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# ON A SPECIAL CLASS OF NONLINEAR INTEGRAL EQUATIONS

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ABSTRACT. We consider nonlinear integral equations

$$\varphi(t, x(t)) + \int_0^1 \psi(t, s, x(t), x(s))\, ds = 0 \quad \text{for } t \in [0,1]$$

with a certain monotonicity in the argument $y = x(t)$ and a certain compactness in the argument $x = x(s)$.

**1. Introduction.** In [**8**], the nonlinear integral equation

$$(1.1) \qquad (Fx)(t) := x^2(t) - t^2 + \frac{J}{4\pi}\int_0^1 k(t, s, x(t), x(s))\, ds = 0$$

was investigated. Here $J \geq 0$ is a parameter, $t \in [0,1]$, and the kernel $k$ is nonlinear in all four arguments with $\partial_y k(t,s,y,x) \geq 0$; cf. (3.11) below for the explicit form of $k$. Since $k$ is $x(t)$-dependent in a nontrivial way, (1.1) is not of standard type. For example, the theory of Hammerstein or Volterra equations, cf. e.g. [**9, 10, 20, 21, 22**] or [**6**], does not include (1.1).

In fact integral equations which show the same characteristic features (described more precisely later on) as (1.1), and whose general form is

$$(1.2) \qquad \varphi(t, x(t)) + \int_0^1 \psi(t, s, x(t), x(s))\, ds = 0 \quad \text{for } t \in [0,1],$$

arises in a variety of other settings, e.g., in neutron transport theory; cf. [**16**] and [**2**]. As a result, it is of interest to obtain a general existence theory for equation (1.2).

Now the main observation concerning (1.1) is that a difference can be made between $x$ with argument $t$ and $x$ with argument $s$. To make

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this clearer, define a nonlinear operator depending on two functions $x, y \in C(J)$ with $J = [0,1]$ by

$$G(x,y)(t) = y^2(t) - t^2 + \frac{J}{4\pi} \int_0^1 k(t,s,y(t),x(s))\,ds;$$

$G$ is obtained from $F$ by replacing $x(t)$ with $y(t)$. Then $Fx = G(x,x)$. Moreover, if $x \in C(J)$ is fixed, then $G(x,\cdot) : C(J) \to C(J)$ will exhibit some kind of monotonicity, uniformly in $x$, since $\partial_y k(t,s,y,x) \geq 0$. Finally, if $y \in C(J)$ is fixed, then there is some hope that $G(\cdot,y) : C(J) \to C(J)$ satisfies a certain compactness condition, possibly uniformly in $y$, because we removed $x(t)$ in the integral part.

By making this more precise, the definition of the class of strongly semicondensing operators is arrived at quite naturally, cf., Definition 2.3 below. It turns out that strongly semicondensing operators are semicondensing, and that for the latter class [**15**] gave an excellent presentation of a mapping degree which generalizes the degree for compact or condensing perturbations of the identity. Therefore we can use a mapping degree as a basic tool in the investigation of (1.2).

It should be mentioned that a variety of maps having diagonal representations $Fx = G(x,x)$ were investigated in Chapters 12 and 13 of [**4**]; cf. also [**5**].

We shall start with some preliminaries in Section 2. After that, in the main Section 3 we shall prove various results on the existence of solutions to (1.2) in the case that the equation shows a certain compactness in $x$ and a certain monotonicity in $y$ in the sense motivated above. Many examples will show that our results are easily applicable.

**2. Preliminaries.** Let us first recall some basic concepts from nonlinear analysis. Throughout this paper, $X$ will denote an infinite dimensional Banach space with norm $|\cdot|$ and dual space $X^*$. Let $B \subset X$ be bounded. Then

$$\alpha(B) = \inf \Big\{ d > 0 : \text{ there is a finite system } B_1, \dots, B_n \subset X \\ \text{with } B = \bigcup_{i=1}^n B_i \text{ and } \operatorname{diam} B_i \leq d,\ 1 \leq i \leq n \Big\},$$

is the Kuratowski measure of noncompactness. Properties of $\alpha$ may be found in [**6**, Proposition 7.2]. By means of the (in general multivalued) duality mapping $\mathcal{F}$ of $X$, where

$$\mathcal{F}x = \{x^* \in X^* : |\,x^*\,| = |\,x\,|,\ x^*(x) = |\,x\,|^2\},$$

the semi-inner products $(\cdot,\cdot)_{\pm} : X \times X \to \mathbf{R}$ are defined by

$$(x,y)_+ = \max\{y^*(x) : y^* \in \mathcal{F}y\} \quad \text{and} \quad (x,y)_- = \min\{y^*(x) : y^* \in \mathcal{F}y\}$$

for $x, y \in X$. Some essential properties of $(\cdot,\cdot)_{\pm}$ are collected in [**6**, Paragraph 13]; cf. also [**15**, Proposition 0.3.1]. Additionally, let us note that

$$\left(\int_a^b x(s)\,ds, y\right)_+ \le \int_a^b (x(s),y)_+\,ds \tag{2.3}$$

for $a, b \in \mathbf{R}$ with $a < b$, $y \in X$, and a Bochner integrable $x : [a,b] \to X$ (cf. [**7**] for the terminology) by Definition of $(\cdot,\cdot)_+$; cf. [**11**, Lemma 1.5].

In particular, if $J \subset \mathbf{R}$ is compact and $X := C(J)$, then

$$(x,y)_+ = \max\{x(t)y(t) : t \in J_y\} \quad \text{and} \quad (x,y)_- = \min\{x(t)y(t) : t \in J_y\} \tag{2.4}$$

for $x, y \in X$ with $J_y = \{t \in J : |\,y(t)\,| = |\,y\,|\}$; cf. [**6**, Example 13.1(c)].

In general Banach spaces, the nonnegativity of elements may be described by means of cones. The following example collects some properties of the standard cone in $X = C(J)$.

**Example 2.1.** Let $J \subset \mathbf{R}$ be compact, $X := C(J)$. Then $C^+(J) := \{x \in X : x(t) \ge 0 \text{ on } J\}$ is a cone, i.e., a closed convex set such that $\lambda K \subset K$ for all $\lambda \ge 0$ and $K \cap (-K) = \{0\}$, with $\mathring{K} \ne \varnothing$, since, e.g., $x_0 \in \mathring{K}$ for $x_0(t) := 1$ on $J$. Moreover, the dual cone $K^* := \{x^* \in X^* : x^*(x) \ge 0 \text{ on } K\}$ may be identified with the set of all positive regular Borel measures on $J$; cf. [**3**, Chapter 29]. With $K_r := K \cap \overline{B}_r(0)$, we have $K_r = \{x \in C(J) : 0 \le x(t) \le r \text{ on } J\}$ for $r > 0$.

Now we turn our attention to nonlinear operators. Let $\mathcal{M} = \{\omega : [0,\infty) \to [0,\infty) : \omega$ is continuous and strictly increasing with $\omega(0)=0$ and $\omega(r)\to\infty$ as $r\to\infty\}$, $\omega\in\mathcal{M}$, and $D\subset X$. A map $F:D\to X$ is called accretive, if $(Fx-Fy,x-y)_+\geq 0$ for $x,y\in D$, and $F$ is called $\omega$-accretive, if $(Fx-Fy,x-y)_+\geq\omega(|\,x-y\,|)|\,x-y\,|$ for $x,y\in D$. Furthermore, $F$ is called strongly accretive, if $F$ is $\omega$-accretive for some $\omega\in\mathcal{M}$.

In particular, if $J\subset\mathbf{R}$ is compact, $X=C(J)$, $D\subset C^+(J)$, and $n\in\mathbf{N}$, then

$$\Phi:D\to X,\quad(\Phi x)(t):=(x(t))^n\quad\text{on } J,\tag{2.5}$$

is strongly accretive since we can choose $\omega(r):=r^n$ on $[0,\infty)$; note that $(a^n-b^n)(a-b)\geq|\,a-b\,|^{n+1}$ for $a,b\geq 0$.

The following definition is taken from [**15**]; cf. Definition 1.1.

**Definition 2.2.** Let $\Omega\subset X$ be open bounded. A continuous $F:\overline{\Omega}\to X$ is called *semicondensing* if there exist a continuous bounded map $G:\Omega\times\Omega\to X$ and an $\omega\in\mathcal{M}$ such that

(a) $Fx=G(x,x)$ for $x\in\Omega$;

(b) $\{G(\cdot,y):y\in\Omega\}$ is equicontinuous (pointwise);

(c) for all $A\subset\Omega$ with $\alpha(A)>0$ there exist an $\varepsilon$ with $0\leq\varepsilon<\omega(\alpha(A))$ and a finite covering $A=\bigcup_{k=1}^K A_k$ of $A$ such that

$$\omega(|\,y-\bar{y}\,|)\,|\,y-\bar{y}\,|\leq(G(x,y)-G(\bar{x},\bar{y}),y-\bar{y})_+ +\varepsilon\,|\,y-\bar{y}\,|$$

for all $y,\bar{y}\in\Omega$ and all $x,\bar{x}$ in the same $A_k$.

In this case $(G,\omega)$ is called a representation of $F$.

Next we shall introduce another important class of nonlinear operators which have a diagonal representation $Fx=G(x,x)$; cf. the introduction.

**Definition 2.3.** Let $\Omega\subset X$ be open bounded and $F:\overline{\Omega}\to X$ continuous. $F$ is called *strongly semicondensing* if there exist a continuous bounded map $G:\Omega\times\Omega\to X$ and an $\omega\in\mathcal{M}$ such that

(a) $Fx=G(x,x)$ for $x\in\Omega$;

(b) $G_x:=G(x,\cdot):\Omega\to X$ is $\omega$-accretive for $x\in\Omega$;

(c) The map $\Omega \ni x \overset{\Gamma}{\longmapsto} G_x \in C(\Omega; X)$ satisfies $\alpha(\Gamma(A)) < \omega(\alpha(A))$ for all $A \subset \Omega$ with $\alpha(A) > 0$.

In this case $(G, \omega)$ is called a representation of $F$.

It is easily seen that strongly semicondensing maps are semicondensing; cf. [**11**, Lemma 1.23(b)].

To solve integral equations of the type (1.2), we shall need two abstract results on the existence of zeros of semicondensing operators. The first one contains two special cases in which the Leray-Schauder boundary condition is satisfied; cf. [**4**, Theorem 13.15 (b), (c), (d); **15**, Corollary 3.2], and [**11**, Corollary 1.18].

**Theorem 2.4.** *Let $\Omega \subset X$ be open bounded, $F : \overline{\Omega} \to X$ semicondensing. In addition, suppose that*

(a) *$\Omega$ is convex and $(I - F)(\partial\Omega) \subset \overline{\Omega}$, or*

(b) *there exists $x_0 \in \Omega$ for all $x \in \partial\Omega : (Fx, x - x_0)_+ \geq 0$.*

*Then $F$ has a zero in $\overline{\Omega}$.*

The following theorem is taken from [**12**].

**Theorem 2.5.** *Let $K \underline{\subset} X$ be a cone with $\overset{\circ}{K} \neq \varnothing$, $r > 0$, $K_r := K \cap \overline{B}_r(0)$, and $F : \overset{\circ}{K}_r = K_r \to X$ semicondensing. Suppose that*

(a) *$x \in K$, $|x| = r$, $Fx + \lambda x = 0 \Longrightarrow \lambda \leq 0$, and*

(b) *$x \in \partial K$, $|x| \leq r$, $x^* \in K^*$, $x^*(x) = 0 \Longrightarrow x^*(Fx) \leq 0$.*

*Then $F$ has a zero in $K_r$.*

Finally, let us state a lemma from the theory of Banach spaces which will be needed later. For a proof, we refer to [**11**, Lemma 2.6]; cf. also [**19**, Theorem 1.3] or [**14**, Remark 2.4] for similar results.

**Lemma 2.6.** *Let $X$ and $Y$ be Banach spaces, $D \subset X$ bounded, $(\Omega, \Sigma, \mu)$ a measurable space with a positive finite measure $\mu$. Suppose* *that, for all* $\omega \in \Omega$*, there is an operator* $U(\omega) : D \to Y$ *such that the following conditions are satisfied.*

(a) $U(\omega) : D \to Y$ *is compact, i.e., continuous, and* $U(\omega)(D) \subset Y$ *is relatively compact.*

(b) $U(\cdot)(x) : \Omega \to Y$ *is strongly measurable for all* $x \in D$;

(c) $\sup_{x \in D} \int_\Omega |\, U(\omega)(x) \,|_Y \mu(d\omega) < \infty$;

(d) $\lim_{\mu(\Omega_0) \to 0} \sup_{x \in D} \int_{\Omega_0} |\, U(\omega)(x) \,|_Y \mu(d\omega) = 0$.

*Then*

$$\bar{U} : D \ni x \mapsto \int_\Omega U(\omega)(x) \mu(d\omega) \in Y$$

*is a compact operator.*

Note that assumptions (c) and (d) of the above lemma hold if $|\, U(\omega)(x) \,|_Y \leq M$ on $\Omega \times D$ for some $M > 0$.

**3. Existence of solutions and applications.** The following theorem will turn out to be quite useful and easily applicable when we look for nonnegative continuous solutions of integral equations of the type (1.2). This will be illustrated by the Examples 3.3 and 3.9 below.

**Theorem 3.1.** *Let* $0 \leq x_0 \in X = C([0,1])$*,* $K = C^+([0,1])$*, and* $r = |\, x_0 \,| > 0$*. Suppose that the following conditions are satisfied.*

(a) $\varphi : [0,1] \times [0,r] \to \mathbf{R}$ *is continuous and* $\psi : [0,1] \times [0,1] \times [0,r] \times [0,r] \to \mathbf{R}$ *is measurable.*

(b) *There exists an* $\omega \in \mathcal{M}$ *such that*

(i) $\Phi : \overset{\circ}{K}_r \to X, (\Phi x)(t) := \varphi(t, x(t))$*, is* $\omega$*-accretive, and*

(ii) $K_r \ni x \overset{\Gamma_1}{\mapsto} (K_r \ni y \mapsto \int_0^1 \psi(\cdot, s, y(\cdot), x(s))\, ds \in X) \in C(K_r; X)$ *is a well-defined, continuous, and bounded operator such that* $\alpha(\Gamma_1(A)) < \omega(\alpha(A))$ *for all* $A \subset \overset{\circ}{K}_r$ *with* $\alpha(A) > 0$*.*

(c) *If* $t \in [0,1]$ *and* $x \in K_r$*, then*

$$[0,r] \ni y \longmapsto \Psi_1^x(t,y) := \int_0^1 \psi(t,s,y,x(s))\, ds \in \mathbf{R}$$

*is increasing.*

(d) *If* $t \in [0,1]$ *and* $x \in K_r$, *then*

$$\Psi^x(t,\cdot) := \varphi(t,\cdot) + \Psi_1^x(t,\cdot) : [0,r] \longrightarrow \mathbf{R}$$

*satisfies* $\Psi^x(t,0) \leq 0$ *as well as* $\Psi^x(t,x_0(t)) \geq 0$.

*Then* $F : K_r \to X$, *defined by*

(3.6)
$$(Fx)(t) := (\Phi x)(t) + (\Psi x)(t) := \varphi(t,x(t)) + \int_0^1 \psi(t,s,x(t),x(s))\,ds,$$

*has a zero* $x \in K_r$ *such that* $0 \leq x \leq x_0$.

*Proof.* First we show that $F : \overline{\mathring{K}_r} = K_r \to X$ is strongly semicondensing. Define

$$G_1 : K_r \times K_r \to X, \qquad G_1(x,y)(t) := \int_0^1 \psi(t,s,y(t),x(s))\,ds \quad \text{on } [0,1].$$

Then $G_1(x,y) = (\Gamma_1(x))(y)$, and $Fx = \Phi x + G_1(x,x)$ for $x,y \in K_r$ together with the assumptions on $\Gamma_1$ and $\varphi$ show that $F : K_r \to X$ is continuous and bounded. Furthermore,

$$(\Phi y - \Phi \bar{y}, y - \bar{y})_- \geq \omega(|\,y-\bar{y}\,|)|\,y-\bar{y}\,| \quad \text{for } y,\bar{y} \in \mathring{K}_r \tag{3.7}$$

by (b)(i) and the properties of $(\cdot,\cdot)_\pm$, since $\mathring{K}_r$ is open. Let $G : \mathring{K}_r \times \mathring{K}_r \to X$, $(x,y) \mapsto \Phi y + G_1(x,y)$. Then $(G,\omega)$ may be used as a representation of $F$ in the sense of Definition 2.3. Indeed, $G$ is continuous and bounded, and $Fx = G(x,x)$ for $x \in \mathring{K}_r$. Moreover, with $\Gamma : \mathring{K}_r \to C(\mathring{K}_r; X)$, $x \mapsto G_x := G(x,\cdot)$, we have $\Gamma(x) = \Phi + \Gamma_1(x)$ on $\mathring{K}_r$. Hence $\alpha(\Gamma(A)) = \alpha(\Phi + \Gamma_1(A)) = \alpha(\Gamma_1(A)) < \omega(\alpha(A))$ for $A \subset \mathring{K}_r$ with $\alpha(A) > 0$. Fix $x \in \mathring{K}_r$. To show that $G_x$ is $\omega$-accretive, we can use (3.7) and (2.4) to obtain

$$
\begin{aligned}
(G(x,y)-G(x,\bar y),y-\bar y)_+ &\ge (G(x,y)-G(x,\bar y),y-\bar y)_- \\
&\ge (\Phi y-\Phi\bar y,y-\bar y)_- \\
&\quad + (G_1(x,y)-G_1(x,\bar y),y-\bar y)_- \\
&\ge \omega(|\,y-\bar y\,|)\,|\,y-\bar y\,| + \min\{(G_1(x,y)(t) \\
&\quad - G_1(x,\bar y)(t))(y(t)-\bar y(t)) : t\in J_{y-\bar y}\} \\
&= \omega(|\,y-\bar y\,|)|\,y-\bar y\,| + \min\{(\Psi_1^x(t,y(t)) \\
&\quad - \Psi_1^x(t,\bar y(t)))\,(y(t)-\bar y(t)) : t\in J_{y-\bar y}\} \\
&\ge \omega(|\,y-\bar y\,|)\,|\,y-\bar y\,|
\end{aligned}
$$

for $y,\bar y\in \overset{\circ}{K}_r$; recall the properties of $(\cdot,\cdot)_\pm$. Therefore, $F$ is strongly semicondensing, hence semicondensing, and we only have to make sure that conditions (a) and (b) from Theorem 2.5 are satisfied. For note first that $\Psi_1^x(t,\cdot):[0,r]\to\mathbf{R}$ is continuous if $t\in[0,1]$ and $x\in K_r$ are fixed. Furthermore, if $y_0,\ \bar y_0\in(0,r)$, then (3.7) may be applied to $y,\ \bar y\in\overset{\circ}{K}_r$, defined by $y(t):=y_0$, $\bar y(t):=\bar y_0$ on $[0,1]$, to get $(\varphi(t,y_0)-\varphi(t,\bar y_0))(y_0-\bar y_0)\ge\omega(|\,y_0-\bar y_0\,|)\,|\,y_0-\bar y_0\,|$. Therefore, by assumptions (c) and (d), $\Psi^x(t,\cdot):[0,r]\to\mathbf{R}$ is continuously increasing with $\Psi^x(t,0)\le 0$ and $\Psi^x(t,x_0(t))\ge 0$ for all $t\in[0,1]$ and $x\in K_r$. To verify (a) from Theorem 2.5, suppose that there are $x\in C^+([0,1])$ with $|\,x\,|=r$ and $\lambda>0$ such that $Fx+\lambda x=0$. For $t\in[0,1]$ we define the continuous and increasing function

$$g_t:[0,r]\to\mathbf{R},\qquad y\mapsto\Psi^x(t,y)+\lambda y.$$

Then $g_t(0)\le 0$, $g_t(x(t))=0$, and $g_t(x_0(t))\ge\lambda x_0(t)\ge 0$, hence $0\le x(t)\le x_0(t)$ on $J$. Because of $x(t_0)=r$ for some $t_0\in[0,1]$, $r=|\,x_0\,|$ implies $x_0(t_0)=r$. Consequently, $0=g_{t_0}(x(t_0))=g_{t_0}(x_0(t_0))\ge\lambda x_0(t_0)$ yields the contradiction $0=x_0(t_0)=r$. This proves $\lambda\le 0$. To show (b) from Theorem 2.5, fix $x\in\partial K\subset K=C^+([0,1])$ with $|\,x\,|\le r$ and $x^*\in K^*$ with $x^*(x)=0$. It follows from Example 2.1 that $\int_0^1 x(t)\mu(dt)=0$ for the positive regular Borel measure $\mu$ which represents $x^*$. Consequently, $x = 0$, $\mu$–almost everywhere. We obtain

$$\begin{aligned}
x^*(Fx) &= \int_0^1 (Fx)(t)\mu(dt) \\
&= \int_0^1 \Big(\varphi(t, x(t)) + \int_0^1 \psi(t, s, x(t), x(s))\, ds\Big)\mu(dt) \\
&= \int_0^1 \Big(\varphi(t, 0) + \int_0^1 \psi(t, s, 0, x(s))\, ds\Big)\mu(dt) \\
&= \int_0^1 \Psi^x(t, 0)\mu(dt) \le 0.
\end{aligned}$$

Therefore we can apply Theorem 2.5; cf. Example 2.1. □

**Corollary 3.2.** *Suppose that in the setting of Theorem* 3.1 *the assumptions* (a), (c), *and* (d) *are satisfied. Furthermore, suppose that*

(b') (i) $\Phi : \overset{\circ}{K}_r \to X, (\Phi x)(t) := \varphi(t, x(t))$, *is strongly accretive, and*

(ii) $K_r \ni x \overset{\Gamma_1}{\mapsto} (K_r \ni y \mapsto \int_0^1 \psi(\cdot, s, y(\cdot), x(s))\, ds \in X) \in C(K_r; X)$ *is a well-defined compact operator.*

*Then* $F : K_r \to X$ *has a zero* $x \in K_r$ *such that* $0 \le x \le x_0$.

Let us consider an example.

**Example 3.3.** Nonlinear integral equations of the form

$$1 = x(t) + x(t) \int_0^1 \frac{k(t, s)}{t^2 - s^2} x(s)\, ds, \qquad t \in [0, 1], \tag{3.8}$$

are of interest in neutron transport theory; cf. [**16, 13**], and Example 3.12 below. In [**16**], (3.8) was dealt with by exploiting the special shape of (3.8) which allows the substitution $y(t) := x(t)^{-1}$. Now we want to show how equations being more general than (3.8) can be solved (nearly) at a glance by using Corollary 3.2. We obtain (cf. [**16**, Theorem 2]) the following

**Theorem 3.4.** *Let* $\varrho, \nu > 0$ *and* $n, m \in \mathbf{N}$. *If* $k : [0, 1] \times [0, 1] \to \mathbf{R}$ *is continuous and such that*

(a) $k(t,s)(t-s) \geq 0$ *for* $t,s \in [0,1]$ *and*

(b) $|\,k(t,s)\,| \leq \varrho |\,t-s\,|^{\nu}(t+s)$ *for* $t,s \in [0,1]$,

*then*

$$(x(t))^n - 1 + (x(t))^m \int_0^1 \frac{k(t,s)}{t^2-s^2}x(s)\,ds = 0, \qquad t \in [0,1], \tag{3.9}$$

*has a solution* $x \in C([0,1])$ *with* $0 < x(t) \leq 1$ *on* $J$.

*Proof.* Of course we want to apply Corollary 3.2 to $x_0(t) := 1$ on $[0,1]$, hence $r=1$, $\varphi : [0,1]\times[0,1] \to \mathbf{R}$, $(t,y) \mapsto y^n - 1$, and

$$\begin{aligned}&\psi : [0,1]\times[0,1]\times[0,1]\times[0,1] \longrightarrow \mathbf{R},\\ &(t,s,y,x) \longmapsto xy^m \frac{k(t,s)}{t^2-s^2}\,\mathbf{1}_{\{s\neq t\}}(t,s).\end{aligned}$$

The main observation is that

$$T : C([0,1]) \longrightarrow C([0,1]), \qquad (Tx)(t) := \int_0^1 \frac{k(t,s)}{t^2-s^2}x(s)\,ds$$

defines a positive compact linear operator. For, note that (a) implies

$$\frac{k(t,s)}{t^2-s^2}x(s) = \frac{k(t,s)(t-s)}{(t+s)(t-s)^2}x(s) \geq 0$$

for $t,s \in [0,1]$ with $t \neq s$ and $x \in K = C^+([0,1])$. Hence, $T$ is positive, i.e., $x \geq 0$ implies $Tx \geq 0$. Notice that (b), together with $1-\nu < 1$, yields

$$\left|\frac{k(t,s)}{t^2-s^2}\right| \leq \varrho\,\frac{1}{|\,t-s\,|^{1-\nu}}$$

for $t,s \in [0,1]$ with $t \neq s$. This shows $T$ is compact; cf. [**1**, Chapter 8.10].

Retaining the notation from Theorem 3.1, we have $\Psi_1^x(t,y) = y^m(Tx)(t)$ for $t,y \in [0,1]$ and $x \in K_1 \subset K$. Therefore $\Psi_1^x(t,\cdot)$ is increasing on $[0,1]$. Moreover, $(\Phi y)(t) := (y(t))^n - 1$ is strongly accretive by (2.5). Since $T$ is compact and

$$|\,\Gamma_1 x - \Gamma_1 \bar{x}\,| \leq |\,Tx - T\bar{x}\,| \quad \text{for } x, \bar{x} \in K_1,$$

$\Gamma_1 : K_1 \to C(K_1; C([0,1]))$ is compact, too. Finally, conditions (d) from Theorem 3.1 are satisfied. Indeed, note that $\Psi^x(t,y) = y^n - 1 + y^m(Tx)(t)$ on $[0,1]^2$ for $x \in K_1$. Consequently, $\Psi^x(t,0) = -1 \le 0$ and $\Psi^x(t, x_0(t)) = \Psi^x(t,1) = (Tx)(t) \ge 0$. Therefore Corollary 3.2 can be applied to give a solution of (3.9). Taking into account that a solution of (3.9) can have no zero, the proof of the theorem is complete. □

Let us remark that solving (3.9) by substituting $y(t) := x(t)^{-1}$ is at best possible in the case $n = m$.

The reader will wonder whether Corollary 3.2, respectively, Theorem 3.1, remains true, if (3.6) is only monotone, but not strongly monotone in $y = x(t)$. To be more precise, we ask whether Corollary 3.2 still holds in the case that $G_x : \overset{\circ}{K}_r \to C([0,1])$ is only accretive, but not $\omega$-accretive, for all $x \in \overset{\circ}{K}_r$; recall the notation from Theorem 3.1. However, simple examples show that this is not necessarily the case. We include such an example as an aside.

**Example 3.5.** Consider

(3.10)
$$(Fx)(t) := \chi(t)x(t) + \theta(t)(x(t)-1)\int_0^1 (1-x(s))\,ds = 0, \qquad t \in [0,1],$$

where $\chi(t) = \mathbf{1}_{[1/2,1]}(t)(2t-1)$ and $\theta(t) = \mathbf{1}_{[0,1/2]}(t)(1-2t)$ on $[0,1]$. Analogously to Example 3.3, it is easily checked that all assumptions besides (b')(i) from Corollary 3.2 are satisfied; here we let $x_0(t) := 1$ on $[0,1]$, hence $r = 1$, $\varphi : [0,1] \times [0,1] \to \mathbf{R}$, $(t,y) \mapsto \chi(t)y$, and

$$\psi : [0,1]^4 \to \mathbf{R}, \qquad (t,s,y,x) \longmapsto \theta(t)(y-1)(1-x).$$

But (3.10) has no solution $x \in C([0,1])$ with $0 \le x(t) \le 1$ on $[0,1]$, since this would imply $x(t) = 0$ on $(1/2, 1]$ and $(x(t)-1)\int_0^1 (1 - x(s))\,ds = 0$ on $[0, 1/2)$ by definition of $\chi$ and $\theta$. Clearly, the latter condition means $x(t) = 1$ on $[0,1/2)$, a contradiction.

Our next objective is to provide a sufficient condition for assumption (b')(ii) of Corollary 3.2 to hold. This criterion is useful when dealing with integral equations (1.2) with a weakly singular kernel $\psi$; cf.

Example 3.9 below. It can be applied in the case that $\psi$ may be approximated by "better" kernels $\psi_\varepsilon$ which may be thought of to be cut-off versions of $\psi$.

**Definition 3.6.** Let $a < b \in \mathbf{R}$, $D \subset X = C([0,1])$ such that $a \le x(t) \le b$ on $[0,1]$ for $x \in D$, and $\psi : [0,1] \times [0,1] \times [a,b] \times [a,b] \to \mathbf{R}$ measurable. $\psi$ is said to have property (A) with respect to $([a,b], D)$ if the following conditions are satisfied.

(a) $D \ni x \overset{\Gamma_1}{\mapsto} (D \ni y \mapsto \int_0^1 \psi(\cdot, s, y(\cdot), x(s))\, ds \in X) \in C(D;X)$ is a well-defined operator.

(b) There exist an $\varepsilon_0 > 0$ and a family $(\psi_\varepsilon)_{\varepsilon \in (0,\varepsilon_0]}$ of continuous functions $\psi_\varepsilon : [0,1] \times [0,1] \times [a,b] \times [a,b] \to \mathbf{R}$ such that

(i) $U_\varepsilon(s)(D) \subset C(D;X)$ is relatively compact for all $s \in [0,1]$ and $\varepsilon \in (0, \varepsilon_0]$; here $((U_\varepsilon(s)(x))(y))(t) := \psi_\varepsilon(t, s, y(t), x(s))$ for $x, y \in D$ and $t \in [0,1]$.

(ii) $\sup\{|\,\Gamma_1^\varepsilon x - \Gamma_1 x\,|_{C(D;X)} : x \in D\} \to 0$ as $\varepsilon \to 0^+$, where $\Gamma_1^\varepsilon$ with kernel $\psi_\varepsilon$ is defined analogously to $\Gamma_1$ with kernel $\psi$.

Now we can prove the following useful

**Lemma 3.7.** *Let $r > 0$ and $\psi : [0,1] \times [0,1] \times [0,r] \times [0,r] \to \mathbf{R}$ be measurable. If $\psi$ has property* (A) *with respect to $([0,r], K_r)$, then*

$$K_r \ni x \overset{\Gamma_1}{\longmapsto} \left( K_r \ni y \longmapsto \int_0^1 \psi(\cdot, s, y(\cdot), x(s))\, ds \in X \right) \in C(K_r; X)$$

*is a well-defined compact operator.*

*Proof.* Let $D = K_r$. Since $\Gamma_1^\varepsilon \to \Gamma_1$ uniformly as $\varepsilon \to 0^+$, we only have to make sure that $\Gamma_1^\varepsilon$ is compact, for fixed $\varepsilon \in (0, \varepsilon_0]$. We want to apply Lemma 2.6 to $X = C([0,1])$, $Y = C(D;X)$, and $(\Omega, \Sigma, \mu) = ([0,1], \mathcal{L}([0,1]), \lambda|_{[0,1]})$, where $\mathcal{L}([0,1])$, respectively, $\lambda|_{[0,1]}$ denotes the Lebesgue $\sigma$-algebra, respectively, the Lebesgue measure restricted to $[0,1]$. For $s \in [0,1]$, let $U_\varepsilon(s) : D \to Y$ be defined by $((U_\varepsilon(s)(x))(y))(t) := \psi_\varepsilon(t, s, y(t), x(s))$ for $x \in D$; note that $U_\varepsilon(s)(x) \in Y = C(K_r; X)$. Clearly $U_\varepsilon(s) : D \to Y$ is compact for $s \in [0,1]$

and $U_\varepsilon(\cdot)(x)$ is strongly measurable for $x \in D$. Moreover, since $\psi_\varepsilon$ is continuous,

$$|\, U_\varepsilon(s)(x)\,|_Y = \sup_{y\in D} |\,(U_\varepsilon(s)(x))(y)\,|_X = \sup_{y\in D} |\,\psi_\varepsilon(\cdot, s, y(\cdot), x(s))\,|_X \le M_\varepsilon$$

for some $M_\varepsilon > 0$ and all $s \in [0,1]$ and $x \in D$. Therefore $\Gamma_1^\varepsilon$ is compact by Lemma 2.6. □

*Remark* 3.8. Suppose that we are in the setting of Corollary 3.2. If (a), (b′)(i), (c), and (d) are satisfied, and if $\psi$ has property (A) with respect to $([0,r], K_r)$, then (3.6) has a solution $x \in K_r$ such that $0 \le x \le x_0$, by Corollary 3.2 and Lemma 3.7.

The following example illustrates possible applications of this remark.

**Example 3.9.** We consider (1.1) from the introduction. This equation is used to describe the so-called plasma corners mathematically; cf. [**8**] and [**11**], and the explicit form of the kernel is

$$\begin{aligned} k(t,s,y,x) := {} & x \ln\left(\frac{[(x-y)^2+(s-t)^2][(x+y)^2+(s-t)^2]}{(x^2+s^2)^2}\right) \\ & + y \ln\left(\frac{(x+y)^2+(s-t)^2}{(x-y)^2+(s-t)^2}\right) \\ & + 2(s-t)\left(\arctan\frac{x-y}{s-t} + \arctan\frac{x+y}{s-t} - 2\arctan\frac{x}{s}\right). \end{aligned} \tag{3.11}$$

By applying Remark 3.8 we shall give a proof of the following theorem from [**8**, p. 168] which fits the general existence theory developed above. In particular, the estimate on the lower part of the boundary will become much simpler; cf. [**11**, Remark 4.2].

**Theorem 3.10.** *If* $J \in [0,2]$*, then* (1.1) *has a solution* $x \in C^+([0,1])$ *such that* $0 \le x(t) \le t$.

Before going on to the proof of Theorem 3.10, let us first state a lemma which contains the properties of the kernel $k$ which will be needed in the sequel.

**Lemma 3.11.** *Let* $D_k := \{(t,s,y,x) \in [0,1]^2 \times [0,\infty)^2 : s > 0$ *and* $s \neq t\}$ *and let* $k$ *be defined by* (3.11).

(a) $k(t,s,t,x) \geq 0$ *for all* $(t,s,x) \in [0,1]^3$ *with* $s > 0$ *and* $s \neq t$.

(b) $k(t,s,0,x) \leq 0$ *for all* $(t,s,x) \in [0,1]^3$ *with* $s > 0$ *and* $t < s \leq 1$, *and* $k(t,s,0,x) \leq 4\pi(t-s)$ *for all* $(t,s,x) \in [0,1]^3$ *with* $0 < s < t$.

(c) *If* $(t,s,y,x) \in D_k$, *then*

$$\partial_y k(t,s,y,x) = \ln\left(\frac{(x+y)^2 + (s-t)^2}{(x-y)^2 + (s-t)^2}\right) \geq 0.$$

(d) *For* $r > 0$ *there is a* $C(r) > 0$ *such that* $|\,k(t,s,y,x)\,| \leq C(r)$ *for* $(t,s,y,x) \in D_k$ *with* $y, x \leq r$.

(e) *For* $r > 0$ *and* $\varepsilon > 0$ *there is a* $C(r,\varepsilon) > 0$ *such that* $|\,\partial_x k(t,s,y,x)\,| \leq C(r,\varepsilon)$ *for all* $(t,s,y,x) \in D_k$ *with* $y, x \leq r$, $s \geq \varepsilon$, *and* $|\,t-s\,| \geq \varepsilon$. *Here*

$$\partial_x k(t,s,y,x) = \ln\left(\frac{[(x+y)^2 + (s-t)^2][(x-y)^2 + (s-t)^2]}{(x^2+s^2)^2}\right) + \frac{4ts}{x^2+s^2}$$

*for* $(t,s,y,x) \in D_k$.

(f) $\psi_0 := k\,\mathbf{1}_{D_k}$ *has property* (A) *with respect to* $([0,1], K_1)$.

*Proof.* Since (a), (b), (c), (d), and (e) may be obtained by direct calculation, cf. the appendix of [**11**] and [**8**], let us concentrate on the main parts of (f). Let $X = C([0,1])$. Clearly, $\Gamma_1 : K_1 \to C(K_1; X)$ is well defined; note that

$$\begin{aligned}
\int_0^1 &\big|k(t,s,y(t),x(s)) - k(t,s,\bar{y}(t),x(s))\big|\,ds \\
&\leq |\,y-\bar{y}\,| \int_0^1 \ln\left(\frac{5}{(s-t)^2}\right) ds \\
&= |\,y-\bar{y}\,|\big(\ln 5 + 2 - 2t\ln t - 2(1-t)\ln(1-t)\big) \\
&\leq 5|\,y-\bar{y}\,|
\end{aligned}$$

for $y, \bar{y} \in K_1$ and $t \in [0,1]$ by (c). Let $\varepsilon_0 := 1/4$ and fix $\varepsilon \in (0, \varepsilon_0]$. Define $\chi_\varepsilon(\tau) := 0$ for $\tau \in [0,\varepsilon]$, $\chi_\varepsilon(\tau) := 1$ for $\tau \in [2\varepsilon, 1]$, and $\chi_\varepsilon$ linear in between. Furthermore, define the continuous $\psi_\varepsilon : [0,1]^4 \to \mathbf{R}$ by

$$\psi_\varepsilon(t,s,y,x) := k(t,s,y,x)\chi_\varepsilon\big(|\,t-s\,|\big)\chi_\varepsilon(s).$$

We are going to show that $U_\varepsilon(s)(K_1)$ is a relatively compact subset of $C(K_1;X)$ for every $s \in [0,1]$. For, we can assume that $s \in [\varepsilon, 1]$. Fix $x, \bar{x} \in K_1$ and $t \in [0,1]$. Then

$$\begin{aligned}
&\big|\big((U_\varepsilon(s)(x))(y)\big)(t) - \big((U_\varepsilon(s)(\bar{x}))(y)\big)(t)\big| \\
&\qquad\qquad \le \big|k(t,s,y(t),x(s)) - k(t,s,y(t),\bar{x}(s)\big|\chi_\varepsilon\big(|\,t-s\,|\big) \\
&\qquad\qquad \le C(1,\varepsilon)|\,x(s)-\bar{x}(s)|
\end{aligned}$$

by (e), and therefore

$$|\,U_\varepsilon(s)(x) - U_\varepsilon(s)(\bar{x})\,|_{C(K_1;X)} \le C(1,\varepsilon)|\,x(s)-\bar{x}(s)|.$$

Hence $U_\varepsilon(s)(K_1) \subset C(K_1;X)$ is relatively compact.

Finally, $\Gamma_1^\varepsilon \to \Gamma_1$ uniformly as $\varepsilon \to 0^+$. Indeed, for $x, y \in K_1$, $t \in [0,1]$, and $\varepsilon \in (0, \varepsilon_0]$ we obtain by means of (d)

$$\begin{aligned}
&\big|\big((\Gamma_1^\varepsilon(x))(y)\big)(t) - \big((\Gamma_1(x))(y)\big)(t)\big| \\
&= \left|\int_0^1 \Big(k(t,s,y(t),x(s))\chi_\varepsilon\big(|\,t-s\,|\big)\chi_\varepsilon(s) - k(t,s,y(t),x(s))\Big)\,ds\right| \\
&\le C\left(1 - \int_0^1 \chi_\varepsilon\big(|\,t-s\,|\big)\chi_\varepsilon(s)\,ds\right) \\
&\le C\left(1 - \int_0^1 \chi_\varepsilon\big(|\,t-s\,|\big)\,ds + 2\varepsilon\right).
\end{aligned}$$

From the definition of $\chi_\varepsilon$ and $|\cdot|_{C(K_1;X)}$ it follows that

$$\sup\{|\,\Gamma_1^\varepsilon x - \Gamma_1 x\,|_{C(K_1;X)} : x \in K_1\} \le C\varepsilon$$

for some $C > 0$. The proof is thus complete. $\square$

Now we turn to the

*Proof of Theorem* 3.10. Fix $J \in [0,2]$. We want to apply Remark 3.8 to $x_0(t) := t$ on $[0,1]$, hence $r = 1$, $\varphi : [0,1]\times[0,1] \to \mathbf{R}$, $(t,y) \mapsto y^2 - t^2$, and $\psi := (J/(4\pi))\psi_0 : [0,1]^4 \to \mathbf{R}$ with $\psi_0$ from Lemma 3.11(f). Since $\psi$ has property (A) with respect to $([0,1], K_1)$, because $\Phi : \overset{\circ}{K}_1 \to X$, $(\Phi y)(t) := (y(t))^2 - t^2$, is strongly accretive by (2.5), and since

$$[0,\infty) \ni y \longmapsto \Psi_1^x(t,y) := \int_0^1 \psi(t,s,y,x(s))\,ds \in \mathbf{R}$$

has derivative $\partial_y \Psi_1^x(t,\cdot) \geq 0$ on $[0,\infty)$ for fixed $t \in [0,1]$ and $x \in K = C^+([0,1])$ by Lemma 3.11(c), we only have to check that the conditions from (d) of Theorem 3.1 are satisfied at $y = x_0(t) = t$ and $y = 0$. For, fix $t \in [0,1]$ and $x \in K_1$. Then Lemma 3.11(a) implies

$$\Psi^x(t,t) = \varphi(t,t) + \Psi_1^x(t,t) = \frac{J}{4\pi}\int_0^1 k(t,s,t,x(s))\,ds \geq 0,$$

and it follows from Lemma 3.11(b) that

$$\begin{aligned}\Psi^x(t,0) = \varphi(t,0) + \Psi_1^x(t,0) &= -t^2 + \frac{J}{4\pi}\int_0^1 k(t,s,0,x(s))\,ds \\ &\leq -t^2 + \frac{J}{4\pi}\int_0^t 4\pi(t-s)\,ds \\ &= t^2\left(\frac{J}{2} - 1\right) \leq 0\end{aligned}$$

since $J \leq 2$. Therefore Remark 3.8 applies. □

In general, the existence of nonnegative solutions of (1.2) cannot be expected. However, some results are possible in certain other situations.

**Example 3.12.** In [**2**] it was investigated for which $\lambda \in \mathbf{R}$

$$x(t) - x_0(t) - \frac{\lambda}{2}x(t)\int_t^1 k(t,s)x(s)\,ds = 0, \qquad t \in [0,1], \tag{3.12}$$

has a solution in $X = C([0,1])$, provided the kernel $k$ is sufficiently nice, and $0 \neq x_0 \in C([0,1])$. We are going to prove

**Theorem 3.13.** *Let $k : [0,1] \times [0,1] \to \mathbf{R}$ be measurable and such that*

$$K : X \to X, \qquad (Kx)(t) := \int_t^1 k(t,s)x(s)\,ds,$$

*is a well-defined compact linear operator $K \neq 0$. Then the following conclusions hold.*

(a) *Fix $r > 0$. If $\lambda \in \mathbf{R}$ is such that $|\,\lambda\,| \leq 2r|\,K\,|^{-1}(r+|\,x_0\,|)^{-2}$, then* (3.12) *has a solution $x \in C([0,1])$ with $|\,x(t) - x_0(t)\,| \leq r$ on $[0,1]$.*

(b) *If $x_0 \geq 0$ and if $k(t,s) \geq 0$ on $[0,1]^2$, then* (3.12) *has a solution $x \in C([0,1])$ with $x \geq 0$ whenever $\lambda \in (-\infty, 0]$.*

(c) *For the special choice $x_0(t) = 1$ on $[0,1]$ and $k(t,s) := t/(t+s)$ for $t,s \in [0,1]$ with $t+s \neq 0$,* (3.12) *has a solution $x \in C([0,1])$ with $x \geq 0$ whenever $\lambda \in (-\infty, 1/(2M)]$, where $M := \max\{t \ln((1+t)/(2t)) : t \in [0,1]\}$.*

*Proof.* (a) Fix $r > 0$, let $\Omega := B_r(x_0) \subset X$ and $\omega(s) := \delta s$ for $s \in [0,\infty)$, where

$$\delta := \frac{|\,x_0\,|}{r + |\,x_0\,|} > 0.$$

Then $\omega \in \mathcal{M}$. Furthermore,

$$|\,\lambda\,|\,|\,Kx\,| \leq \frac{2r}{r+|\,x_0\,|} = 2(1-\delta) \quad \text{and} \quad |\,\lambda\,|\,|\,Kx\,|\,|\,x\,| \leq 2r$$

for $x \in \overline{\Omega}$ by definition of $\delta$. Choosing the obvious $F$ and $G$, we obtain

$$\begin{aligned}(G_x(y) - G_x(\bar y), y - \bar y)_+ &= \left(y - x_0 - \frac{\lambda}{2} y Kx - \bar y + x_0 + \frac{\lambda}{2}\bar y Kx, y - \bar y\right)_+ \\ &\geq \left(1 - \frac{|\,\lambda\,|}{2}\,|\,Kx\,|\right)|\,y - \bar y\,|^2 \\ &\geq \delta|\,y - \bar y\,|^2 = \omega(|\,y - \bar y\,|)|\,y - \bar y\,|\end{aligned}$$

for $x, y, \bar y \in \Omega$. Consequently, $G_x : \Omega \to X$ is $\omega$-accretive for fixed $x \in \Omega$. Moreover, the compactness of $K$ implies the compactness of $\Gamma : \Omega \to C(\Omega; X)$. This shows that $F$ is strongly semicondensing, hence semicondensing. To apply Corollary 2.4(b), we therefore only have to note that

$$(Fx, x - x_0)_+ = \left(x - x_0 - \frac{\lambda}{2}x(Kx), x - x_0\right)_+$$
$$\geq r^2 - \frac{|\,\lambda\,|}{2}\,|\,x\,|\,|\,Kx\,|, \qquad r \geq 0$$

for all $x \in \partial\Omega$, i.e., $|\,x - x_0\,| = r$, by (3.13). This yields a zero of $F$ in $\overline{\Omega}$.

(b) follows from Corollary 3.2. (c) follows from (a) and (b), since the integral operator $K$ with kernel $k$ clearly is compact by the Arzelà-Ascoli theorem. $\square$

One may have noticed that up to now we always had $X = C([0,1])$. Clearly, applications of the theory of semicondensing operators to (1.2) are not restricted to this choice. So let us finally give an application of Corollary 2.4(a) in the spaces $X = L_p([0,1]; Y)$ of Bochner integrable functions for $p \in (1, \infty)$ and some Banach space $Y$; then

$$(3.14) \qquad (x, y)_{\pm, X} = |\,y\,|_X^{2-p} \int_0^1 (x(t), y(t))_{\pm, Y}\,|\,y(t)\,|_Y^{p-2}\,dt$$

for $x, y \in X$ and $y \neq 0$; cf. [**11**, Lemma A.2].

In [**18**] and [**17**], similar results were obtained in the case that the kernel $\psi$ in (1.2) is only dependent on $t, s$, and $x(s)$.

**Theorem 3.14.** *Let $Y$ be a Banach space, $J = [0,1]$, $p \in (1, \infty)$, $X = L_p(J; Y)$, and $q = p/(p-1)$. Let $\psi : [0,1] \times [0,1] \times Y \times Y \to Y$ be a function such that the following conditions are satisfied.*

(a) (i) $\psi(t, s, \cdot, \cdot) : Y \times Y \to Y$ *is continuous for almost all* $(t, s) \in J \times J$.

(ii) $\psi(\cdot, \cdot, y, x) : J \times J \to Y$ *is strongly measurable for all* $(y, x) \in Y \times Y$.

(b) *There are $a \in L_p^+(J)$, $b \in L_1^+(J)$ and $c \in L_q^+(J)$ such that* $|\,a\,|_p\,|\,c\,|_q < 1$ *and*

$$|\,\psi(t, s, y, x)\,|_Y \leq a(t)\big(b(s) + c(s)|\,x\,|_Y\big)$$

*for all* $y, x \in Y$ *and almost all* $(t,s) \in J^2$.

(c) *There are* $0 \neq \bar{a} \in L_p^+(J)$ *and* $0 \neq \bar{c} \in L_q^+(J)$ *such that*

$$|\,\psi(t,s,y,x) - \psi(t,s,y,\bar{x})\,|_Y \leq \bar{a}(t)\bar{c}(s)|\,x - \bar{x}\,|_Y$$

*for all* $x, \bar{x}, y \in Y$ *and almost all* $(t,s) \in J^2$.

(d) *There is a* $0 \neq \bar{b} \in L_1^+(J)$ *such that* $|\,\bar{a}\,|_p\,|\,\bar{c}\,|_q + |\,\bar{b}\,|_1 < 1$ *and*

$$(\psi(t,s,y,x) - \psi(t,s,\bar{y},x), y - \bar{y})_{-,Y} \leq \bar{b}(s)|\,y - \bar{y}\,|_Y^2$$

*for all* $x, y, \bar{y} \in Y$ *and almost all* $(t,s) \in J^2$. *Then*

$$x(t) = \int_0^1 \psi(t,s,x(t),x(s))\,ds, \qquad t \in J, \tag{3.15}$$

*has a solution in* $X = L_p(J;Y)$.

*Proof.* By (b) we have

$$r^{-1}|\,a\,|_p|\,b\,|_1 + |\,a\,|_p|\,c\,|_q \leq 1, \quad \text{i.e. } |\,a\,|_p\big(|\,b\,|_1 + |\,c\,|_q r\big) \leq r \tag{3.16}$$

for all sufficiently large $r \in (0,\infty)$. Let $\Omega = B_r(0) \subset X$ for such a fixed $r$, choose $\delta > 0$ such that

$$\big(|\,\bar{a}\,|_p + \delta\big)|\,\bar{c}\,|_q \leq 1 - |\,\bar{b}\,|_1, \tag{3.17}$$

and define $\omega(s) := \big(|\,\bar{a}\,|_p + \delta\big)|\,\bar{c}\,|_q s$ on $[0,\infty)$. Furthermore, let

$$G : \overline{\Omega} \times \overline{\Omega} \to X, \qquad (x,y) \longmapsto y + G_1(x,y),$$

with

$$G_1 : \overline{\Omega} \times \overline{\Omega} \to X, \qquad (G_1(x,y))(t) := -\int_0^1 \psi(t,s,y(t),x(s))\,ds.$$

Then $Fx = G(x,x)$ is strongly semicondensing with representation $(G,\omega)$. For, note that

$$\sup_{x,y\in\overline{\Omega}} |\,G_1(x,y)\,|_X^p \leq |\,a\,|_p^p\big(|\,b\,|_1 + |\,c\,|_q r\big)^p \tag{3.18}$$

by (b). Moreover, if $x \in \Omega$ is fixed, then $G_x = G(x, \cdot) : \Omega \to X$ is $\omega$-accretive. Indeed, let $y, \bar{y} \in \Omega$, and $f(t,s) := \psi(t,s,y(t),x(s)) - \psi(t,s,\bar{y}(t),x(s))$ for $(t,s) \in J^2$. Then $f(t,\cdot) : J \to Y$ is Bochner integrable for almost all $t \in J$. Hence (d) and (2.3) imply

$$\begin{aligned}\left(\int_0^1 f(t,s)\,ds, y(t)-\bar{y}(t)\right)_{+,Y} &\le \int_0^1 \big(f(t,s), y(t)-\bar{y}(t)\big)_{+,Y}\,ds \\ &= |\,\bar{b}\,|_1 |\,y(t)-\bar{y}(t)\,|_Y^2\end{aligned}$$

for almost all $t \in J$. This in turn yields

$$\begin{aligned}(G(x,y)-&G(x,\bar{y}), y-\bar{y})_{+,X} \\ &\ge (G(x,y)-G(x,\bar{y}), y-\bar{y})_{-,X} \\ &= |\,y-\bar{y}\,|_X^2 - |\,y-\bar{y}\,|_X^{2-p} \\ &\quad \cdot \int_0^1 \left(\int_0^1 f(t,s)\,ds, y(t)-\bar{y}(t)\right)_{+,Y} |\,y(t)-\bar{y}(t)\,|_Y^{p-2}\,dt \\ &\ge \big(1-|\,\bar{b}\,|_1\big)|\,y-\bar{y}\,|_X^2 \\ &\ge \omega(|\,y-\bar{y}\,|_X)|\,y-\bar{y}\,|_X\end{aligned}$$

by (3.14) and (3.17). Furthermore, for $x, \bar{x}, y \in \Omega$, (c) and Hölder's inequality give

$$|\,G(x,y)-G(\bar{x},y)\,|_X^p \le |\,\bar{a}\,|_p^p |\,\bar{c}\,|_q^p |\,x-\bar{x}\,|_X^p.$$

Consequently, $|\,\Gamma x - \Gamma\bar{x}\,|_{C(\Omega;X)} \le |\,\bar{a}\,|_p |\,\bar{c}\,|_q |\,x-\bar{x}\,|_X$ for $x, \bar{x} \in \Omega$, where $\Gamma : \Omega \to C(\Omega; X)$ is defined by $\Gamma(x) = G_x$. Standard covering arguments imply

$$\alpha_{C(\Omega;X)}(\Gamma(A)) \le |\,\bar{a}\,|_p |\,\bar{c}\,|_q \alpha_X(A) < \omega(\alpha_X(A))$$

for $A \subset \Omega$ with $\alpha_X(A) > 0$; recall the definition of $\omega$, (3.17), and $\bar{c} \ne 0$. Therefore $F : \overline{\Omega} \to X$ is strongly semicondensing. To apply Corollary 2.4(a), we only have to note that

$$|\,(I-F)(x)\,|_X^p \le |\,a\,|_p^p \big(|\,b\,|_1 + |\,c\,|_q r\big)^p \le r^p$$

for $x \in X$ with $|\,x\,|_X = r$ by (3.18) and (3.16). $\square$

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