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# ALMOST SURE CONVERGENCE OF AQSI SEQUENCES IN DOUBLE ARRAYS

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ABSTRACT. For double arrays of constants $\{a_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ and a sequence $\{X_n,\ n \ge 1\}$ of asymptotically quadrant sub-independent (AQSI) random variables the almost sure convergence of $\sum_{i=1}^{k_n} a_{ni}X_i/\log k_n$ is derived. The Marcinkiewicz strong law of large numbers for AQSI sequence is also obtained by applying this result.

**1. Introduction.** Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\{X_n,\ n \ge 1\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, P)$.

Lehmann [**5**] introduced the notion of positive quadrant dependence: A sequence $\{X_n,\ n \ge 1\}$ is said to be pairwise positive quadrant dependent if, for $s, t \in \mathbf{R}$,

$$P\{X_i > s,\, X_j > t\} - P\{X_i > s\}P\{X_j > t\} \ge 0, \tag{0.a}$$

or

$$P\{X_i < s,\, X_j < t\} - P\{X_i < s\}P\{X_j < t\} \ge 0. \tag{0.b}$$

Dropping the assumption of positive dependence, but using the magnitude of the lefthand sides in (0.a) and (0.b) as a measure of dependence, Birkel [**1**] introduced the notion of asymptotic quadrant independence: A sequence $\{X_n\}$ of random variables is called asymptot-

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ically quadrant independent (AQI) if there exists a nonnegative sequence $\{q(m)\}$ such that, for all $i \neq j$ and $s, t \in \mathbf{R}$,

$$\text{(1.a)} \quad |P\{X_i > s, X_j > t\} - P\{X_i > s\}P\{X_j > t\}| \leq q(|i-j|)\alpha_{ij}(s,t),$$

$$\text{(1.b)} \quad |P\{X_i < s, X_j < t\} - P\{X_i < s\}P\{X_j < t\}| \leq q(|i-j|)\beta_{ij}(s,t),$$

where $q(m) \to 0$ and $\alpha_{ij}(s,t) \geq 0$, $\beta_{ij}(s,t) \geq 0$.

Chandra and Ghosal [**3**] considered a dependence condition which is a useful weakening of this definition of AQI proposed by Birkel [**1**]: A sequence $\{X_n,\ n \geq 1\}$ of random variables is said to be asymptotically quadrant sub-independent (AQSI) if there exists a nonnegative sequence $\{q(m)\}$ such that $q(m) \to 0$, and for all $i \neq j$,

$$\text{(2.a)} \quad P\{X_i > s, X_j > t\} - P\{X_i > s\}P\{X_j > t\} \leq q(|i-j|)\,\alpha_{ij}(s,t), \quad s, t > 0,$$

$$\text{(2.b)} \quad P\{X_i < s, X_j < t\} - P\{X_i < s\}P\{X_j < t\} \leq q(|i-j|)\,\beta_{ij}(s,t), \quad s, t < 0,$$

where $\alpha_{ij}(s,t)$ and $\beta_{ij}(s,t)$ are nonnegative numbers. This AQSI condition is satisfied by AQI sequences as well as by pairwise $m$-dependent and pairwise negative quadrant dependent sequences.

There are two well-known results; namely, the Kolmogorov strong law of large numbers and the Rademacher-Mensov strong law of large numbers, e.g., [**7**, p. 114], [**6**, Section 36], [**8**, Chapter 3], Hall and Heyde [**4**, p. 22]. Chandra and Ghosal [**3**] proved the strong law of large numbers for weighted averages of AQSI sequences by using an extension of the well-known Rademacher-Mensov inequality, see Lemma 2.1 in Section 2.

In this paper we obtain the almost-sure convergence of a triangular array of weighted sum of AQSI random variables. A result of this type has not been established in the literature.

We will use the following concept in this paper. Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables, and let $X$ be a nonnegative random variable. If there exists a constant $C$, $0 < C < \infty$, satisfying

$\sup_{n\geq 1} P(|X_n| > t) \leq CP(X \geq t)$ for any $t \geq 0$, then $\{X_n,\ n \geq 1\}$ is said to be stochastically dominated by $X$ (briefly $\{X_n,\ n \geq 1\} \prec X$).

Throughout the remainder of this paper, $C$ will stand for a constant whose value may vary from line to line.

**2. Results.** The following result is an extension of the well-known Rademacher-Mensov inequality. A proof of this result can be found in Theorem 10 of [**2**].

**Lemma 2.1** [**3**]. *Let $X_1, \dots, X_n$ be square integrable random variables such that there exist numbers $c_1^2, \dots, c_n^2$ satisfying*

$$E(X_{m+1} + \cdots + X_{m+p})^2 \leq c_{m+1}^2 + \cdots + c_{m+p}^2, \quad \forall\, m, p. \tag{3}$$

*Then we have*

$$E\biggl(\max_{1\leq k\leq n}\biggl(\sum_{i=1}^{k} X_i\biggr)^2\biggr) \leq ((\log n/\log 3) + 2)^2 \sum_{i=1}^{n} c_i^2. \tag{4}$$

**Lemma 2.2.** *Let $\{X_n,\ n \geq 1\}$ be a sequence of mean zero, square integrable and asymptotically quadrant sub-independent random variables with $\sum_{m=1}^{\infty} q(m) < \infty$ and, for all $i \neq j$,*

$$\int_0^\infty \int_0^\infty \alpha_{ij}(s,t)\, ds\, dt \leq D(1 + EX_i^2 + EX_j^2), \tag{5}$$

$$\int_0^\infty \int_0^\infty \beta_{ij}(s,t)\, ds\, dt \leq D(1 + EX_i^2 + EX_j^2). \tag{6}$$

*Then we have*

$$E\biggl(\sum_{i=1}^{n} X_i\biggr)^2 \leq C \sum_{i=1}^{n} (1 + EX_i^2), \tag{7}$$

*and*

$$E\biggl(\max_{1\leq k\leq n}\biggl(\sum_{i=1}^{k} X_i\biggr)^2\biggr) \leq ((\log n/\log 3) + 2)^2 \sum_{i=1}^{n} (1 + EX_i^2). \tag{8}$$

*Proof*. By Lemma 2 of [**5**] we have

$$\mathrm{Cov}\,(X_i^+, X_j^+) \le Dq(|i-j|)(1 + EX_i^2 + EX_j^2).$$

So

$$\mathrm{Var}\left(\sum_{i=1}^n X_i^+\right) \le C \sum_{i=1}^n (1 + EX_i^2) \quad \text{for all } n.$$

Similarly

$$\mathrm{Var}\left(\sum_{i=1}^n X_i^-\right) \le C \sum_{i=1}^n (1 + EX_i^2) \quad \text{for all } n.$$

Thus

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^n X_i\right) &\le 2\ \mathrm{Var}\left(\sum_{i=1}^n X_i^+\right) + 2\ \mathrm{Var}\left(\sum_{i=1}^n X_i^-\right)\\ &\le C \sum_{i=1}^n (1 + EX_i^2) \quad \text{for all } n.\end{aligned}$$

Hence the proof of (7) is complete. Equation (8) follows from (7) and Lemma 2.1 .

From (8) of Lemma 2.2 we have the following maximal inequality.

**Theorem 2.3.** *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of mean zero, square integrable* AQSI *random variables with* $\sum_{m=1}^{\infty} q(m) < \infty$*, satisfying* (5) *and* (6). *Then*

$$(9)\quad P\left\{\max_{1\le k\le n} |\sum_{i=1}^k X_i| \ge \varepsilon\right\} \le C((\log n/\log 3) + 2)^2 \sum_{i=1}^n (1 + EX_i^2).$$

The following theorem is the main result:

**Theorem 2.4.** *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of mean zero, square integrable* AQSI *random variables with* $\sum_{m=1}^{\infty} q(m) < \infty$ *and satisfying* (5) *and* (6). *Let* $\{X_n,\ n \ge 1\}$ *be stochastically dominated by* *a nonnegative random variable $X$ with $EX^r < \infty$ for $0 < r < 2$. Let $\{k_n,\ n \geq 1\}$ be an increasing sequence of integers. If $\{a_{ni},\ 1 \leq i \leq k_n,\ n \geq 1\}$ is an array of constants satisfying*

$$\sum_{i=1}^{k_n} |a_{ni} - a_{n,i+1}| = O\left(\frac{1}{k_n^{1/r}}\right), \tag{10}$$

*where $a_{n,k_n+1} = 0$, then, as $n \to \infty$,*

$$\frac{1}{\log k_n} \sum_{i=1}^{k_n} a_{ni} X_i \longrightarrow 0 \quad a.s. \tag{11}$$

**3. Proof of Theorem 2.4.** Without loss of generality, we suppose that $a_{ni} \geq 0, i \geq 1, n \geq 1$. Otherwise we assume that $a_{ni_1}, \dots, a_{ni_m}$ are nonnegative, while $a_{ni_{m+1}}, \dots, a_{ni_{k_n}}$ are negative. It is easy to check that if $\{a_{ni_j},\ 1 \leq j \leq m\}$ and $\{a_{ni_j},\ m+1 \leq j \leq k_n\}$ satisfy (10), then we only have to consider $\sum_{j=1}^m a_{ni_j} X_{ni_j}$ and $\sum_{j=m+1}^{k_n} a_{ni_j} X_{ni_j}$. Let

$$X_i' = (-i^{1/r}) \vee (X_i \wedge i^{1/r}), \quad X_i'' = X_i - X_i'.$$

Since $X_i'$ and $X_i''$ are increasing functions of $X_i$, both $\{X_n' - EX_n'\}$ and $\{X_n'' - EX_n''\}$ also form mean zero AQSI sequences. Let

$$S_n' = \sum_{i=1}^{k_n} a_{ni}(X_i' - EX_i'), \quad S_n'' = \sum_{i=1}^{k_n} a_{ni}(X_i'' - EX_i''),$$
$$A_k = \sum_{i=1}^{k} (X_i' - EX_i')$$

and assume $0 < r < 2$. For fixed $n$, there exists $t \in N$ such that $2^t < k_n \leq 2^{t+1}$. Then from (10) we easily get

$$|S_n'| \leq C(2^t)^{-1/r} \max_{1 \leq i \leq 2^{t+1}} |A_i|$$

by applying the Abelian transformation. Noticing that $\{X'_n - EX'_n, n \geq 1\}$ is an AQSI sequence and applying Theorem 2.3, and for each $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{t=1}^{\infty} P(|S'_n| \geq \varepsilon \log k_n \quad \text{for some} \quad k_n \in (2^t, 2^{t+1}]) \\
&\qquad \leq \sum_{t=1}^{\infty} P\bigg\{ \frac{1}{(2^t)^{1/r}} \max_{1 \leq i \leq 2^{t+1}} |A_i| > \frac{\varepsilon}{C} t \log 2 \bigg\} \\
&\qquad \leq C \sum_{m=1}^{\infty} q(m) \sum_{t=1}^{\infty} (t+3)^2 2^{-2t/r} (t \log 2)^{-2} \sum_{i=1}^{2^{t+1}} (1 + EX_i'^2) \\
&\qquad \leq C \sum_{t=1}^{\infty} 2^{-2t/r} \sum_{i=1}^{2^{t+1}} (1 + EX_i'^2) \\
&\qquad \leq C \sum_{t=1}^{\infty} 2^{-2t/r} 2^{t+1} + C \sum_{t=1}^{\infty} 2^{-2t/r} \sum_{i=1}^{2^{t+1}} EX_i'^2 \\
&\qquad \leq C \bigg\{ \sum_{t=1}^{\infty} 2^{-(2t/r)+t+1} + \sum_{i=1}^{\infty} P(|X_i| > i^{1/r}) \\
&\qquad\qquad + \sum_{i=1}^{\infty} i^{-2/r} EX_i^2 I(|X_i| \leq i^{1/r}) \bigg\} \\
&\qquad \leq C \bigg\{ \sum_{t=1}^{\infty} 2^{-(2t/r)+t+1} + \sum_{i=1}^{\infty} P(X > i^{1/r}) \\
&\qquad\qquad + \sum_{i=1}^{\infty} \frac{EX_i^2 I(|X_i| \leq i^{1/r})}{i^{2/r}} \bigg\},
\end{aligned}$$

where $C$ depends only on $\varepsilon$. Obviously, $\sum_{t=1}^{\infty} 2^{t+1-(2t/r)} < \infty$, and it follows from the condition $EX^r < \infty$ that $\sum_{i=1}^{\infty} P(X > i^{1/r}) < \infty$ and $\sum_{i=1}^{\infty} i^{-2/r} EX^2 I(X \leq i^{1/r}) < \infty$, see the Appendix. Thus we have

$$\sum_{t=1}^{\infty} P(|S'_n| \geq \varepsilon \log k_n \quad \text{for some} \quad k_n \in (2^t, 2^{t+1}]) < \infty.$$

By the Borel-Cantelli lemma we conclude that

$$\frac{S'_n}{\log k_n} \longrightarrow 0 \quad \text{a.s.} \tag{12}$$

On the other hand, since

$$\sum_{i=1}^{\infty} P(|X_i| \geq i^{1/r}) < C \sum_{i=1}^{\infty} P(X \geq i^{1/r}) < \infty,$$

we have

$$P(|X_i| > i^{1/r} \ i.o) = 0. \tag{13}$$

From (10) and (13), we have

$$\begin{aligned}\left|\sum_{i=1}^{k_n} a_{ni} X_i''\right| &\leq \left(\max_{1\leq i\leq k_n}\left|\sum_{j=1}^{n} X_j''\right|\right)\left(\sum_{i=1}^{n}|a_{ni}-a_{n,i+1}|\right)\\ &\leq \frac{C}{k_n^{1/r}}\sum_{i=1}^{k_n}|X_i| I(|X_i|\geq i^{1/r}) \longrightarrow 0 \quad \text{a.s.}\end{aligned}$$

By applying the Abelian transformation, that is, we have

$$\sum_{i=1}^{k_n} a_{ni} X_i'' \longrightarrow 0 \quad \text{a.s.} \tag{14}$$

(a) If $1 < r < 2$, since $\{X_n\} \prec X$ and $\sum_{i=1}^{\infty} i^{-1/r} EXI(X > i^{1/r}) < \infty$ we get that

$$\sum_{i=1}^{\infty} \frac{1}{i^{1/r}} E|X_i''| \leq C \sum_{i=1}^{\infty} i^{-1/r} EXI(X > i^{1/r}) < \infty.$$

By Kronecker's lemma, we get

$$\frac{1}{k_n^{1/r}} \sum_{i=1}^{k_n} E|X_i''| \longrightarrow 0. \tag{15}$$

(b) If $r = 1$,

$$E|X_i''| \leq CEXI(X > i) \longrightarrow 0 \quad \text{as} \quad i \to \infty,$$

thus we have as well

$$\frac{1}{k_n}\sum_{i=1}^{k_n} E|X_i''| \longrightarrow 0. \tag{16}$$

From (10),(15) and (16) we have, for $1 \le r \le 2$,

$$\begin{aligned} \Big|\sum_{i=1}^{k_n} a_{ni}EX_i''\Big| &\le \frac{C}{{k_n}^{1/r}}\bigg(\max_{1\le i\le k_n}\Big|\sum_{j=1}^{i} EX_j''\Big|\bigg) \\ &\le \frac{C}{{k_n}^{1/r}}\sum_{i=1}^{k_n} E|X_i''| \longrightarrow 0 \end{aligned} \tag{17}$$

by applying the Abelian transformation. From (14) and (17) it follows that, for $1 \le r < 2$,

$$S_n'' \longrightarrow 0 \quad \text{a.s.}$$

Since $S_n = S_n' + S_n''$, we obtain (11) for $1 \le r < 2$.

(c) If $0 < r < 1$, since (12) and (14) hold it remains to show that

$$\sum_{i=1}^{k_n} a_{ni}EX_i' \longrightarrow 0.$$

From the Appendix we have

$$\begin{aligned} &\sum_{i=1}^{\infty}\frac{1}{i^{1/r}}\,E|X_i'| \\ &\qquad \le C\bigg\{\sum_{i=1}^{\infty} P(X \ge i^{1/r}) + \sum_{i=1}^{\infty}\frac{1}{i^{1/r}}\,EXI(X \le i^{1/r})\bigg\} < \infty. \end{aligned}$$

Consequently, by the Kronecker lemma

$$\frac{1}{k_n^{1/r}}\sum_{i=1}^{k_n} E|X_i'| \longrightarrow 0 \quad \text{as} \quad i \to \infty.$$

It follows that

$$\left|\sum_{i=1}^{k_n} a_{ni}EX_i'\right| \le \frac{C}{{k_n}^{1/r}}\left(\max_{1\le i\le k_n}\left|\sum_{j=1}^{i} EX_j'\right|\right)$$
$$\le \frac{C}{{k_n}^{1/r}}\sum_{i=1}^{k_n} E|X_i'| = 0(1).$$

Thus

$$(\log k_n)^{-1}\sum_{i=1}^{k_n} a_{ni}X_i \longrightarrow 0 \quad \text{a.s.},$$

that is, (11) holds for $0 < r < 1$. The proof is complete. □

From Theorem 2.4 we get the following strong law of large number for AQSI sequence.

**Corollary 2.4.** *Assume that* $\{X, X_n,\ n \ge 1\}$ *is a sequence of identically distributed, mean zero and square integrable* AQSI *random variables with* $\sum_{m=1}^{\infty} q(m) < \infty$ *and satisfying* (5) *and* (6). *If* $E|X|^r < \infty$ *for* $0 < r < 2$, *then*

$$n^{-1/r}(\log n)^{-1}\sum_{i=1}^{n} X_i \longrightarrow 0 \quad a.s.$$

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## Appendix

**Lemma A.** *If* $\{X_n\}$ *is stochastically dominated by a nonnegative random variable* $X(\{X_n\} \prec X)$ *with* $EX^r < \infty$ *for* $0 < r < 2$ *then we have*

(a) $\sum_{i=1}^{\infty} i^{-2/r}E(X_i^2 I\{|X_i|^r \le i\}) < \infty$,

(b) $\sum_{i=1}^{\infty} i^{-1/r}E(|X_i| I\{|X_i|^r \le i\}) < \infty$, if $0 < r < 1$.

*Proof*. The proof is based on certain ideas in [**3**]. Note that, for some $0 < r < 2$

$$E|X|^r < \infty \Longleftrightarrow \int_0^\infty y^{r-1} P\{|X| > y\}\, dy < \infty \tag{A.1}$$

and

$$E|X|^r < \infty \Longleftrightarrow \sum_{n=1}^\infty P\{|X|^r > n\} < \infty. \tag{A.2}$$

*The proof of* (a). Since $\{X_n\}$ is stochastically dominated by a nonnegative random variable $X$, we obtain

$$\begin{aligned}
&\sum_{i=1}^\infty i^{-2/r} E(X_i^2 I\{|X_i|^r \le i\}) \\
&\qquad \le C \sum_{i=1}^\infty \sum_{k=i}^\infty k^{-(2/r)-1} E(X_i^2 I\{|X_i|^r \le i\}) \\
&\qquad \le C \sum_{i=1}^\infty \sum_{k=i}^\infty k^{-(2/r)-1} \int_0^{i^{1/r}} yP(\{|X_i| > y\})\, dy \\
&\qquad \le C \sum_{k=1}^\infty \sum_{i=1}^k k^{-(2/r)-1} \sum_{n=1}^i \int_{(n-1)^{1/r}}^{n^{1/r}} yP\{|X_i| > y\}\, dy \\
&\qquad \le C \sum_{k=1}^\infty \sum_{n=1}^k k^{-2/r} \int_{(n-1)^{1/r}}^{n^{1/r}} y\Big(k^{-1} \sum_{i=1}^k P\{X > y\}\Big)\, dy \\
&\qquad \le C \sum_{n=1}^\infty \sum_{k=n}^\infty k^{-2/r} \int_{(n-1)^{1/r}}^{n^{1/r}} yP\{X > y\}\, dy \\
&\qquad \le C \sum_{n=1}^\infty n^{1-(2/r)} \int_{(n-1)^{1/r}}^{n^{1/r}} yP\{X > y\}\, dy \\
&\qquad \le C \sum_{n=1}^\infty \int_{(n-1)^{1/r}}^{n^{1/r}} y^{r-1} P\{X > y\}\, dy \\
&\qquad \le CEX^r < \infty.
\end{aligned}$$

*The proof of* (b). The proof is similar to that of (a).

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