + q(x)y(x)] \tag{2.1}$$
$$x \in \mathbf{I} := [a, b)\,, -\infty < a < b \leq +\infty,$$

where $p, q$ and $w$ are real valued, Lebesgue measurable functions on $\mathbf{I}$, and $p^{-1}, q, w \in L^1_{loc}(\mathbf{I})$, $w > 0$ almost everywhere on $\mathbf{I}$. These conditions for $p, q$ and $w$ are minimal; note that there is no sign restriction on the coefficient $p$.

To pass from the differential expression to operators, we introduce the Hilbert space $L^2_w(\mathbf{I})$ consisting of all complex valued functions $y$ such that

$$\int_a^b w(x)|y(x)|^2\,dx < \infty$$

with the inner product

$$(y, z) = \int_a^b w(x)y(x)\overline{z(x)}\,dx.$$

Denote by $D$ the linear set of all functions $y \in L^2_w(\mathbf{I})$ such that $y$ and $py'$ are locally absolutely continuous functions on $\mathbf{I}$ and $\ell y \in L^2_w(\mathbf{I})$. We define the operator $L$ on $D$ by the equality $Ly = \ell y$.

For two arbitrary functions $y, z \in D$, we have Green's formula

$$\int_a^x w(\xi)(\ell y)(\xi)\overline{z(\xi)}\,d\xi - \int_a^x w(\xi)y(\xi)\overline{(\ell z)(\xi)}\,d\xi = [y, z]\,(x) - [y, z]\,(a), \tag{2.2}$$

where $[y, z]\,(x) := W_x(y, \overline{z}) := (yp\overline{z}' - py'\overline{z})\,(x)$, $x \in \mathbf{I}$. It is clear from (2.2) that limit $[y, z]\,(b) := \lim_{x\to b^-}[y, z]\,(x)$ exists and is finite for all $y, z \in D$. For any function $y \in D$, $y\,(a)$ and $(py')\,(a)$ can be defined by $y(a) := \lim_{t\to a^+} y(t)$ and $(py')\,(a) := \lim_{t\to a^+}(py')(t)$. These limits exist and are finite, since $y$ and $py'$ are absolutely continuous functions on $[a, c]$, $\forall\, c \in (a, b)$.

We denote by $D_0$ the set of all functions $y$ in $D$ which satisfy the conditions

$$y\,(a) = (py')\,(a) = 0, \quad [y, z]\,(b) = 0, \quad \forall\, z \in D. \tag{2.3}$$

Further, we denote the restriction of the operator $L$ to $D_0$ by $L_0$. The operator $L_0$ is a closed, symmetric operator with deficiency indices $(1,1)$ or $(2,2)$, and $L = L_0^*$ [**1, 4, 5, 6, 10, 16, 20, 21**]. The operators $L_0$ and $L$ are called the *minimal* and *maximal* operators, respectively.

Let symmetric operator $L_0$ have deficiency indices $(1,1)$, so the case of Weyl's limit-point occurs for $\ell$ or $L_0$. Then, all the self-adjoint extensions $\mathbf{L}_\alpha$ of the operator $L_0$ are described by the boundary conditions: $y\,(a)\cos\alpha + (py')\,(a)\sin\alpha = 0$, $\alpha \in [0,\pi)$, $y \in D$, [**1, 4, 5, 16, 20, 21**].

Recall that a linear operator $S$, with dense domain $D(S)$, acting in some Hilbert space $H$ is called *dissipative* (*accretive*) if $\operatorname{Im}(Sf,f) \geq 0$, $\operatorname{Im}(Sf,f) \leq 0$, for all $f \in D(S)$ and *maximal dissipative* (*maximal accretive*) if it does not have a proper dissipative (accretive) extension.

All the maximal dissipative (accretive) extensions $\mathbf{L}_h$ of the operator $L_0$ are described by the boundary conditions: $(py')\,(a) - hy\,(a) = 0$, where $\operatorname{Im} h \geq 0$ or $h = \infty$, $\operatorname{Im} h \leq 0$ or $h = \infty$, $y \in D$. For $h = \infty$, the corresponding boundary condition has the form $y\,(a) = 0$.

Further, we assume that $L_0$ has deficiency indices $(2,2)$, so that the Weyl limit-circle case holds for the differential expression $\ell$ or the operator $L_0$, see [**1, 4, 5, 6, 10, 16, 20, 21**].

Denote by $u(x)$ and $v(x)$ the solutions (real-valued) of the equation

$$\ell y = 0, \quad x \in \mathbf{I} \tag{2.4}$$

satisfying the initial conditions

$$u(a) = 1, \quad (pu')(a) = 0, \quad v(a) = 0, \quad (pv')(a) = 1. \tag{2.5}$$

It follows from the conditions (2.5) and the constancy of the Wronskian that

$$W_x(u,v) = W_a(u,v) = 1, \quad a \leq x \leq b. \tag{2.6}$$

Consequently, $u$ and $v$ form a fundamental system of solutions of (2.4). Since $L_0$ has deficiency indices $(2,2)$, $u, v \in L_w^2(\mathbf{I})$ and, moreover, $u, v \in D$.

**Lemma 2.1.** **The Plücker identity.** *For arbitrary functions $y, z \in D$, we have the equality*

$$[y,z]\,(x) = \det\begin{pmatrix} [y,u]\,(x) & [y,v]\,(x) \\ [\bar{z},u]\,(x) & [\bar{z},v]\,(x) \end{pmatrix}, \quad a \leq x \leq b. \tag{2.7}$$

*Proof.* Since the functions $u$ and $v$ are real-valued and since $[u,v](x)=1$ $(a\le x\le b)$, one obtains

$$\begin{aligned}
&([y,u]\,[\bar z,v]-[y,v]\,[\bar z,u])(x)\\
&\quad=((ypu'-py'u)\,(\bar zpv'-p\bar z'v))(x)-((ypv'-py'v)\,(\bar zpu'-p\bar z'u))(x)\\
&\quad=(ypu'\bar zpv'-ypu'p\bar z'v-py'u\bar zpv')(x)\\
&\qquad+(py'up\bar z'v-ypv'\bar zpu'+ypv'p\bar z'u+py'v\bar zpu'-py'vp\bar z'u)(x)\\
&\quad=((-yp\bar z'+py'\bar z)\,(pu'v-upv'))(x)=[y,z]\,(x).
\end{aligned}$$

The lemma is proved. □

**Theorem 2.2.** *The domain $D_0$ of the operator $L_0$ consists of precisely those functions $y\in D$ satisfying the following boundary conditions*

$$y(a)=(py')(a)=0,\quad [y,u]\,(b)=[y,v]\,(b)=0. \tag{2.8}$$

*Proof.* As noted above, the domain $D_0$ of $L_0$ coincides with the set of all functions $y\in D$ satisfying (2.3). By virtue of Lemma 2.1, (2.3) is equivalent to

$$y(a)=(py')(a)=0,\quad [y,u]\,(b)\,[\bar z,v]\,(b)-[y,v]\,(b)\,[\bar z,u]\,(b)=0. \tag{2.9}$$

Further $[\bar z,v]\,(b)$ and $[\bar z,u]\,(b)$, $z\in D$, can be arbitrary, therefore equality (2.9) for all $z\in D$ is possible if and only if the conditions (2.8) hold. The theorem is proved. □

An important role in the theory of extensions is played by the concept of the space of boundary values of the symmetric operator. The triplet $(\mathcal{H},\Gamma_1,\Gamma_2)$, where $\mathcal{H}$ is a Hilbert space and $\Gamma_1$ and $\Gamma_2$ are linear mappings of $D\,(A^*)$ into $\mathcal{H}$, is called, see [**3, 9**, p. 152], [**11**], a *space of boundary values* of a closed symmetric operator $A$, acting in a Hilbert space $H$ with equal (finite or infinite) deficiency indices if

i) $(A^*f,g)_H-(f,A^*g)_H=(\Gamma_1f,\Gamma_2g)_{\mathcal{H}}-(\Gamma_2f,\Gamma_1g)_{\mathcal{H}}$, for all $f,g\in D\,(A^*)$, and

ii) for every $F_1,F_2\in\mathcal{H}$, there exists a vector $f\in D\,(A^*)$ such that $\Gamma_1f=F_1$ and $\Gamma_2f=F_2$.

Let us adopt the notation $E := \mathbf{C}^2$, and denote by $\Gamma_1$ and $\Gamma_2$ the linear mappings of $D$ into $E$ defined by

$$\Gamma_1 y = \begin{pmatrix} -y(a) \\ [y,u]\,(b) \end{pmatrix}, \quad \Gamma_2 y = \begin{pmatrix} (py')(a) \\ [y,v]\,(b) \end{pmatrix}. \tag{2.10}$$

Then we have

**Theorem 2.3.** *The triplet $(E, \Gamma_1, \Gamma_2)$ defined according to (2.10) is a space of boundary values of the operator $L_0$.*

*Proof.* The first requirement of the definition of a space of boundary values holds in view of (2.2) and Lemma 2.1:

$$\begin{aligned}(\Gamma_1 y, \Gamma_2 z)_E - (\Gamma_2 y, \Gamma_1 z)_E &= -y\,(a)\,(p\bar{z}')\,(a) + (py')\,(a)\,\bar{z}\,(a) \\ &\quad + [y,u]\,(b)\,[\bar{z},v]\,(b) - [y,v]\,(b)\,[\bar{z},u]\,(b) \\ &= [y,z]\,(b) - [y,z]\,(a) \\ &= (Ly,z) - (y,Lz)\,, \quad \forall\, y,z \in D.\end{aligned}$$

The second requirement will be proved as the following lemma.

**Lemma 2.4.** *For any complex numbers $\alpha_0, \alpha_1, \beta_0$ and $\beta_1$, there is a function $y \in D$ satisfying the boundary conditions*:

$$y\,(a) = \alpha_0, \quad (py')\,(a) = \alpha_1, \quad [y,u]\,(b) = \beta_0, \quad [y,v]\,(b) = \beta_1. \tag{2.11}$$

*Proof.* Let $f$ be an arbitrary function in $L^2_w\,(\mathbf{I})$ satisfying

$$(f,u) = \beta_0 + \alpha_1, \quad (f,v) = \beta_1 - \alpha_0. \tag{2.12}$$

There is such an $f$ even among the linear combination of $u$ and $v$. Indeed, if we set $f = c_1 u + c_2 v$, then conditions (2.12) are a system of equations in the constants $c_1$ and $c_2$ whose determinant is the Gram determinant of the linearly independent functions $u$ and $v$ and is, therefore, nonzero.

Denote by $y\,(x)$ the solution of the equation $\ell y = f\,(x)$, $x \in \mathbf{I}$, satisfying the initial conditions $y\,(a) = \alpha_0$, $(py')(a) = \alpha_1$. We claim that $y(x)$ is the desired function. We first observe that $y(x)$ is expressed by

$$y(x) = \alpha_0 u(x) + \alpha_1 v(x) + \int_a^x \{u(x)v(\xi) - u(\xi)v(x)\} w(\xi) f(\xi)\, d\xi,$$

Observing that $u, v \in L_w^2(\mathbf{I})$, we have $y \in L_w^2(\mathbf{I})$ and, moreover, $y \in D$. Further, applying Green's formula (2.2) to $y$ and $u$, we obtain $(f, u) = (\ell y, u) = [y, u](b) - [y, u](a) + (y, \ell u)$. But $\ell u = 0$, and thus $(y, \ell u) = 0$. Moreover, since $y(a) = \alpha_0, (py')(a) = \alpha_1$, we have $[y, u](a) = y(a)(pu')(a) - (py')(a)u(a) = -\alpha_1$. Therefore,

$$(f, u) = [y, u](b) + \alpha_1. \tag{2.13}$$

Then, from (2.12) and (2.13) we obtain $[y, u](b) = \beta_0$.

Analogously,

$$(f, v) = (\ell y, v) = [y, v](b) - [y, v](a) + (y, \ell v) = [y, v](b) - \alpha_0. \tag{2.14}$$

Then, from (2.12) and (2.14) we obtain $[y, v](b) = \beta_1$. Lemma 2.4 is proved and consequently, so is Theorem 2.3.

Using Theorem 2.3 and [**3, 9**, Theorem 1.6, p. 156], [**11**], we can state the following theorem.

**Theorem 2.5.** *For any contraction $K$ in $E$, $(= \mathbf{C}^2)$, i.e., $\|K\|_E \le 1$, the restriction of the operator $L$ to the set of functions $y \in D$ satisfying the boundary condition*

$$(K - I)\Gamma_1 y + i(K + I)\Gamma_2 y = 0 \tag{2.15}$$

*or*

$$(K - I)\Gamma_1 y - i(K + I)\Gamma_2 y = 0 \tag{2.16}$$

*is, respectively, a maximal dissipative or a maximal accretive extension of the operator $L_0$. Conversely, every maximal dissipative (maximal accretive) extension of $L_0$ is the restriction of $L$ to the set of vectors $y \in D$ satisfying (2.15) ((2.16)), and the contraction $K$ is uniquely determined by the extensions. These conditions give self-adjoint extension if $K$ is unitary. In the latter case (2.15) and (2.16) are equivalent* *to the condition* $(\cos A)\,\Gamma_1 y - (\sin A)\,\Gamma_2 y = 0$, *where* $A$ *is a self-adjoint operator* (*hermitian matrix*) *in* $E$. *The general form of the dissipative and accretive extensions of the operator* $L_0$ *is given by the conditions*

$$K\left(\Gamma_1 y + i\Gamma_2 y\right) = \Gamma_1 y - i\Gamma_2 y, \quad \Gamma_1 y + i\Gamma_2 y \in D\left(K\right) \tag{2.17}$$

$$K\left(\Gamma_1 y - i\Gamma_2 y\right) = \Gamma_1 y + i\Gamma_2 y, \quad \Gamma_1 y - i\Gamma_2 y \in D\left(K\right) \tag{2.18}$$

*respectively, where* $K$ *is a linear operator, with domain* $D\left(K\right) \subseteq E$, *in* $E$ *with* $\|Kf\| \le \|f\|$, $f \in D\left(K\right)$. *The general form of symmetric extensions is given by the formulae* (2.17) *and* (2.18), *where* $K$ *is an isometric operator.*

*In particular, the boundary conditions*

$$\left(py'\right)(a) - h_1 y\left(a\right) = 0 \tag{2.19}$$

$$\left[y,v\right](b) + h_2\left[y,u\right](b) = 0 \tag{2.20}$$

*with* $\operatorname{Im} h_1 \ge 0$ *or* $h_1 = \infty$, *and* $\operatorname{Im} h_2 \ge 0$ *or* $h_2 = \infty$, $\operatorname{Im} h_1 \le 0$ *or* $h_1 = \infty$, *and* $\operatorname{Im} h_2 \le 0$ *or* $h_2 = \infty$ *describe all the maximal dissipative* (*maximal accretive*) *extensions of* $L_0$ *with separated boundary conditions. The self-adjoint extensions of* $L_0$ *are obtained precisely when* $\operatorname{Im} h_1 = 0$ *or* $h_1 = \infty$, *and* $\operatorname{Im} h_2 = 0$ *or* $h_2 = \infty$. *Here for* $h_1 = \infty$, $(h_2 = \infty)$, *condition* (2.19) ((2.20)) *should be replaced by* $y\left(a\right) = 0$, $(\left[y,u\right](b) = 0)$.

In the sequel we shall study the maximal dissipative operator $L_K$, where $K$ is the strict contraction in $E$, i.e., $\|K\|_E < 1$, generated by the expression $\ell$ and boundary condition (2.15). It is obvious that the boundary condition, generally speaking, may be coupled. In particular, if we consider separated boundary conditions (2.19) and (2.20), then at points $a$ and $b$ there are simultaneously nonself-adjoint (dissipative) boundary conditions, i.e., $\operatorname{Im} h_1 > 0$ and $\operatorname{Im} h_2 > 0$.

Since $K$ is a strict contraction, the operator $K+I$ must be invertible, and the boundary condition (2.15) is equivalent to the condition

$$\Gamma_2 y + T\Gamma_1 y = 0, \tag{2.21}$$

where $T = -i\left(K+I\right)^{-1}\left(K-I\right)$, $\operatorname{Im} T > 0$, and $-K$ is the Cayley transform of the dissipative operator $T$. We denote by $\tilde{L}_T\,(= L_K)$ the maximal dissipative operator generated by the expression $\ell$ and the boundary condition (2.21).

Let

$$T = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix},$$

where $\operatorname{Im} h_1 > 0$, $\operatorname{Im} h_2 > 0$. Then the boundary condition (2.21) coincides with the separated boundary conditions (2.19) and (2.20).

We should note that a numerous research is devoted to spectral analysis of the self-adjoint singular Sturm-Liouville operators in Weyl's limit-circle case, see [**1, 2, 4, 5, 12, 16, 20, 21, 23**].

**3. Self-adjoint dilation of the maximal dissipative operator.** Let us add to space $H := L^2_w(\mathbf{I})$ the 'incoming' and 'outgoing' channels $D_- := L^2((-\infty, 0); E)$ and $D_+ := L^2((0,\infty); E)$. We form the orthogonal sum $\mathcal{H} = D_- \oplus H \oplus D_+$ and call it the *main Hilbert space of the dilation*. The elements of $\mathcal{H}$ are three-component vector-valued functions $f = \langle \varphi_-, y, \varphi_+ \rangle$. In $\mathcal{H}$ we shall consider the operator $\mathcal{L}_T$ generated by

$$\mathcal{L}\langle \varphi_-, y, \varphi_+\rangle = \left\langle i\,\frac{d\varphi_-}{d\xi},\, \ell\, y,\, i\,\frac{d\varphi_+}{d\varsigma}\right\rangle \tag{3.1}$$

on the set of vectors $D\,(\mathcal{L}_T)$ satisfying the conditions: $\varphi_- \in W_2^1((-\infty,0); E)$, $\varphi_+ \in W_2^1((0,\infty); E)$, $y \in D$,

$$\Gamma_2 y + T\Gamma_1 y = C\varphi_-\,(0)\,, \quad \Gamma_2 y + T^*\Gamma_1 y = C\varphi_+\,(0)\,, \tag{3.2}$$

where $C^2 := 2\operatorname{Im} T$, $C > 0$, and $W_2^1$ is the Sobolev space. Then we have

**Theorem 3.1.** *The operator $\mathcal{L}_T$ is self-adjoint in $\mathcal{H}$ and it is a self-adjoint dilation of the maximal dissipative operator $\tilde{L}_T\,(= L_K)$.*

*Proof.* Let $f, g \in D\,(\mathcal{L}_T)$ and $f = \langle \varphi_-, y, \varphi_+\rangle$, $g = \langle \psi_-, z, \psi_+\rangle$. Then we have

$$\begin{aligned}(\mathcal{L}_T f, g)_{\mathcal{H}} - (f, \mathcal{L}_T g)_{\mathcal{H}} &= i\,(\varphi_-\,(0)\,, \psi_-\,(0))_E - i\,(\varphi_+\,(0)\,, \psi_+\,(0))_E \\ &\quad + [y,z]\,(b) - [y,z]\,(a).\end{aligned} \tag{3.3}$$

Using the boundary conditions (3.2) and (2.7), we obtain by direct computation

$$i\left(\varphi_-(0),\psi_-(0)\right)_E - i\left(\varphi_+(0),\psi_+(0)\right)_E + [y,z](b) - [y,z](a) = 0.$$

Thus, the operator $\mathcal{L}_T$ is symmetric, and $D(\mathcal{L}_T) \subseteq D(\mathcal{L}_T^*)$.

It is easy to check that $\mathcal{L}_T$ and $\mathcal{L}_T^*$ are generated by the same expression (3.1). Let us describe the domain of $\mathcal{L}_T^*$. We shall compute the terms outside the integral sign, which are obtained by integration by parts in bilinear form $(\mathcal{L}_T f, g)_{\mathcal{H}}$, $f \in D(\mathcal{L}_T)$, $g \in D(\mathcal{L}_T^*)$. Their sum is equal to zero:

$$[y,z](b) - [y,z](a) + i\left(\varphi_-(0),\psi_-(0)\right)_E - i\left(\varphi_+(0),\psi_+(0)\right)_E = 0. \tag{3.4}$$

Further, solving the boundary conditions (3.2) for $\Gamma_1 y$ and $\Gamma_2 y$, we find that

$$\begin{aligned}
\Gamma_1 y &= -iC^{-1}\left(\varphi_-(0) - \varphi_+(0)\right),\\
\Gamma_2 y &= C\varphi_-(0) + iTC^{-1}\left(\varphi_-(0) - \varphi_+(0)\right).
\end{aligned}$$

Therefore, using (2.7) and (2.10), we find that (3.4) is equivalent to the equality

$$\begin{aligned}
& i\left(\varphi_+(0),\psi_+(0)\right)_E - i\left(\varphi_-(0),\psi_-(0)\right)_E \\
&\quad = [y,z](b) - [y,z](a) \\
&\quad = [y,u](b)\,[\bar{z},v](b) - [y,v](b)\,[\bar{z},u](b) - [y,u](a)\,[\bar{z},v](a) \\
&\qquad + [y,v](a)\,[\bar{z},u](a) \\
&\quad = (\Gamma_1 y, \Gamma_2 z)_E - (\Gamma_2 y, \Gamma_1 z)_E \\
&\quad = -i\left(C^{-1}\left(\varphi_-(0) - \varphi_+(0)\right), \Gamma_2 z\right)_E - \left(C\varphi_-(0), \Gamma_1 z\right)_E \\
&\qquad - i\left(TC^{-1}\left(\varphi_-(0) - \varphi_+(0)\right), \Gamma_1 z\right)_E.
\end{aligned}$$

Since the values $\varphi_\pm(0)$ can be arbitrary vectors, a comparison of the coefficients of $\varphi_{i\pm}(0)$, $i = 1,2$, on the left and right of the last equality proves us that the vector $g = \langle \psi_-, z, \psi_+ \rangle$ satisfies the boundary conditions (3.2): $\Gamma_2 z + T\Gamma_1 z = C\psi_-(0)$, $\Gamma_2 z + T^*\Gamma_1 z = C\psi_+(0)$. Therefore, $D(\mathcal{L}_T^*) \subseteq D(\mathcal{L}_T)$, and hence, $\mathcal{L}_T = \mathcal{L}_T^*$.

The self-adjoint operator $\mathcal{L}_T$ generates the unitary group $U_t = \exp[i\mathcal{L}_T t]$, $t \in \mathbf{R} := (-\infty,\infty)$, on $\mathcal{H}$. Denote by $P : \mathcal{H} \to H$

and $P_1 : H \to \mathcal{H}$ the mappings acting according to the formulae $P : \langle \varphi_-, y, \varphi_+ \rangle \to y$ and $P_1 : y \to \langle 0, y, 0 \rangle$, respectively. Let us define $Z_t = PU_tP_1, t \geq 0$. The operator family $\{Z_t\}_{t \geq 0}$ is a strictly continuous semigroup of completely nonunitary contractions on $H$, see [**13, 14, 15**]. Let us define the generator of this semigroup by $B_T$. We shall show that $\tilde{L}_T = B_T$.

To do this, we first prove that the equality

$$P\left(\mathcal{L}_T - \lambda I\right)^{-1} P_1 y = \left(\tilde{L}_T - \lambda I\right)^{-1} y, \quad y \in H, \quad \operatorname{Im}\lambda < 0 \tag{3.5}$$

holds. Let us define $g$ by $\left(\mathcal{L}_T - \lambda I\right)^{-1} P_1 y = g = \langle \psi_-, z, \psi_+ \rangle$. Then $\left(\mathcal{L}_T - \lambda I\right) g = P_1 y$, and $\ell z - \lambda z = y$, $\psi_-(\xi) = \psi_-(0)\, e^{-i\lambda\xi}$, $\psi_+(\varsigma) = \psi_+(0)\, e^{-i\lambda\varsigma}$. Since $g \in D\left(\mathcal{L}_T\right)$, hence $\psi_- \in W_2^1\left((-\infty, 0); E\right)$ and so $\psi_-(0) = 0$ and, consequently, $z$ satisfies the boundary condition $\Gamma_2 z + T\Gamma_1 z = 0$. Therefore, $z \in D(\tilde{L}_T)$, and since a point $\lambda$ with $\operatorname{Im}\lambda < 0$ cannot be an eigenvalue of dissipative operator, then $z = (\tilde{L}_T - \lambda I)^{-1} y$. Thus for $y \in H$ and $\operatorname{Im}\lambda < 0$ we have

$$\left(\mathcal{L}_T - \lambda I\right)^{-1} P_1 y = \left\langle 0, \left(\tilde{L}_T - \lambda I\right)^{-1} y, C^{-1}(\Gamma_2 y + T^*\Gamma_1 y)\, e^{-i\lambda\varsigma} \right\rangle.$$

Next, applying the mapping $P$ to this equality, we obtain (3.5). In view of (3.5) we get

$$\begin{aligned} \left(\tilde{L}_T - \lambda I\right)^{-1} &= P\left(\mathcal{L}_T - \lambda I\right)^{-1} P_1 = -iP \int_0^\infty U_t e^{-i\lambda t}\, dt P_1 \\ &= -i \int_0^\infty Z_t e^{-i\lambda t}\, dt = \left(B_T - \lambda I\right)^{-1}, \quad \operatorname{Im}\lambda < 0. \end{aligned}$$

Hence $\tilde{L}_T = B_T$, and the theorem is proved. $\square$

**4. Scattering theory of the dilation and functional model of the maximal dissipative operator.** The unitary group $\{U_t\}$ has an important property which allows us to apply the Lax-Phillip's scheme [**14**], namely, it has the orthogonal 'incoming' and 'outgoing' subspaces $D_- = \langle L^2\left((-\infty, 0); E\right), 0, 0 \rangle$ and $D_+ = \langle 0, 0, L^2\left((0, \infty); E\right) \rangle$ with the following properties:

(1) $U_t D_- \subset D_-,\ t \leq 0$; $U_t D_+ \subset D_+,\ t \geq 0$;

(2) $\cap_{t\leq 0}U_tD_- = \cap_{t\geq 0}U_tD_+ = \{0\}$;

(3) $\overline{\cup_{t\geq 0}U_tD_-} = \overline{\cup_{t\leq 0}U_tD_+} = \mathcal{H}$;

(4) $D_- \perp D_+$.

Property (4) is obvious. We shall prove the property (1) for $D_+$ (for $D_-$, the proof is analogous). Let $R_\lambda = (\mathcal{L}_T - \lambda I)^{-1}$. For all $\lambda$ with $\operatorname{Im}\lambda < 0$ and for all $f = \langle 0, 0, \varphi_+\rangle \in D_+$, we have

$$R_\lambda f = \left\langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\varphi_+(s)\, ds\right\rangle.$$

So we have $R_\lambda f \in D_+$. Therefore, if $g \perp D_+$, then

$$0 = (R_\lambda f, g)_{\mathcal{H}} = -i\int_0^\infty e^{-i\lambda t}(U_t f, g)_{\mathcal{H}}\, dt, \quad \operatorname{Im}\lambda < 0.$$

From this we have $(U_t f, g)_{\mathcal{H}} = 0$ for all $t \geq 0$. Consequently, $U_tD_+ \subset D_+$ for $t \geq 0$ and the property (1) is proved.

To prove the property (2) we define $P^+ : \mathcal{H} \mapsto L^2(\mathbf{R}_+; E)$ $(bfR_+ := [0,\infty))$ and $P_1^+ : L^2(\mathbf{R}_+; E) \mapsto D_+$ as mappings with $P^+ : \langle \varphi_- y, \varphi_+\rangle \mapsto \varphi_+$ and $P_1^+ : \varphi \mapsto \langle 0, 0, \varphi\rangle$ respectively. Observe that the semigroup of isometries $U_t^+ = P^+U_tP_1^+$, $t \geq 0$, is the one-side shift in $L^2(\mathbf{R}_+; E)$. Indeed, the generator of the semigroup of the shift $V_t$ in $L^2(\mathbf{R}_+; E)$ is the differential operator $id/d\xi$ with the boundary condition $\varphi(0) = 0$. On the other hand, the generator $B$ of semi-group of isometries $U_t^+$, $t \geq 0$, is the operator $B\varphi = P^+\mathcal{L}_TP_1^+\varphi = P^+\mathcal{L}_T\langle 0, 0, \varphi\rangle = P^+\langle 0, 0, i(d\varphi/d\xi)\rangle = i(d\varphi/d\xi)$, where $\varphi \in W_2^1(\mathbf{R}_+; E)$ and $\varphi(0) = 0$. But since a semi-group is uniquely determined by its generator, it follows that $U_t^+ = V_t$, and hence,

$$\cap_{t\geq 0}U_tD_+ = \langle 0, 0, \cap_{t\geq 0}V_tL^2(\mathbf{R}_+; E)\rangle = \{0\},$$

i.e., the property (2) is proved.

In the scheme of the Lax-Phillips scattering theory, the scattering matrix is defined in terms of the theory of spectral representations. We proceed to construct them. Along the way, we also prove property (3) of the incoming and outgoing subspaces.

We recall that the linear operator $A$, with domain $D(A)$, acting in the Hilbert space $H$ is called totally nonself-adjoint (or simple) if invariant subspace $M \subseteq D(A)$, $M \neq 0$, of the operator $A$ on which restriction $A$ on $M$ is self-adjoint does not exist.

Then we have

**Lemma 4.1.** *The operator $\tilde{L}_T$ $(L_K)$ is totally nonself-adjoint (simple).*

*Proof.* Let $H_0 \subset H$ be a subspace where the operator $\tilde{L}_T$ indicates the self-adjoint operator $\tilde{L}'_T$, i.e., the subspace $H_0$ is invariant with respect to semigroup of isometries $V_t = \exp\left(i\tilde{L}'_T t\right)$, $V_t^* = \exp\left(-i\tilde{L}'_T t\right)$, $V_t^{-1} = V_t^*$, $t > 0$. If $f \in H_0 \cap D\left(\tilde{L}_T\right)$, then $f \in D\left(\tilde{L}_T^*\right)$, and

$$\begin{aligned} 0 &= \frac{d}{dt}\left\|\exp\left(i\tilde{L}_T t\right) f\right\|_H^2 \\ &= -2\left(\operatorname{Im} T\Gamma_1\left(\exp\left(i\tilde{L}_T t\right) f, \Gamma_1 \exp\left(i\tilde{L}_T t\right) f\right)\right)_E, \quad \operatorname{Im} T > 0. \end{aligned}$$

Consequently, we have $\Gamma_1\left(\exp\left(i\tilde{L}_T t\right) f\right) = 0$. For eigenvectors $y_\lambda \in H_0$ of the operator $\tilde{L}_T$, we have $\Gamma_1 y_\lambda = 0$. Using this result with boundary condition $\Gamma_2 y + T\Gamma_1 y = 0$, we have $\Gamma_2 y_\lambda = 0$, i.e., $y_\lambda(a) = 0, (py'_\lambda)(a) = 0$. Then by the uniqueness theorem of the Cauchy problem for the equation $\ell y = \lambda y$, $x \in \mathbf{I}$, we have $y_\lambda \equiv 0$. Since all solutions of $\ell y = \lambda y$ $(x \in \mathbf{I})$ belong to $L_w^2(\mathbf{I})$, it can be concluded that the resolvent $R_\lambda(\tilde{L}_T)$ of the operator $\tilde{L}_T$ is a Hilbert-Schmidt operator, and hence the spectrum of $\tilde{L}_T$ is purely discrete. Hence, by the theorem on expansion in eigenfunctions of the self-adjoint operator $\tilde{L}'_T$, we have $H_0 = \{0\}$, i.e., the operator $\tilde{L}_T$ is simple. The lemma is proved. □

To prove the property (3), we set

$$H_- = \overline{\cup_{t \geq 0} U_t D_-}, \quad H_+ = \overline{\cup_{t \leq 0} U_t D_+},$$

and first prove

**Lemma 4.2.** *The equality $H_- + H_+ = \mathcal{H}$ holds.*

*Proof.* Taking into account the property (1) of the subspaces $D_\pm$, we shall show that the subspace $\mathcal{H}' = \mathcal{H} \ominus (H_- + H_+)$ is invariant with respect to the group $\{U_t\}$ and has the form $\mathcal{H}' = \langle 0, H', 0\rangle$, where $H'$ is a subspace of $H$. Therefore, if the subspace $\mathcal{H}'$, and hence also $H'$, were nontrivial, then the unitary group $\{U_t'\}$ restricted to this subspace, would be a unitary part of the group $\{U_t\}$, and therefore, the restriction $L_T'$ of the operator $L_T$ to $H'$ would be the self-adjoint operator of $H'$. It follows from simplicity of the operator $L_T'$ that $H' = \{0\}$, i.e., $\mathcal{H}' = \{0\}$. So, the lemma is proved. □

We denote by $\varphi$ and $\psi$ those solutions of the equation $\ell y = \lambda y$, $x \in \mathbf{I}$, with initial conditions

(4.1)

$$\varphi(a,\lambda) = 0, \quad (p\varphi')(a,\lambda) = -1, \quad \psi(a,\lambda) = 1, \quad (p\psi')(a,\lambda) = 0.$$

We denote by $M(\lambda)$ the matrix-valued function satisfying the conditions

$$M(\lambda)\Gamma_1\varphi = \Gamma_2\varphi, \quad M(\lambda)\Gamma_1\psi = \Gamma_2\psi. \tag{4.2}$$

It can be directly verified that $M(\lambda)$ has the form

$$M(\lambda) = \begin{pmatrix} m_\infty(\lambda) & -1/[\varphi,u](b) \\ -1/[\varphi,u](b) & [\varphi,v](b)/[\varphi,u](b) \end{pmatrix}, \tag{4.3}$$

where $m_\infty(\lambda)$ is the Titchmarsh-Weyl function of the self-adjoint operator $L_\infty$ generated by the expression $\ell$ with the boundary conditions $y(a) = 0$ and $[y,u](b) = 0$. Then we have

$$m_\infty(\lambda) = -\frac{[\psi,u](b)}{[\varphi,u](b)}.$$

It is easy to show that the matrix-valued function $M(\lambda)$ is meromorphic in $\mathbf{C}$ with all its poles on real axis $\mathbf{R}$, and that it has the following properties:

(a) $\operatorname{Im} M(\lambda) \le 0$ if $\operatorname{Im}\lambda > 0$, and $\operatorname{Im} M(\lambda) \ge 0$ if $\operatorname{Im}\lambda < 0$;

(b) $M^*(\lambda) = M(\lambda)$ for all $\lambda \in \mathbf{R}$, except for the poles of $M(\lambda)$.

We denote by $\chi_j(x)$ and $\theta_j(x)$, $j = 1, 2$, the solutions of the equation $\ell y = \lambda y$, $x \in \mathbf{I}$, which satisfy the conditions

$$\begin{aligned} \Gamma_1 \chi_j &= (M(\lambda) + T)^{-1} C e_j, \\ \Gamma_1 \theta_j &= (M(\lambda) + T^*)^{-1} C e_j, \quad j = 1, 2, \end{aligned} \tag{4.4}$$

where $e_1$ and $e_2$ are the orthonormal basis for $E$.

Let $U_{\lambda j}^-$, $j = 1, 2$, be defined by

$$\begin{aligned} &U_{\lambda j}^-(x, \xi, \varsigma) \\ &\qquad = \left\langle e^{-i\lambda\xi} e_j,\, \chi_j(x),\, C^{-1}(M(\lambda) + T^*)(M(\lambda) + T)^{-1} C e^{-i\lambda\varsigma} e_j \right\rangle. \end{aligned}$$

It must be noted that vectors $U_{\lambda j}^-$, $j = 1, 2$, for all $\lambda \in \mathbf{R}$ do not belong to $\mathcal{H}$. However, $U_{\lambda j}^-$, $j = 1, 2$, satisfies the equation $\mathcal{L}U = \lambda U$ and the boundary conditions (3.2).

The transformation $F_- : f \to \tilde{f}_-(\lambda)$ for the vectors $f = \langle \varphi_-, y, \varphi_+ \rangle$ is determined using the vectors $U_{\lambda j}^-$, $j = 1, 2$, by the formula

$$(F_- f)(\lambda) := \tilde{f}_-(\lambda) := \sum_{j=1}^{2} \tilde{f}_j^-(\lambda) e_j,$$

where $\varphi_-(\xi), \varphi_+(\varsigma)$ and $y(x)$ are smooth, compactly supported functions, and

$$\tilde{f}_j^-(\lambda) = \frac{1}{\sqrt{2\pi}} \left( f, U_{\lambda j}^- \right)_{\mathcal{H}}, \quad j = 1, 2.$$

**Lemma 4.3.** *The transformation $F_-$ isometrically maps $H_-$ onto $L^2(\mathbf{R}; E)$. For all vectors $f, g \in H_-$, the Parseval equality*

$$(f, g)_{\mathcal{H}} = \left( \tilde{f}_-, \tilde{g}_- \right)_{L^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2} \tilde{f}_j^-(\lambda) \overline{\tilde{g}_j^-(\lambda)}\, d\lambda,$$

*and the inversion formula*

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^{2} U_{\lambda j}^- \tilde{f}_j^-(\lambda)\, d\lambda,$$

*hold, where* $\tilde{f}_-(\lambda) = (F_- f)(\lambda)$, $\tilde{g}_-(\lambda) = (F_- g)(\lambda)$.

*Proof.* We shall show that the transformation $F_-$ maps $D_-$ to $H_-^2(E)$. Here and below, $H_\pm^2(E)$ denote the Hardy classes in $L^2(\mathbf{R};E)$ consisting of the vector-valued functions analytically extendible to the upper and lower half-planes, respectively. For $f,g \in D_-$, $f = \langle f_-,0,0\rangle$, $g = \langle g_-,0,0\rangle$, $f_-,g_- \in L^2((-\infty,0);E)$, we have

$$\tilde{f}_j^-(\lambda) = \frac{1}{\sqrt{2\pi}}\left(f, U_{\lambda j}^-\right)_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\infty}^{0}\left(f_-(\xi), e^{-i\lambda\xi}e_j\right)_E d\xi \in H_-^2,$$

$$\tilde{f}_-(\lambda) = \sum_{j=1}^{2}\tilde{f}_j^-(\lambda)\, e_j \in H_-^2(E),$$

and the Parseval equality:

$$(f,g)_{\mathcal{H}} = \left(\tilde{f}_-, \tilde{g}_-\right)_{L^2} = \int_{-\infty}^{\infty}\sum_{j=1}^{2}\tilde{f}_j^-(\lambda)\,\overline{\tilde{g}_j^-(\lambda)}\, d\lambda.$$

Now, we want to extend this equality to the whole $H_-$. To this end consider in $H_-'$ the dense set $H_-$ of vectors, obtained on smooth, compactly supported functions belonging to $D_-$ in the following way: $f \in H_-'$, $f = U_{t_f} f_0$, $f_0 = \langle \varphi_-,0,0\rangle$, $\varphi_- \in C_0^\infty((-\infty,0);E)$. For these vectors, noting $\mathcal{L}_T = \mathcal{L}_T^*$ and using the fact that $U_{-t}f \in \langle C_0^\infty((-\infty,0);E),0,0\rangle$, and $(U_{-t}f, U_{\lambda j}^-)_{\mathcal{H}} = e^{-i\lambda t}(f, U_{\lambda j}^-)_{\mathcal{H}}$, $j=1,2$, for $t > t_f, t_g$, we have

$$\begin{aligned}
(f,g)_{\mathcal{H}} &= (U_{-t}f, U_{-t}g)_{\mathcal{H}} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{2}\left(U_{-t}f, U_{\lambda j}^-\right)_{\mathcal{H}}\overline{\left(U_{-t}g, U_{\lambda j}^-\right)_{\mathcal{H}}}\, d\lambda \\
&= \int_{-\infty}^{\infty}\sum_{j=1}^{2}\tilde{f}_j^-(\lambda)\,\overline{\tilde{g}_j^-(\lambda)}\, d\lambda.
\end{aligned}$$

By taking the closure, we obtain the Parseval equality for the whole space $H_-$. The inversion formula follows from the Parseval equality if all integrals in it are understood as limits in the mean of the integrals on a finite interval. Finally,

$$F_- H_- = \overline{\cup_{t\geq 0} F_- U_t D_-} = \overline{\cup_{t\geq 0} e^{i\lambda t} H_-^2(E)} = L^2(\mathbf{R};E),$$

i.e., $F_-$ maps $H_-$ onto whole $L^2(\mathbf{R};E)$. So, the lemma is proved. $\square$

Let us define $U^+_{\lambda j}$, $j=1,2$, by

$$U^+_{\lambda j}(x,\xi,\varsigma)=\left\langle S_T(\lambda)\,e^{-i\lambda\xi}e_j,\theta_j(x),e^{-i\lambda\varsigma}e_j\right\rangle,$$

where

$$S_T(\lambda)=C^{-1}(M(\lambda)+T)(M(\lambda)+T^*)^{-1}C. \tag{4.5}$$

Using the vectors $U^+_{\lambda j}$ $(j=1,2)$, we will see that the transformation $F_+ : f\to\tilde f_+(\lambda)$ for the vectors $f=\langle\varphi_-,y,\varphi_+\rangle$ is determined by the formula

$$(F_+f)(\lambda):=\tilde f_+(\lambda):=\sum_{j=1}^{2}\tilde f_j^+(\lambda)\,e_j,$$

where $\varphi_-(\xi),\varphi_+(\varsigma)$, and $y(x)$ are smooth, compactly supported functions, and

$$\tilde f_j^+(\lambda)=\frac{1}{\sqrt{2\pi}}\left(f,U^+_{\lambda j}\right)_{\mathcal H},\quad j=1,2.$$

The proof of the next result is analogous to that of Lemma 4.3.

**Lemma 4.4.** *The transformation $F_+$ isometrically maps $H_+$ onto $L^2(\mathbf{R};E)$. For all vectors $f,g\in H_+$, the Parseval equality*

$$(f,g)_{\mathcal H}=\left(\tilde f_+,\tilde g_+\right)_{L^2}=\int_{-\infty}^{\infty}\sum_{j=1}^{2}\tilde f_j^+(\lambda)\,\overline{\tilde g_j^+(\lambda)}\,d\lambda,$$

*and the inversion formula*

$$f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{j=1}^{2}U^+_{\lambda j}\tilde f_j^+(\lambda)\,d\lambda,$$

*are valid, where $\tilde f_j^+(\lambda)=(F_+f)(\lambda)$, $\tilde g_+(\lambda)=(F_+g)(\lambda)$.*

It is obvious that the matrix-valued function $S_T(\lambda)$ is meromorphic in $\mathbf{C}$ and all poles are in the lower half-plane. Then, it is trivial from

(4.5) that $\|S_T(\lambda)\|_E \leq 1$ for $\operatorname{Im}\lambda > 0$ and $S_T(\lambda)$ is the unitary matrix for all $\lambda \in \mathbf{R}$.

Since $S_T(\lambda)$ is the unitary matrix for $\lambda \in \mathbf{R}$, then, it follows from the definitions of the vectors $U^+_{\lambda j}$ and $U^-_{\lambda j}$ that

$$U^+_{\lambda j} = \sum_{k=1}^{2} S_{jk}(\lambda)\, U^-_{\lambda k}, \quad j = 1, 2,$$

where $S_{jk}(\lambda)$, $j, k = 1, 2$, are entries of the matrix $S_T(\lambda)$. According to Lemma 4.2, from the last equality, it then follows that $H_- = H_+ = \mathcal{H}$. Hence, property (3) of the incoming and outgoing subspaces presented above has been established.

Thus the transformation $F_-$ maps

(i) $\mathcal{H}$ isometrically onto $L^2(\mathbf{R}; E)$,

(ii) the subspace $D_-$ onto $H^2_-(E)$, and

(iii) the operators $U_t$ are carried into the operators of multiplication by $e^{i\lambda t}$.

It means that $F_-$ is the 'incoming' spectral representation of group $U_t$. Similarly, $F_+$ is the 'outgoing' spectral representation of the group $U_t$. It follows from the formulas for $U^-_{\lambda j}$ and $U^+_{\lambda j}$, $j = 1, 2$, that the transition from $F_-$ -representation of the vector $f \in \mathcal{H}$ to its $F_+$ -representation is realized as follows: $\tilde{f}_+(\lambda) = S_T^{-1}(\lambda)\, \tilde{f}_-(\lambda)$. According to [**14**], we have now proved the following theorem.

**Theorem 4.5.** *The matrix $S_T^{-1}(\lambda)$ is the scattering matrix of the group $U_t$ (of the operator $\mathcal{L}_T$).*

Let $S(\lambda)$ be an arbitrary inner matrix-valued function [**15**] on the upper half-plane. Define $\mathcal{K} = H^2_+ \ominus SH^2_+$. Then $\mathcal{K} \neq \{0\}$ is a subspace of the Hilbert space $H^2_+$. We consider the semigroup of the operators $\mathbf{Z}_t$, $t \geq 0$, acting in $\mathcal{K}$ according to the formula $\mathbf{Z}_t\varphi = P\left[e^{i\lambda t}\varphi\right]$, $\varphi := \varphi(\lambda) \in \mathcal{K}$, where $P$ is the orthogonal projection from $H^2_+$ onto $\mathcal{K}$. The generator of the semigroup $\{\mathbf{Z}_t\}$ is denoted by $B : B\varphi = \lim_{t\to+0}(it)^{-1}(\mathbf{Z}_t\varphi - \varphi)$, which is a maximal dissipative operator acting in $\mathcal{K}$ and with the domain $D(B)$ consisting of all vectors $\varphi \in \mathcal{K}$, such that the limit exists. The operator $B$ is called a model dissipative operator (we remark that this model dissipative operator, associated with the names of Lax and Phillips [**14**], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foias [**15**]). The basic assertion is that $S(\lambda)$ is the characteristic function of the operator $B$.

From the explicit form of the unitary transformation $F_-$ that, under the mapping $F_-$, we have:

$$\mathcal{H} \longrightarrow L^2(\mathbf{R}; E), \quad f \to \tilde{f}_-(\lambda) := (F_- f)(\lambda), \quad D_- \to H^2_-(E),$$
$$D_+ \longrightarrow S_T H^2_+(E), \quad \mathcal{H} \ominus (D_- \oplus D_+) \to H^2_+(E) \ominus S_T H^2_+(E),$$
$$U_t f \longrightarrow \left(F_- U_t F_-^{-1} \tilde{f}_-\right)(\lambda) = e^{i\lambda t} \tilde{f}_-(\lambda).$$

These formulas show that the operator $\tilde{L}_T\,(L_K)$ is a unitary equivalent to the model dissipative operator with the characteristic function $S_T(\lambda)$. Thus we have proved the following theorem.

**Theorem 4.6.** *The characteristic function of the maximal dissipative operator $\tilde{L}_T$ coincides with the matrix-valued function $S_T(\lambda)$ determined by formula* (4.5). *The matrix-valued function $S_T(\lambda)$ is meromorphic in the complex plane $\mathbf{C}$ and is an inner function in the upper half-plane.*

**5. The spectral analysis of a maximal dissipative operator.** Questions of the spectral analysis of the maximal dissipative operator $L_K\left(\tilde{L}_T\right)$ can be solved in terms of the characteristic function. Thus, for example, the absence of the singular factor $s(\lambda)$in the factorization $\det S_T(\lambda) = s(\lambda)\mathcal{B}(\lambda)$ ($\mathcal{B}(\lambda)$ is the Blaschke product) ensures the completeness of the system of eigenfunctions and associated functions of the operator $\tilde{L}_T\,(L_K)$ in the space $L^2_w(\mathbf{I})$, see [**7, 13, 15**].

We first use the following lemma.

**Lemma 5.1.** *The characteristic function $\tilde{S}_K(\lambda)$ of the operator $L_K$ has the form*

$$\begin{aligned}\tilde{S}_K(\lambda) &:= S_T(\lambda)\\ &= X_1 (I - K_1K_1^*)^{-1/2} (\theta(\xi) - K_1)(I - K_1^*\theta(\xi))^{-1}\\ &\quad \cdot (I - K_1^*K_1)^{1/2} X_2,\end{aligned}$$

*where* $K_1 = -K$ *is the Cayley transformation of the dissipative operator* $T$*, and* $\theta(\xi)$*is the Cayley transformation of the matrix-valued function* $M(\lambda)$*,* $\xi = (\lambda - i)(\lambda + i)^{-1}$*, and*

$$X_1 := (\operatorname{Im} T)^{-1/2} (I - K_1)^{-1} (I - K_1 K_1^*)^{1/2},$$
$$X_2 := (I - K_1^* K_1)^{-1/2} (I - K_1^*) (\operatorname{Im} T)^{1/2}, \quad |\det X_1| = |\det X_2| = 1.$$

*Proof.* In view of Theorem 4.6, we have

$$S_T(\lambda) = (\operatorname{Im} T)^{-1/2} (M(\lambda) + T)(M(\lambda) + T^*)^{-1} (\operatorname{Im} T)^{1/2}.$$

Then

(5.1)
$$\begin{aligned}
\operatorname{Im} T &= \frac{1}{2i}(T - T^*) \\
&= \frac{1}{2}\left[(I - K_1)^{-1}(I + K_1) + (I + K_1^*)(I - K_1^*)^{-1}\right] \\
&= \frac{1}{2}\left[(I - K_1)^{-1} + (I - K_1)^{-1}K_1 + (I - K_1^*)^{-1} + K_1^*(I - K_1^*)^{-1}\right] \\
&= \frac{1}{2}\left[(I - K_1)^{-1} + (I - K_1)^{-1} - I + (I - K_1^*)^{-1} + (I - K_1^*)^{-1} - I\right] \\
&= (I - K_1)^{-1} + (I - K_1^*)^{-1} - I \\
&= (I - K_1)^{-1}\left[I - K_1^* + I - K_1 - (I - K_1)(I - K_1^*)\right](I - K_1^*)^{-1} \\
&= (I - K_1)^{-1}(I - K_1 K_1^*)(I - K_1^*)^{-1}.
\end{aligned}$$

Similarly,

$$\operatorname{Im} T = (I - K_1^*)^{-1}(I - K_1^* K_1)(I - K_1)^{-1}. \tag{5.2}$$

Let us denote by $\theta_1(\lambda)$ the Cayley transformation of the accretive operator $M(\lambda)$ for $\operatorname{Im}\lambda > 0$. Then we have

$$M(\lambda) = -i(I - \theta_1(\lambda))^{-1}(I + \theta_1(\lambda)),$$

so, we obtain

$$
\begin{aligned}
M(\lambda)+T &= -i\left[(I-\theta_1(\lambda))^{-1}(I+\theta_1(\lambda)) - (I-K_1)^{-1}(I+K_1)\right] \\
&= -i\left[-(I-\theta_1(\lambda))^{-1}(I-\theta_1(\lambda)-2I)+(I-K_1)^{-1}(I-K_1-2I)\right] \\
&= -i\left[-I+2(I-\theta_1(\lambda))^{-1}+I-2(I-K_1)^{-1}\right] \\
&= -2i\left[(I-\theta_1(\lambda))^{-1}-(I-K_1)^{-1}\right] \\
&= -2i(I-K_1)^{-1}(\theta_1(\lambda)-K_1)(I-\theta_1(\lambda))^{-1}.
\end{aligned} \tag{5.3}
$$

Similarly,

$$M(\lambda)+T^* = -2i(I-K_1^*)^{-1}(I-K_1^*\theta_1(\lambda))(I-\theta_1(\lambda))^{-1}$$

and

$$(M(\lambda)+T^*)^{-1} = -\frac{1}{2i}(I-\theta_1(\lambda))(I-K_1^*\theta_1(\lambda))^{-1}(I-K_1^*). \tag{5.4}$$

In view of (5.1)–(5.4), we have

$$
\begin{aligned}
\tilde{S}_K(\lambda) &:= S_T(\lambda) \\
&= X_1(I-K_1K_1^*)^{-1/2}(\theta(\xi)-K_1)(I-K_1^*\theta(\xi)) \\
&\quad \cdot (I-K_1^*K_1)^{1/2}X_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta(\xi) &:= \theta_1\left(-i(\xi+1)(\xi-1)^{-1}\right), X_1 \\
&:= (\operatorname{Im} T)^{-1/2}(I-K_1)(I-K_1^*K_1)^{1/2}, \\
X_2 &:= (I-K_1^*K_1)^{-1}(I-K_1^*)(\operatorname{Im} T)^{1/2}.
\end{aligned}
$$

It is evident that $|\det X_1| = |\det X_2| = 1$. Hence, the lemma is proved. □

It is known [**7, 15**] that the inner matrix-valued function $\tilde{S}_K(\lambda)$ is a Blaschke-Potopov product if and only if $\det \tilde{S}_K(\lambda)$ is a Blaschke product. Then it follows from Lemma 5.1 that the characteristic function $\tilde{S}_K(\lambda)$ is a Blaschke-Potopov product if and only if the matrix-valued function

$$X_K(\xi) = (I - K_1 K_1^*)^{-1/2} (\theta(\xi) - K_1)(I - K_1^* \theta(\xi))^{-1} (I - K_1^* K_1)^{1/2}$$

is a Blaschke-Potopov product in a unit disk.

In order to state the completeness theorem, we will first define a suitable form for the $\Gamma$-capacity [**7, 19**].

Let $\tilde{E}$ be an $m$-dimensional ($m < \infty$) Euclidean space. In $\tilde{E}$, we fix an orthonormal basis $e_1, e_2, \ldots, e_m$ and denote by $E_k$, $k = 1, 2, \ldots, m$, the linear span of vectors $e_1, e_2, \ldots, e_k$. If $M \subset E_k$, then the set of $x \in E_{k-1}$ with the property $\operatorname{Cap}\{\lambda : \lambda \in \mathbf{C}, (x + \lambda e_k) \in M\} > 0$ will be denoted by $\Gamma_{k-1} M$. ($\operatorname{Cap} G$ is the inner logarithmic capacity of the set $G \subset \mathbf{C}$). The $\Gamma$-*capacity of the set* $M \subset \tilde{E}$ is a number $\Gamma - \operatorname{Cap} M := \sup \operatorname{Cap}\{\lambda : \lambda \in \mathbf{C}, \lambda e_1 \subset \Gamma_1 \Gamma_2 \cdots \Gamma_{m-1} M\}$, where the sup is taken with respect to all orthonormal basics in $\tilde{E}$, see [**7, 19**]. It is known [**19**] that every set $M \subset \tilde{E}$ of zero $\Gamma$-capacity has zero $2m$-dimensional Lebesgue measure (in the decomplexified space $\tilde{E}$), however, the converse is false.

Denote by $[E]$ the set of all linear operators in $E\ (= \mathbf{C}^2)$. To convert $[E]$ into the 4-dimensional Euclidean space, we introduce the inner product $\langle T, S \rangle = \operatorname{tr} S^* T$ for $T, S \in [E]$ ($\operatorname{tr} S^* T$ is the trace of the operator $S^* T$). Hence, we may introduce the $\Gamma$-capacity of a set of $[E]$.

We will utilize the following important result of [**7**].

**Lemma 5.2.** *Let $X(\xi)$, $|\xi| < 1$, be a holomorphic function with the values to be contractive operators in $[E]$, i.e., $\|X(\xi)\|_E \leq 1$. Then for $\Gamma$ -quasi-every strictly contractive operators $K$ in $[E]$, i.e., for all strictly contractive $K \in [E]$ with the possible exception of a set of $\Gamma$-capacity zero, the inner part of the contractive function*

$$X_K(\xi) := (I - K K^*)^{-1/2} (X(\xi) - K)(I - K^* X(\xi))^{-1} (I - K^* K)^{1/2}$$

*is a Blaschke-Potopov product.*

Summarizing all the obtained results for the maximal dissipative operators $L_K\ (\tilde{L}_T)$, we have proved the following theorem.

**Theorem 5.3.** *For $\Gamma$-quasi-every strictly contractive $K \in [E]$, i.e., for all strictly contractive $K \in [E]$ with the possible exception of a set of $\Gamma$-capacity zero, the characteristic function $\tilde{S}_K(\lambda)$ of the maximal dissipative operator $L_K$ is a Blaschke-Potopov product, and the spectrum of $L_K$ is purely discrete and belongs to the open upper half-plane. For $\Gamma$-quasi-every strictly contractive $K \in [E]$, the operator $L_K$ has a countable number of isolated eigenvalues with finite multiplicity and limit point at infinity, and the system of eigenfunctions and associated functions of this operator is complete in the space $L^2_w(\mathbf{I})$.*

*Remarks.* 1. Since a linear operator $S$ in Hilbert space $H$ is maximal accretive if and only if $-S$ is maximal dissipative, all results concerning maximal dissipative operators can be immediately transferred to maximal accretive operators.

2. The results are valid for regular Sturm-Liouville operators (with regular end points $a$ and $b$). In this case space of boundary values for minimal symmetric operator $L_0$ has the form $(\mathbf{C}^2, \Gamma_1, \Gamma_2)$, where

$$\Gamma_1 y = \begin{pmatrix} -y(a) \\ y(b) \end{pmatrix}, \quad \Gamma_2 y = \begin{pmatrix} (py')(a) \\ (py')(b) \end{pmatrix}, \quad y \in D.$$

RETRACTED

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