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SUPERCONVERGENCE OF THE ITERATED HYBRID COLLOCATION METHOD FOR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS

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Dedicated to Professor K.E. Atkinson on the occasion
of his 65th birthday with friendship and esteem

ABSTRACT. A hybrid collocation method for Volterra integral equations with weakly singular kernels was introduced in [**12**]. The main purpose of this paper is to study *superconvergence* properties of the *iterated* hybrid collocation solution. It is proved that the iterated collocation solution has an improvement on order of convergence for the original collocation solution provided that suitable collocation parameters are chosen. Moreover, we apply the hybrid collocation and the associated iterated method to solving Volterra integro-differential equations with weakly singular kernels. Numerical examples are presented to confirm the superconvergence results of the iterated collocation methods.

1. Introduction. The main purpose of this paper is to study superconvergence properties of iterated solutions of Volterra integral equations of the second kind with weakly singular kernels, based on the hybrid collocation method developed in [**12**].

It is well known that the solution of the equations exhibits a singularity near the left end-point of the domain because of the weak singularity in the kernel. When we develop a numerical method for solving equations of this type we should take this into account in order to produce an approximate solution with high order convergence. The collocation

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method using a graded mesh is one of the most commonly used methods, see, e.g., [**6, 9, 11, 20**]. However, as pointed out in [**11, 12**], the use of a graded mesh may cause serious round-off errors when high order polynomials are used since the first subinterval near the singular point is very small in length. Aiming at overcoming such a difficulty, a hybrid collocation method is developed in [**12**] where the length of the first subinterval is the same as those in a quasi-uniform partition. To compensate the use of a large subinterval, nonpolynomial functions reflecting the singular behavior of the solution are employed in the first subinterval. As a result, an optimal order of global convergence may be obtained, and the stability of the method is guaranteed as well. Nonpolynomial spline collocation for weakly singular Volterra equations was probably first studied in [**4**]. The β-polynomial collocation methods for weakly singular Volterra equations and integro-differential equations were developed in [**14–16**]. In recent papers [**18, 19**], a transformation was used to convert the equation to the one whose solution is smooth so that a standard piecewise polynomial collocation method may be used to obtain an optimal order of convergence.

The iterated solution may improve order of convergence for the original solution of the equations. This phenomenon is called superconvergence. Superconvergence of iterated collocation methods for Volterra integral equations with *smooth* kernels was studied in [**3, 5, 8**]. In this paper, we investigate the superconvergence property of the iterated solution for Volterra equations with weakly singular kernels, based on the hybrid collocation method. It will be shown that the iterated solution has an improvement on the original collocation solution if we choose suitable collocation parameters. By identifying the values of the iterated collocation solution at the collocation points with those of the original collocation solution, we obtain a *local* superconvergence of the collocation solution at the collocation points by using the global superconvergence of the iterated solution.

We will apply the results for Volterra integral equations to Volterra integro-differential equations since the error estimate of a Volterra integro-differential equation is closely related to the error estimate of the iterated solution of the corresponding Volterra integral equations. Piecewise collocation methods using graded meshes for solving Volterra integro-differential equations were extensively studied, see, e.g., [**7, 10, 21, 22**].

This paper is organized as follows. In Section 2, we first recall the hybrid interpolation operators which serve as a base for the collocation method and then describe the hybrid collocation methods in terms of the hybrid interpolation operator and their iterated methods. In Section 3 superconvergence properties of the iterated solution are investigated. We first present a *global* superconvergence property of the iterated collocation solution of the Volterra integral equation of the second kind with weakly singular kernels. As a consequence of this result, we observe a *local* superconvergence property of the collocation solution at the collocation points, as their values are equal to those of the iterated solution at the collocation points. Section 4 is devoted to a study of an application of these results to the iterated collocation method for solving weakly singular integro-differential equations. The equation is then reformulated as a Volterra integral equation of the second kind with a *mildly* singular kernel and apply the superconvergence result of the iterated solution of the Volterra integral equation to the resulting equation and use it to obtain a superconvergence property of the corresponding approximate solution of the original integro-differential equation. We also observe *local* superconvergence of the iterated solution of the reformulated equation at the knots of the piecewise polynomial. In the final section two numerical examples are presented to confirm the superconvergence estimates of the iterated collocation methods.

2. Iterated hybrid collocation methods. In this section, we review the hybrid collocation method developed in [**12**] and describe its iterated method.

We begin with an introduction of the equations. Let $\triangle := \{(t,s) : 0 \leq s < t \leq 1\}$ and $I := [0,1]$. For given kernels $K, L \in C(\triangle)$ and a given parameter $\alpha \in (0,1)$, we define a Volterra integral operator $\mathcal{T} : C(I) \to C(I)$ by

$$(\mathcal{T}u)(t) := \int_0^t M(t,s)u(s)\,ds, \quad t \in I,$$

where

$$M(t,s) := (t-s)^{\alpha-1}K(t,s) + L(t,s), \quad (t,s) \in \triangle.$$

Consider the Volterra integral equations of the second kind

$$u(t) - (\mathcal{T}u)(t) = f(t), \quad t \in I, \tag{2.1}$$

where $f \in C(I)$ is a given function and $u \in C(I)$ is the unknown function to be determined.

Since the kernel M has a singularity along the diagonal $t = s$, the solution of equation (2.1) exhibits, in general, singularities at the zero in its derivatives even in the case of a smooth forcing term f, see, e.g., [**11, 12**]. Let us recall a result on the singularity of the solution of equation (2.1). Specifically, let $\mathbf{N}_0$ be the set of nonnegative integers. For a given positive integer m, we define a finite dimensional subspace W_m of $C(I)$ by

$$W_m := \operatorname{span}\{t^{i+j\alpha} : \ i,j \in \mathbf{N}_0 \ \text{ and } \ i + j\alpha < m\}. \tag{2.2}$$

This space captures the singularity of the solution of equation (2.1). We next make comments on the dimension of W_m. To this end, we set $l_m := \dim W_m$ and for a real number x, we denote by $\lfloor x \rfloor$ the largest integer less than x. Let

$$\mathbf{U}_m := \left\{(i,j) : \ j = 0, 1, \ldots, \left\lfloor \frac{m-i}{\alpha} \right\rfloor, \ i = 0, 1, \ldots, m-1 \right\}.$$

It is straightforward that $i, j \in \mathbf{N}_0$ with $i + j\alpha < m$ for any $(i,j) \in \mathbf{U}_m$. Also, we introduce a subset $\mathbf{V}_m$ of $\mathbf{U}_m$ by the following way: $(i_1, j_1) \in \mathbf{V}_m$ if there exists a lattice point $(i_2, j_2) \in \mathbf{U}_m$ such that

$$(i_2, j_2) \neq (i_1, j_1) \quad \text{and} \quad i_1 + j_1\alpha = i_2 + j_2\alpha.$$

When α is an irrational number, it is easy to verify that $\mathbf{V}_m = \varnothing$ and thus, l_m is equal to the cardinality of $\mathbf{U}_m$, that is,

$$l_m = \sum_{i=0}^{m-1} \left\lfloor \frac{m-i}{\alpha} \right\rfloor + m.$$

When α is a rational number, we let l'_m be one if $\mathbf{V}_m = \varnothing$, or the cardinality of $\mathbf{V}_m$ otherwise, and hence

$$l_m = \sum_{i=0}^{m-1} \left\lfloor \frac{m-i}{\alpha} \right\rfloor + m + 1 - l'_m.$$

We denote by U_m the sum of two subspaces W_m and $C^m(I)$, that is,

$$U_m := W_m \oplus C^m(I). \tag{2.3}$$

It is shown in [**11, 12**] that, if $K, L \in C^m(\triangle)$ and $f \in U_m$, then equation (2.1) has a unique solution $u \in U_m$. In general, the solution of equation (2.1) will have a component in the space W_m and thus, it is *not* in $C^m(I)$. As a result, the piecewise polynomial approximation on uniform meshes will not produce the optimal order of convergence.

To describe iterated hybrid collocation methods for equation (2.1), we recall the hybrid interpolation operators. We first introduce a graded partition of the interval I in terms of parameters m and α. To this end, for a given positive integer N and $q := m/\alpha$, we let i_0 be an integer satisfying the condition

$$\left(\frac{N}{i_0}\right)^q \le N \quad \text{and} \quad \left(\frac{N}{i_0 - 1}\right)^q > N.$$

It is easy to see that such an integer i_0 exists and satisfies the following estimate

$$N^{1-1/q} \le i_0 < N^{1-1/q} + 1.$$

Set $N' := N - i_0 + 1$ and choose $N' + 1$ points t_i, $i = 0, 1, \ldots, N'$, in I with

$$t_0 = 0, \quad t_i = \left(\frac{i_0 + i - 1}{N}\right)^q, \quad i = 1, 2, \ldots, N'. \tag{2.4}$$

The points t_i, $i = 0, 1, \ldots, N'$, divide the interval I into N' subintervals $\sigma_i := [t_{i-1}, t_i]$, $i = 1, 2, \ldots, N'$. We use the notations $h_i := t_i - t_{i-1}$ and $h := \max_{1 \le i \le N'} h_i$.

We next introduce a finite dimensional space corresponding to the partition (2.4). To this end, we denote by P_m the space of polynomials of degree $\le m-1$ on I. For a subinterval J of I and a function $f \in C(I)$, we use $f|_J$ for the restriction of f on J. Also, for $V \subseteq C(I)$, we let

$$V|_J := \{v|_J : \ v \in V\}.$$

Associated with the partition (2.4), the space S_h is defined by

$$S_h := \{u : \ u|_{\sigma_1} \in W_m|_{\sigma_1}, \ u|_{\sigma_i} \in P_m|_{\sigma_i}, \ i = 2, 3, \ldots, N'\}. \tag{2.5}$$

This space couples the piecewise polynomial space with the singular function space W_m, which allows us to capture the singularity of the exact solution u. Thus, the piecewise polynomial space is to approximate the smooth part of the exact solution which will have an optimal order of convergence. It is easy to verify that the dimension of S_h is given by

$$\dim S_h = l_m + (N' - 1)m.$$

We now define the hybrid interpolation operator $\mathcal{Q}_h$ from $C(I)$ to S_h. For this purpose, we introduce a notation

$$m_k := \begin{cases} l_m, & k = 1, \\ m, & k = 2, 3, \dots, N'. \end{cases}$$

Choosing l_m parameters τ'_j, $j = 1, 2, \dots, l_m$ and m parameters τ_j, $j = 1, 2, \dots, m$ such that

$$0 < \tau'_1 < \tau'_2 < \cdots < \tau'_{l_m} < 1, \quad \text{and} \quad 0 < \tau_1 < \tau_2 < \cdots < \tau_m < 1,$$

we obtain the interpolation points in every subinterval σ_k, $k = 1, 2, \dots, N'$ by setting

$$t_{1j} := t_0 + \tau'_j h_1, \quad j = 1, 2, \dots, l_m,$$

and

$$t_{kj} := t_{k-1} + \tau_j h_k, \quad k = 2, 3, \dots, N', \quad j = 1, 2, \dots, m.$$

It will be seen later that, in order for the iterated collocation solution to have superconvergence properties, the collocation parameters τ_j, $j = 1, 2, \dots, m$ have to be appropriately chosen. The hybrid interpolation operator $\mathcal{Q}_h : C(I) \to S_h$ is defined by

$$(\mathcal{Q}_h v)(t_{kj}) = v(t_{kj}), \quad k = 1, 2, \dots, N', \quad j = 1, 2, \dots, m_k. \tag{2.6}$$

It is known that the interpolation operator $\mathcal{Q}_h$ is well defined, see [**1, 12**].

For a function v, continuous on $[t_{i-1}, t_i]$, $i = 1, 2, \dots, N'$, with possible jump discontinuities at t_i, we define its maximum norm on $[t_l, t_n]$ with $0 \le l < n \le N'$ by

$$\|v\|_{[t_l, t_n]} = \max_{l \le i \le n} \max_{t_{i-1} \le t \le t_i} |v(t)|.$$

We will simply use $\|v\|$ when $[t_l, t_n] = I$. It is proved in [**12**] that there exist a positive constant c and a positive integer N_0 such that, for all $N > N_0$,

$$\|v - \mathcal{Q}_h\, v\| \le c\, N^{-m}, \tag{2.7}$$

for all $v \in U_m$. We remark that the set S_h is not a subspace of $C(I)$, and therefore $\mathcal{Q}_h$ is not a projection on $C(I)$. We may, however, take any norm preserving extension of $\mathcal{Q}_h$ to $L^\infty(I)$, see [**2**].

We next describe the hybrid collocation and iterated hybrid collocation methods for equation (2.1). The hybrid collocation method for solving equation (2.1) is to seek $u_h \in S_h$ such that

$$\begin{gathered} u_h(t_{kj}) - (\mathcal{T} u_h)(t_{kj}) = f(t_{kj}), \\ \text{for} \quad k = 1, 2, \ldots, N', \quad j = 1, 2, \ldots, m_k. \end{gathered} \tag{2.8}$$

Denoting by $\mathcal{I}$ the identity operator and using the interpolation operator $\mathcal{Q}_h$, we may write (2.8) as an operator equation in the form

$$(\mathcal{I} - \mathcal{Q}_h \mathcal{T})\, u_h = \mathcal{Q}_h\, f. \tag{2.9}$$

It is established in [**12**] that there exists a positive integer N_0 such that for all $N > N_0$ equation (2.9) has a unique solution $u_h \in S_h$ and there exists a positive constant c such that for all $N > N_0$

$$\|u - u_h\| \le c\, N^{-m}, \tag{2.10}$$

where u is the exact solution of equation (2.1).

With the collocation approximation u_h at hand, we define the corresponding iterated collocation solution $\hat{u}_h$ by

$$\hat{u}_h = f + \mathcal{T} u_h. \tag{2.11}$$

In the next section, we will analyze the superconvergence property of the iterated collocation solution $\hat{u}_h$. It will be shown that, if the collocation parameters are appropriately chosen, the iterated solution $\hat{u}_h$ gives higher order convergence than the original collocation solution u_h does.

3. Superconvergence. In this section, we present a global superconvergence property of the iterated collocation solution of equation (2.1). In addition, by observing that the collocation solution u_h and its iterated solution $\hat{u}_h$ coincide at the collocation points t_{kj}, $k = 1, 2, \dots, N'$, $j = 1, 2, \dots, m_k$, we obtain a *local* superconvergence property of the collocation solution u_h.

We first describe a choice of the collocation parameters. Let r be a nonnegative integer less than m. We choose collocation parameters τ_j, $j = 1, 2, \dots, m$, satisfying the following condition

$$\int_0^1 \prod_{j=1}^{m} (s - \tau_j) s^i \, ds = 0, \quad i = 0, 1, \dots, r. \tag{3.1}$$

Note that, when $r = m-1$, the values τ_j, $j = 1, 2, \dots, m$, are the zeros of the Legendre polynomial of degree m on interval I.

To establish the superconvergence property of the iterated solution, we first prove two technical lemmas. Let ℓ be an integer ≥ -1. For $\alpha \in (0, 1)$ and $\beta = 0, 1$, we assume that $K^{\ell}_{\alpha,\beta}(t, s)$ is m times continuously differentiable on $\triangle$ and satisfies

$$\left| \frac{\partial^i}{\partial s^i} K^{\ell}_{\alpha,\beta}(t, s) \right| \leq c \begin{cases} (t - s)^{\alpha+\ell-i} & \beta = 0, \\ 1 & \beta = 1, \end{cases} \tag{3.2}$$

with a positive constant c for all $i = 0, 1, \dots, m$. For convenience, we introduce a notation for $v \in L^{\infty}(I)$ and $j = 1, 2, \dots, N'$,

$$I^{\ell,j}_{\alpha,\beta}(t) := \int_{\sigma_j} K^{\ell}_{\alpha,\beta}(t, s)[v(s) - (\mathcal{Q}_h v)(s)] \, ds, \quad t \in I, \tag{3.3}$$

where $\mathcal{Q}_h$ is the interpolation operator defined in (2.6) associated with the partition (2.4). In the next lemma, we will estimate $I^{\ell,j}_{\alpha,\beta}$ when the operator $\mathcal{Q}_h$ is defined in terms of the collocation parameters τ_j determined by (3.1). To state the result, we introduce two hypotheses:

(H1) $\beta = 1$; or $\beta = 0$, $\ell \geq r$.

(H2) $\beta = 0$, $\ell < r$.

Lemma 3.1. *Suppose that the collocation parameters τ_j, $j = 1, 2, \dots, m$, are chosen to satisfy* (3.1). *If $v \in U_{m+r+1}$, then there exists a positive constant c such that for all N and $k = 3, 4, \dots, N'$,*

$$\max_{t\in\sigma_k} \sum_{j=2}^{k-2} |I_{\alpha,\beta}^{\ell,j}(t)| \le c\, N^{-(m+\lambda)}, \tag{3.4}$$

where $\lambda = (r+1)(1-\alpha/m)$ in case (H1), *and $\lambda = \min\{(r+1)(1-\alpha/m),\ \alpha+\ell+1\}$ in case* (H2).

Proof. For $t \in \sigma_k$ and $2 \le j \le k-2$, (see [**1**])

$$I_{\alpha,\beta}^{\ell,j}(t) = \int_{\sigma_j} K_{\alpha,\beta}^{\ell}(t,s)\, v[t_{j1}, t_{j2}, \dots, t_{jm}, s] \prod_{l=1}^{m} (s - t_{jl})\, ds,$$

holds where $v[s_1, s_2, \dots, s_{m+1}]$ denotes the mth order divided difference of v at the nodes $s_1, s_2, \dots, s_{m+1}$. For notational convenience, we let

$$P_{\alpha,\beta}^{\ell,j}(t,s) := K_{\alpha,\beta}^{\ell}(t,s)\, v[t_{j1}, t_{j2}, \dots, t_{jm}, s], \quad t \in \sigma_k, \quad s \in \sigma_j.$$

We expand the function $P_{\alpha,\beta}^{\ell,j}$ as the Taylor polynomial of degree r with respect to the second variable plus the remainder. Making use of the fact that the values τ_j, $j = 1, 2, \dots, m$ satisfy condition (3.1), we obtain that, for $t \in \sigma_k$,

$$I_{\alpha,\beta}^{\ell,j}(t) = \frac{1}{(r+1)!} \int_{\sigma_j} (s - t_{j-1})^{r+1} \prod_{l=1}^{m} (s - t_{jl})\, \frac{\partial^{r+1}}{\partial s^{r+1}} P_{\alpha,\beta}^{\ell,j}(t,s)\Big|_{s=\xi_j} ds, \tag{3.5}$$

for some $\xi_j \in [t_{j-1}, s]$. Noting that for $n = 0, 1, \dots, r+1$,

$$\|v^{(n)}[t_{j1}, t_{j2}, \dots, t_{jm}, s]\|_{\sigma_j} \le \|v^{(m+n)}\|_{\sigma_j} \le c\, t_{j-1}^{\alpha-(m+n)},$$

we have that for $t \in \sigma_k$,

$$\left| \frac{\partial^{r+1}}{\partial s^{r+1}} P_{\alpha,\beta}^{\ell,j}(t,s)\Big|_{s=\xi_j} \right| \le c \sum_{n=0}^{r+1} \left| \frac{\partial^n}{\partial s^n} K_{\alpha,\beta}^{\ell}(t,s)\Big|_{s=\xi_j} \right| t_{j-1}^{\alpha-(m+r+1-n)}. \tag{3.6}$$

It follows from (3.5) and (3.6) that, for $t \in \sigma_k$,

(3.7)
$$|I^{\ell,j}_{\alpha,\beta}(t)| \le c \sum_{n=0}^{r+1} h_j^{m+r+1}\, t_{j-1}^{\alpha-(m+r+1-n)} \int_{\sigma_j} \left| \frac{\partial^n}{\partial s^n}\, K^{\ell}_{\alpha,\beta}(t,s)\Big|_{s=\xi_j} \right| ds.$$

The estimate (3.7) is usually derived in order to study the superconvergence property of the numerical approximation to the exact solution, see [**1, 6, 11**]. Because

$$\begin{aligned} h_j &= [(j+i_0-1)^q - (j+i_0-2)^q]\, N^{-q} \\ &= (j+i_0-2)^q \left\{ \left[1+(j+i_0-2)^{-1}\right]^q - 1 \right\} N^{-q}, \end{aligned}$$

with $q := m/\alpha$, by the mean value theorem, there exists θ with $0 < \theta < (j+i_0-2)^{-1}$ such that

$$h_j = q(j+i_0-2)^{q-1}(1+\theta)^{q-1}N^{-q} \le c(j+i_0-2)^{q-1}N^{-q},$$

for some positive constant c. Since $N^{1-1/q} \le j+i_0-2$ and $h_j \le h_{k-1}$, it is straightforward to show that

$$\begin{aligned} h_j^{m+r+1} t_{j-1}^{\alpha-(m+r+1-n)} &\le c\, h_j^n (j+i_0-2)^{n-r-1} N^{-m} \\ &\le c\, h_{k-1}^n N^{-m-(r+1-n)(1-1/q)}. \end{aligned} \tag{3.8}$$

Notice that in case (H2)

$$\left| \frac{\partial^n}{\partial s^n}\, K^{\ell}_{\alpha,\beta}(t,s)\Big|_{s=\xi_j} \right| \le c,$$

for $s \in \sigma_j$, $\xi_j \in [t_{j-1}, s]$, $t \in \sigma_k$ and $n = 0, 1, \dots, \ell$, and

$$\left| \frac{\partial^n}{\partial s^n}\, K^{\ell}_{\alpha,\beta}(t,s)\Big|_{s=\xi_j} \right| \le c(t-\xi_j)^{\alpha+\ell-n} \le c(t-s)^{\alpha+\ell-n}, \tag{3.9}$$

for $s \in \sigma_j$, $\xi_j \in [t_{j-1}, s]$, $t \in \sigma_k$ and $n = 1+\ell, \dots, r+1$. Hence, in this case we have that for $t \in \sigma_k$,

$$T_n(t) := \sum_{j=2}^{k-2} \int_{\sigma_j} \left| \frac{\partial^n}{\partial s^n}\, K^{\ell}_{\alpha,\beta}(t,s)\Big|_{s=\xi_j} \right| ds \le c, \tag{3.10}$$

for $n = 0, 1, \dots, 1+\ell$, and

$$T_n(t) \le c[(t-t_{k-2})^{\alpha+\ell+1-n} - (t-t_1)^{\alpha+\ell+1-n}] \le c\, h_{k-1}^{\alpha+\ell+1-n}, \tag{3.11}$$

for $n = 2+\ell, \dots, r+1$. It is easily verified that in case (H1), for $t \in \sigma_k$,

$$T_n(t) \le c, \tag{3.12}$$

for all $n = 0, 1, \dots, r+1$. Since $h_{k-1} < cN^{-1}$, we conclude from (3.7), (3.8), (3.10), (3.11) and (3.12) that for $t \in \sigma_k$,

$$\sum_{j=2}^{k-2} |I_{\alpha,\beta}^{\ell,j}(t)| \le c \sum_{n=0}^{r+1} h_{k-1}^{n} N^{-m-(r+1-n)(1-1/q)} \le c\, N^{-(m+\lambda)},$$

in case (H1) and

$$\sum_{j=2}^{k-2} |I_{\alpha,\beta}^{\ell,j}(t)| \le c\Bigg[\sum_{n=0}^{1+\ell} h_{k-1}^{n} N^{-m-(r+1-n)(1-1/q)} + \sum_{n=2+\ell}^{r+1} h_{k-1}^{\alpha+\ell+1} N^{-m-(r+1-n)(1-1/q)}\Bigg] \le c\, N^{-(m+\lambda)},$$

in case (H2). This completes the proof. $\square$

We define the Volterra integral operator $\mathcal{K}_{\alpha,\beta}^{\ell} : L^\infty(I) \to C(I)$ by

$$(\mathcal{K}_{\alpha,\beta}^{\ell}\, u)(t) := \int_0^t K_{\alpha,\beta}^{\ell}(t,s)\, u(s)\, ds, \quad t \in I.$$

In our next result, we translate the estimate in Lemma 3.1 to an estimate of the quantity $\mathcal{K}_{\alpha,\beta}^{\ell}(v - \mathcal{Q}_h\, v)$.

Lemma 3.2. *Let $\mathcal{Q}_h$ be the interpolation operator defined in* (2.6) *associated with the partition* (2.4) *and with the collocation parameters τ_j, $j = 1, 2, \dots, m$ satisfying* (3.1). *If $v \in U_{m+r+1}$, then there exist a positive constant c and a positive integer N_0 such that, for all $N > N_0$,*

$$\|\mathcal{K}_{\alpha,\beta}^{\ell}(\mathcal{I} - \mathcal{Q}_h)\, v\| \le c\, N^{-(m+\nu)}, \tag{3.13}$$

where

$$\nu = \begin{cases} \min\{(r+1)(1-\alpha/m),\ \alpha+\ell+1\} & \beta=0, \\ \min\{(r+1)(1-\alpha/m),\ 1\} & \beta=1. \end{cases}$$

Proof. It follows from (2.7) that

$$|(\mathcal{K}^{\ell}_{\alpha,\beta}(v-\mathcal{Q}_h\, v))(t)| \le c\, N^{-m} \int_0^t |K^{\ell}_{\alpha,\beta}(t,s)|\, ds, \quad t \in I.$$

To estimate the right-hand side of the last inequality, we notice that, by the definition of partition (2.4),

$$h_1 = \left(\frac{i_0}{N}\right)^q < \left(\frac{N^{1-1/q}+1}{N}\right)^q \le \frac{2^q}{N}, \quad \text{and} \quad h_2 < c\, N^{-1}.$$

Thus, by (3.2) we have that for $t \in \sigma_1 \cup \sigma_2$,

$$|(\mathcal{K}^{\ell}_{\alpha,0}(v-\mathcal{Q}_h\, v))(t)| \le c\, N^{-m} t^{\alpha+\ell+1} \le c\, N^{-(m+\alpha+\ell+1)}$$

and

$$|(\mathcal{K}^{\ell}_{\alpha,1}(v-\mathcal{Q}_h\, v))(t)| \le c\, N^{-m} t \le c\, N^{-(m+1)}.$$

This implies that

$$\max_{t\in\sigma_1\cup\sigma_2} |(\mathcal{K}^{\ell}_{\alpha,\beta}(v-\mathcal{Q}_h\, v))(t)| \le c\, N^{-(m+\varepsilon)}, \tag{3.14}$$

where $\varepsilon = \alpha+\ell+1$ if $\beta=0$, and $\varepsilon=1$ if $\beta=1$.

By using Lemma 3.1, we obtain that for $t \in \sigma_k$ and $k = 3, 4, \dots, N'$,

(3.15)

$$\left| \int_{t_1}^{t_{k-2}} K^{\ell}_{\alpha,\beta}(t,s)[v(s)-(\mathcal{Q}_h\, v)(s)]\, ds \right| \le \sum_{j=2}^{k-2} \left| I^{\ell,j}_{\alpha,\beta}(t) \right| \le c\, N^{-(m+\lambda)},$$

where λ is given in Lemma 3.1. Again, by (2.7) and (3.2), we have that

$$\left| \int_{t_{k-2}}^{t} K^{\ell}_{\alpha,\beta}(t,s)[v(s)-(\mathcal{Q}_h\, v)(s)]\, ds \right| \le c\, N^{-(m+\varepsilon)}. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we have the estimate (3.13) with $\nu = \min\{\varepsilon, \lambda\}$. It remains to verify the ν has the desired form. Recall that if $\beta = 1$, then $\varepsilon = 1$ and $\lambda = (r+1)(1-\alpha/m)$, and if $\beta = 0$, then $\varepsilon = \alpha + \ell + 1$ and $\lambda = (r+1)(1-\alpha/m)$ in case (H1) or $\lambda = \min\{(r+1)(1-\alpha/m), \alpha+\ell+1\}$ in case (H2). Therefore, we have the desired form for the parameter ν. $\square$

To analyze the superconvergence property of the iterated collocation solution, we recall the following result which can be found in [**1**].

Lemma 3.3. *Let X be a Banach space and $\mathcal{A}, \mathcal{B}$ be bounded linear operators on X to itself. If $(\mathcal{I} - \mathcal{A}\mathcal{B})^{-1}$ exists from X to itself, then $(\mathcal{I} - \mathcal{B}\mathcal{A})^{-1}$ also exists, and*

$$(\mathcal{I} - \mathcal{B}\mathcal{A})^{-1} = \mathcal{I} + \mathcal{B}(\mathcal{I} - \mathcal{A}\mathcal{B})^{-1}\mathcal{A}. \tag{3.17}$$

We are now ready to prove the main theorem of this section, concerning superconvergence of the iterated hybrid collocation method. In the next theorem and the corollary that follows, we denote by u_h the collocation solution of equation (2.1) associated with the partition (2.4) and with collocation parameters τ_j, $j = 1, 2, \ldots, m$ satisfying (3.1) with $m > 1$ and $r = 1$, and by $\hat{u}_h$ its iterated solutions determined by (2.11).

Theorem 3.4. *Suppose that $K, L \in C^{m+2}(\triangle)$ and $f \in U_{m+2}$. Let u be the exact solution of* (2.1)*. Then, there exist a positive constant c and a positive integer N_0 such that for all $N > N_0$,*

$$\|u - \hat{u}_h\| \le c\, N^{-(m+\alpha)}. \tag{3.18}$$

Proof. Applying $\mathcal{Q}_h$ to both sides of (2.11), we have that

$$\mathcal{Q}_h\, \hat{u}_h = u_h. \tag{3.19}$$

Substituting equation (3.19) into the right-hand side of (2.11) yields that $\hat{u}_h$ satisfies the operator equation

$$\hat{u}_h - \mathcal{T}\mathcal{Q}_h\, \hat{u}_h = f.$$

Subtracting this equation from (2.1) we find that

$$(\mathcal{I} - \mathcal{T}\mathcal{Q}_h)(u - \hat{u}_h) = \mathcal{T}(\mathcal{I} - \mathcal{Q}_h)\, u. \tag{3.20}$$

It is easily verified that $\mathcal{T}$ is compact from $L^\infty(I)$ to $C(I)$ as well as from $L^\infty(I)$ to itself. Since $\mathcal{I} - \mathcal{T}$ is injective from $L^\infty(I)$ to itself (see [**9**]), by the Fredholm alternative theorem, $(\mathcal{I} - \mathcal{T})^{-1}$ exists.

On the other hand, according to the construction of $\mathcal{Q}_h$, we have for any $v \in C(I)$ that (see [**1, 12**])

$$\lim_{N\to\infty} \|v - \mathcal{Q}_h\, v\| = 0. \tag{3.21}$$

It follows from (3.21) and Lemma 3.1.1 of [**1**] that

$$\lim_{N\to\infty} \|\mathcal{T} - \mathcal{Q}_h\, \mathcal{T}\| = 0. \tag{3.22}$$

By Theorem 3.1.1 of [**1**], there exist a positive constant c and a positive integer N_0 such that for all $N > N_0$, $(\mathcal{I} - \mathcal{Q}_h\mathcal{T})^{-1}$ exists and is uniformly bounded by c in the L^∞ norm. By Lemma 3.3 and equation (3.20) we obtain that

$$\|u - \hat{u}_h\| \le c\, \|\mathcal{T}(\mathcal{I} - \mathcal{Q}_h)\, u\|. \tag{3.23}$$

By our assumptions we have that $u \in U_{m+2}$. Recall that the kernel M of $\mathcal{T}$ consists of two parts. One part satisfies (3.2) with $\beta = 1$, and the other satisfies (3.2) with $\beta = 0$ and $\ell = -1$. Note that

$$(r+1)\left(1 - \frac{\alpha}{m}\right) = 2 - \frac{2\alpha}{m} > 1, \quad \text{and} \quad \alpha + \ell + 1 = \alpha.$$

It follows from Lemma 3.2 that

$$\|\mathcal{T}(\mathcal{I} - \mathcal{Q}_h)\, u\| \le c\left[N^{-(m+1)} + N^{-(m+\alpha)}\right] \le c\, N^{-(m+\alpha)}.$$

Combining this and (3.23) shows that (3.18) is satisfied. $\square$

A direct consequence of the global superconvergence property of the iterated solution $\hat{u}_h$ is a local superconvergence property of u_h at the collocation points. We present this result in the next corollary.

Corollary 3.5. *Suppose that* $K, L \in C^{m+2}(\triangle)$ *and* $f \in U_{m+2}$. *Let* u *be the exact solution of* (2.1). *Then, there exist a positive constant* c *and a positive integer* N_0 *such that, for all* $N > N_0$,

$$\max_{1\le k\le N'} \max_{1\le j\le m_k} |u(t_{kj}) - u_h(t_{kj})| \le c\, N^{-(m+\alpha)}. \tag{3.24}$$

Proof. To establish the estimate (3.24), we note by (3.19) that

$$\begin{aligned}\max_{1\le k\le N'} \max_{1\le j\le m_k} |u(t_{kj}) - u_h(t_{kj})| &= \max_{1\le k\le N'} \max_{1\le j\le m_k} |u(t_{kj}) - \hat{u}_h(t_{kj})| \\ &\le \|u - \hat{u}_h\|.\end{aligned}$$

Thus, the estimate (3.24) follows directly from Theorem 3.4. □

We remark that superconvergence occurs when the collocation parameters τ_j, $j = 1, 2, \dots, m$, are chosen to satisfy equation (3.1) with $r = 1$, for an arbitrary set of the collocation points τ_j', $j = 1, 2, \dots, l_m$, in the first subinterval $[t_0, t_1]$.

4. Volterra integro-differential equations. We consider in this section Volterra integro-differential equations with weakly singular kernels. This equation is reformulated as a Volterra integral equation of the second kind with a mildly singular kernel. We apply the superconvergence result of the iterated solution of the Volterra integral equation to the resulting equation and use it to obtain superconvergence properties of the corresponding approximate solution of the original integro-differential equation.

We first describe the Volterra integro-differential equation. For $\alpha \in (0, 1)$ and $K \in C(\triangle)$, we set

$$K_\alpha(t, s) := (t - s)^{\alpha-1} K(t, s), \quad 0 \le s < t \le 1,$$

and define the Volterra integral operator $\mathcal{K}_\alpha : L^\infty(I) \to C(I)$ by

$$(\mathcal{K}_\alpha\, u)(t) := \int_0^t K_\alpha(t, s)\, u(s)\, ds, \quad t \in I.$$

Consider the Volterra integro-differential equations of the form

$$y'(t) = a(t)\,y(t) + b(t) + (\mathcal{K}_\alpha y)(t), \quad t \in I, \tag{4.1}$$

with initial-value condition $y(0) = y_0$, where $a, b \in C(I)$ are given functions and $y \in C^1(I)$ is the unknown function to be determined.

We now reformulate this problem as a standard Volterra integral equation, see [**10**]. To this end, we set

$$u(t) = y'(t), \quad t \in I, \tag{4.2}$$

define the linear operator $\mathcal{H} : L^\infty(I) \to C(I)$ by

$$(\mathcal{H}\,u)(t) := \int_0^t u(s)\,ds, \quad t \in I, \tag{4.3}$$

and introduce the linear operator $\mathcal{G} : L^\infty(I) \to C(I)$ by

$$(\mathcal{G}\,u)(t) := a(t)(\mathcal{H}\,u)(t) + (\mathcal{K}_\alpha \mathcal{H}\,u)(t), \quad t \in I. \tag{4.4}$$

Using the Dirichlet formula, the operator $\mathcal{G}$ is actually the Volterra integral operator from $L^\infty(I)$ to $C(I)$ with the kernel

$$G_\alpha(t,s) := a(t) + \int_s^t K_\alpha(t,\tau)\,d\tau, \quad 0 \le s < t \le 1. \tag{4.5}$$

Therefore, equation (4.1) with the initial-value condition may be written as a Volterra integral equation of the second kind in the form

$$(\mathcal{I} - \mathcal{G})\,u = f, \tag{4.6}$$

where f is defined by

$$f(t) := b(t) + y_0\,a(t) + y_0 \int_0^t K_\alpha(t,s)\,ds, \quad t \in I.$$

The following theorem concerns the solution of equation (4.6) and as well as equation (4.1).

Theorem 4.1. *If $K \in C^m(\triangle)$ and $a, b \in U_m$, then equation* (4.6) *has a unique solution $u \in U_m$, and equation* (4.1) *has a unique solution $y \in U_{m+1} \cap C^1(I)$.*

Proof. Following a similar argument in the proof of Theorem 2.1 of [**12**], we have that equation (4.6) has a unique solution $u \in U_m$. By (4.2) equation (4.1) has a unique solution $y \in U_{m+1} \cap C^1(I)$. □

We will establish numerical methods for solving equation (4.6), from which we obtain numerical methods for equation (4.1). For this purpose, we apply the hybrid collocation method to equation (4.6) which is to find $u_h \in S_h$ such that

$$(4.7) \qquad (\mathcal{I} - \mathcal{Q}_h \, \mathcal{G}) \, u_h = \mathcal{Q}_h \, f.$$

Having determined the solution u_h of (4.7), we may obtain an approximate solution y_h of (4.1) by setting

$$(4.8) \qquad y_h(t) := y_0 + \int_0^t u_h(s) \, ds.$$

The next theorem shows that the hybrid collocation method provides an optimal order of global convergence.

Theorem 4.2. *Let $K \in C^m(\triangle)$ and $a, b \in U_m$, and let u and y be the exact solutions of* (4.6) *and* (4.1)*, respectively. Assume that $\mathcal{Q}_h$ is the interpolation operator defined in* (2.6) *associated with the partition* (2.4)*. Then, there exist a positive constant c and a positive integer N_0 such that, for all $N > N_0$,* (4.7) *has a unique solution $u_h \in S_h$ and*

$$(4.9) \qquad \|u - u_h\| \le c \, N^{-m}.$$

Meanwhile, the approximate solution y_h of (4.1) *is uniquely determined by* (4.8)*, and also there holds*

$$(4.10) \qquad \|y - y_h\| \le c \, N^{-m}.$$

Proof. Note that $\mathcal{G}$ is a compact operator from $L^\infty(I)$ to $C(I)$. By an argument similar to the proof of the existence and boundedness of

$(\mathcal{I}-\mathcal{Q}_h\mathcal{T})^{-1}$, there exist a positive constant c and a positive integer N_0 such that for all $N > N_0$, $(\mathcal{I}-\mathcal{Q}_h\mathcal{G})^{-1}$ exists and is uniformly bounded by c in the L^∞ norm. Therefore, equation (4.7) has a unique solution

$$u_h = \mathcal{Q}_h\,\mathcal{G}\,u_h + \mathcal{Q}_h\,f \in S_h.$$

Noticing that

$$u - u_h = u - \mathcal{Q}_h\,u + \mathcal{Q}_h(u - u_h), \tag{4.11}$$

we conclude from (2.7) that, for all $N > N_0$,

$$\|u - u_h\| = \|(\mathcal{I} - \mathcal{Q}_h\,\mathcal{G})^{-1}(u - \mathcal{Q}_h\,u)\| \le c\,N^{-m}.$$

Combining (4.2) and (4.8) yields that, for $t \in I$,

$$y(t) - y_h(t) = \int_0^t [u(s) - u_h(s)]\,ds.$$

Hence, the estimate (4.10) follows from (4.9). □

We next consider the iterated collocation solution and investigate its superconvergence property. Assume that u_h is the unique solution of (4.7). We define the iterated collocation solution $\hat{u}_h$ by

$$\hat{u}_h = f + \mathcal{G}\,u_h. \tag{4.12}$$

Applying $\mathcal{Q}_h$ to both sides of (4.12) gives

$$\mathcal{Q}_h\,\hat{u}_h = u_h. \tag{4.13}$$

It follows from (4.6), (4.12) and (4.13) that

$$(\mathcal{I} - \mathcal{G}\,\mathcal{Q}_h)(u - \hat{u}_h) = \mathcal{G}(\mathcal{I} - \mathcal{Q}_h)\,u. \tag{4.14}$$

The next lemma gives an error estimate of the approximation y_h to the solution y of equation (4.1).

Lemma 4.3. *Let u and y be the exact solutions of* (4.6) *and* (4.1), *respectively, and let $\hat{u}_h$ and y_h be the corresponding approximations determined by* (4.12) *and* (4.8). *Then, there exist a positive constant c and a positive integer N_0 such that, for all $N > N_0$,*

$$\|y - y_h\| \le c\,(\|\mathcal{H}(\mathcal{I} - \mathcal{Q}_h)\,u\| + \|u - \hat{u}_h\|). \tag{4.15}$$

Proof. By (4.2) and (4.8), we have

$$\|y - y_h\| = \|\mathcal{H}(u - u_h)\| \le \|\mathcal{H}(\mathcal{I} - \mathcal{Q}_h)\,u\| + \|\mathcal{H}\mathcal{Q}_h(u - u_h)\|. \tag{4.16}$$

We denote by L_{kj} the Lagrange function with respect to t_{kj}, $k = 1, 2, \dots, N'$, $j = 1, 2, \dots, m_k$. Hence, we obtain for $t \in I$,

$$[\mathcal{Q}_h(u - u_h)](t) = \sum_{k=1}^{N'} \sum_{j=1}^{m_k} [u\,(t_{kj}) - u_h(t_{kj})] L_{kj}(t). \tag{4.17}$$

It is shown in [**12**] that there exists a positive constant c such that for all positive integers N

$$\sum_{k=1}^{N'} \sum_{j=1}^{m_k} \|L_{kj}\| \le c. \tag{4.18}$$

By (4.13) we have that, for all $k = 1, 2, \dots, N'$, $j = 1, 2, \dots, m_k$,

$$|u(t_{kj}) - u_h(t_{kj})| = |u(t_{kj}) - \hat{u}_h(t_{kj})| \le \|u - \hat{u}_h\|. \tag{4.19}$$

Using (4.17), (4.18), (4.19) and the boundedness of $\mathcal{H}$, we have

$$\|\mathcal{H}\,\mathcal{Q}_h(u - u_h)\| \le c\,\|u - \hat{u}_h\|. \tag{4.20}$$

Combining (4.16) and (4.20) proves (4.15). $\square$

We are now ready to prove superconvergence results for the collocation and iterated collocation solutions. To this end, we denote by u_h the collocation solution of equation (4.7) associated with the partition (2.4) and with collocation parameters τ_j, $j = 1, 2, \dots, m$ satisfying

(3.1) with $m > 1$ and $r = 1$, and by $\hat{u}_h$ its iterated solution determined by (4.12) and by y_h the corresponding approximate solution determined by (4.8) to the solution y of equation (4.1).

Theorem 4.4. *Suppose that $K \in C^{m+2}(\triangle)$ and $a, b \in U_{m+2}$. Let u and y be the exact solution of* (4.6) *and* (4.1)*, respectively. Then, there exist a positive constant c and a positive integer N_0 such that for all $N > N_0$,*

$$\|u - \hat{u}_h\| \le c\, N^{-(m+1)} \tag{4.21}$$

and

$$\|y - y_h\| \le c\, N^{-(m+1)}. \tag{4.22}$$

Proof. By our assumptions we have that $u \in U_{m+2}$. Recall that $\mathcal{H}$ and $\mathcal{K}_\alpha$ are the Volterra integral operators with kernels satisfying (3.2) with $\beta = 1$ and $\beta = 0$, $\ell = 0$, respectively. Note that

$$\alpha + \ell + 1 = \alpha + 1, \quad (r+1)\left(1 - \frac{1}{q}\right) = 2 - \frac{2}{q} > 1.$$

By Lemma 3.2 and (4.4), it is straightforward that

$$\|\mathcal{G}(\mathcal{I} - \mathcal{Q}_h)\, u\| \le c\big[N^{-(m+1)} + N^{-\min\{2-2\alpha/m,\ \alpha+1\}}\big] \le c\, N^{-(m+1)}. \tag{4.23}$$

By Theorem 4.2, there exist a positive constant c and a positive integer N_0 such that for all $N > N_0$, $(\mathcal{I} - \mathcal{G}\mathcal{Q}_h)^{-1}$ exists and is uniformly bounded by the constant c in the L^∞ norm. Combining this with (4.14) and (4.23) proves (4.21). We conclude from Lemma 4.3, (4.21) and Lemma 3.2 with $\mathcal{K}^\ell_{\alpha,\beta} := \mathcal{H}$ that the estimate (4.22) holds. $\square$

The next corollary presents a local superconvergence property of the collocation solution u_h at the collocation points.

Corollary 4.5. *Suppose that $K \in C^{m+2}(\triangle)$ and $a, b \in U_{m+2}$. Let u be the exact solution of* (4.6)*. Then, there exist a positive constant c and a positive integer N_0 such that for all $N > N_0$,*

$$\max_{1 \le k \le N'} \max_{1 \le j \le m_k} |u(t_{kj}) - u_h(t_{kj})| \le c\, N^{-(m+1)}. \tag{4.24}$$

Proof. The estimate (4.24) follows immediately from (4.13) and (4.21). □

We next investigate a *local* superconvergence property of the iterated collocation solution at the knots t_i, $i = 0, 1, \dots, N'$, of the piecewise polynomials. We will show that the order of local superconvergence of the iterated collocation solution may be improved if we use a few more basis functions on the first interval. To this end, we modify the interpolation operator. Associated with the partition (2.4), we define the space $\widetilde{S}_h$ by

$$\widetilde{S}_h := \{u : \ u|_{\sigma_1} \in W_{m+1}|_{\sigma_1}, \ u|_{\sigma_i} \in P_m|_{\sigma_i}, \ i = 2, 3, \dots, N'\}.$$

In comparison with the space S_h, the new space $\widetilde{S}_h$ is S_h with W_m replaced by W_{m+1}. Choosing l_{m+1} parameters τ_j'', $j = 1, 2, \dots, l_{m+1}$ such that

$$0 < \tau_1'' < \tau_2'' < \cdots < \tau_{l_{m+1}}'' < 1,$$

we have l_{m+1} interpolation points in the first subinterval σ_1 by setting

$$\tilde{t}_{1j} := t_0 + \tau_j'' h_1, \quad j = 1, 2, \dots, l_{m+1},$$

The modified interpolation operator $\widetilde{\mathcal{Q}}_h : C(I) \to \widetilde{S}_h$ is defined by

$$(\widetilde{\mathcal{Q}}_h\, v)(\tilde{t}_{1j}) = v(\tilde{t}_{1j}), \quad j = 1, 2, \dots, l_{m+1}.$$

and

$$(\widetilde{\mathcal{Q}}_h\, v)(t_{kj}) = v(t_{kj}), \quad k = 2, \dots, N', \quad j = 1, 2, \dots, m,$$

where t_{kj}, $k = 2, \dots, N'$, $j = 1, 2, \dots, m$ are the same interpolation points used by the operator $\mathcal{Q}_h$, as described in Section 2. By an argument similar to the operator $\mathcal{Q}_h$, we conclude that there exist a positive constant c and a positive integer N_0 such that for all $N > N_0$

$$\|v - \widetilde{\mathcal{Q}}_h\, v\| \le c\, N^{-m}, \tag{4.25}$$

for all $v \in U_m$ and

$$\|v - \widetilde{\mathcal{Q}}_h\, v\|_{\sigma_1} \le c\, N^{-(m+1)}, \tag{4.26}$$

for all $v \in U_{m+1}$. We remark that the purpose of introducing the operator $\widetilde{\mathcal{Q}}_h$ is to obtain an extra one order in the estimate (4.26), in comparison with the convergence order of operator $\mathcal{Q}_h$. Making use of (4.26), we may improve the order of local superconvergence of the iterated collocation solution at t_i, $i = 0, 1, \ldots, N'$. Set

$$R_\alpha(t,s) := \int_s^t K_\alpha(t,\tau)\,d\tau, \quad 0 \le s \le t \le 1,$$

and define the Volterra integral operator $\mathcal{R}_\alpha : L^\infty(I) \to C(I)$ by

$$(\mathcal{R}_\alpha\, u)(t) := \int_0^t R_\alpha(t,s)\, u(s)\, ds, \quad t \in I.$$

Using the operator $\widetilde{\mathcal{Q}}_h$, we give the following estimates.

Lemma 4.6. *Let $\widetilde{\mathcal{Q}}_h$ be the modified interpolation operator associated with the partition (2.4) and with collocation parameters τ_j, $j = 1, 2, \ldots, m$ satisfying (3.1) with $m > 2$ and $r = 2$. Suppose that $a, b \in U_{m+3}$, $K \in C^{m+3}(\triangle)$ and $v \in U_{m+3}$. Then there exists a positive constant c independent of N such that*

$$\|\mathcal{R}_\alpha(\mathcal{I} - \widetilde{\mathcal{Q}}_h)\, v\| \le c\, N^{-(m+1+\alpha)}, \tag{4.27}$$

and

$$\max_{0 \le i \le N'} |(\mathcal{G}\, v)(t_i) - (\mathcal{G}\widetilde{\mathcal{Q}}_h\, v)(t_i)| \le c\, N^{-(m+1+\alpha)}. \tag{4.28}$$

Proof. It is easy to verify that $R_\alpha(t,s)$ satisfies (3.1) with $\beta = 0$ and $\ell = 0$. By the constructions of $\mathcal{Q}_h$ and $\widetilde{\mathcal{Q}}_h$, we have that, for $v \in C(I)$,

$$\widetilde{\mathcal{Q}}_h\, v|_{[t_1,1]} = \mathcal{Q}_h\, v|_{[t_1,1]}. \tag{4.29}$$

Note that

$$\alpha + \ell + 1 = \alpha + 1 \quad \text{and} \quad (r+1)\left(1 - \frac{\alpha}{m}\right) = 3 - \frac{3\alpha}{m} > 2.$$

Recalling (4.25) and Lemma 3.2 with $\beta = 0$, we have that (4.27) is satisfied.

It is easily seen from (4.26) that

$$\|\mathcal{H}(\mathcal{I} - \widetilde{\mathcal{Q}}_h)\, v\|_{\sigma_1} \leq c\, N^{-(m+2)}. \tag{4.30}$$

Combining this with (4.29) and recalling Lemma 3.1 in case (H1), we have that for $i = 2, 3, \ldots, N'$,

(4.31)

$$\big|[\mathcal{H}(\mathcal{I} - \widetilde{\mathcal{Q}}_h)\, v](t_i)\big| \leq c\big[N^{-(m+2)} + N^{-(m+3-3\alpha/m)}\big] \leq c\, N^{-(m+2)}.$$

From (4.4), (4.27), (4.30) and (4.31), we conclude that

$$\max_{0\leq i\leq N'} |(\mathcal{G}(v - \widetilde{\mathcal{Q}}_h\, v))(t_i)| \leq c\big[N^{-(m+2)} + N^{-(m+1+\alpha)}\big] \leq c\, N^{-(m+1+\alpha)}.$$

This completes the proof. □

We denote by $\hat{u}_h$ the iterated solution determined by (4.12) associated with the interpolation operator $\widetilde{\mathcal{Q}}_h$. In the next theorem, we present a *local* superconvergence property of the error of $\hat{u}$ at the knots t_j, $j = 0, 1, \ldots, N'$, of the piecewise polynomials. We will show that at these points, the error has superconvergence of an even higher order.

Theorem 4.7. *Suppose that the conditions of Lemma* 4.6 *hold. Let u be the exact solution of equation* (4.6)*. Then, there exist a positive constant c and a positive integer N_0 such that for all $N > N_0$,*

$$\max_{0\leq i\leq N'} |u(t_i) - \hat{u}_h(t_i)| \leq c\, N^{-(m+1+\alpha)}. \tag{4.32}$$

Proof. By our assumptions we have that $u \in U_{m+3}$. By a similar argument in Theorem 4.4, there exist a positive constant c and a positive integer N_0 such that for all $N > N_0$, $(\mathcal{I} - \mathcal{G}\widetilde{\mathcal{Q}}_h)^{-1}$ exists and is uniformly bounded by the constant c in the L^∞ norm. Therefore, it follows from (4.28) that for $i = 0, 1, \ldots, N'$,

$$|u(t_i) - \hat{u}_h(t_i)| = \big|[(\mathcal{I} - \mathcal{G}\widetilde{\mathcal{Q}}_h)^{-1}\mathcal{G}(\mathcal{I} - \widetilde{\mathcal{Q}}_h)u](t_i)\big| \leq c\, N^{-(m+1+\alpha)},$$

proving the estimate (4.32). □

In general, the local estimate presented in Theorem 4.7 cannot be improved to a global estimate. However, in a special case when $a = 0$ in equation (4.1), this is possible. In the next theorem we present a special *global* superconvergence result for the equation when $a = 0$.

Theorem 4.8. *Let u be the exact solution of* (4.6) *with $a = 0$. Then there exist a positive constant c and a positive integer N_0 such that for all $N > N_0$,*

$$\|u - \hat{u}_h\| \le c\, N^{-(m+1+\alpha)}. \tag{4.33}$$

Proof. Recall that when $a = 0$, the operator $\mathcal{G}$ reduces to $\mathcal{R}_\alpha$. Hence, it follows from (4.14) that

$$\|u - \hat{u}\| \le \|(\mathcal{I} - \mathcal{R}_\alpha \mathcal{Q}_h)^{-1}\| \, \|\mathcal{R}_\alpha(\mathcal{I} - \widetilde{\mathcal{Q}}_h)\, u\|.$$

Using this inequality with (4.27) yields the estimate (4.33). □

5. Numerical examples. In this section, we present two numerical examples to confirm the theoretical analysis obtained in the previous sections. The first example is concerned with superconvergence of the iterated collocation solution for Volterra equations of the second kind while the second example is about superconvergence for integro-differential equations. The weakly singular integrals appearing in the numerical solution of these examples are computed by employing the Gauss-type quadrature formula introduced in [**17**].

Example 1. Consider the Volterra integral equation of the second kind with a weakly singular kernel having the form

$$u(t) - \int_0^t (t-s)^{-1/2}\, u(s)\, ds = f(t), \quad t \in I. \tag{5.1}$$

The kernel has a singularity along the diagonal with a parameter $\alpha = 1/2$. We choose f as

$$f(t) := (t^2 + t)^{1/2} \cos t - \int_0^t (t-s)^{-1/2} (s^2 + s)^{1/2} \cos s\, ds, \quad t \in I,$$

so that

$$u(t) = (t^2 + t)^{1/2} \cos t, \quad t \in I,$$

is the exact solution of equation (5.1). Note that the first derivative of the solution has a singularity at $t = 0$. We choose W_2 as defined in (2.2) on the first subinterval and use the space of piecewise linear polynomials on the other subintervals. The collocation parameters on the first subinterval are chosen as

$$\{\tau'_1, \tau'_2, \tau'_3, \tau'_4\} = \left\{\frac{1}{9}, \frac{1}{4}, \frac{4}{9}, \frac{9}{16}\right\}.$$

The collocation parameters

$$\{\tau_1, \tau_2\} = \left\{\frac{2}{5}, \frac{3}{4}\right\}, \quad \{\tau_1, \tau_2\} = \left\{\frac{1}{3}, 1\right\}$$

and

$$\{\tau_1, \tau_2\} = \left\{\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6}\right\}$$

on the other subintervals are employed, respectively, for the collocation and iterated collocation methods. The collocation parameters in the last two cases correspond to Radau II points and Gauss points on I satisfying the condition (3.1) with $r = 1$ and $r = 0$, respectively. The theoretical orders of convergence for the collocation method and the iterated collocation method are 2 and 2.5, respectively, according to the theory established in the last two sections. Numerical results shown in Table 5.1 are about the errors and orders of convergence for the collocation solution and the iterated collocation solution, where $\hat{u}_{h,1}$ and $\hat{u}_{h,2}$ stand for the iterated collocation solutions corresponding to Radau II points and Gauss points, respectively. It can be seen that the numerical results confirm the theoretical estimates.

TABLE 5.1. Superconvergence with $m = 2$ and $\alpha = 0.5$.

N	16	32	64	128
$\|u - u_h\|$	1.358e-1	3.838e-2	7.429e-3	1.520e-3
Order of conv.	—	1.82	2.37	2.29
$\|u - \hat{u}_{h,1}\|$	1.374e-1	3.417e-2	5.569e-3	8.475e-4
Order of conv.	—	2.01	2.62	2.72
$\|u - \hat{u}_{h,2}\|$	1.088e-1	2.961e-2	5.218e-3	8.572e-4
Order of conv.	—	1.88	2.50	2.61

Example 2. In this example, we consider the Volterra integro-differential equation

$$y'(t) = y(t) + b(t) + \int_0^t (t-s)^{\alpha-1}\, y(s)\, ds, \quad t \in I, \tag{5.2}$$

with initial-value condition $y(0) = 0$, where b is chosen such that

$$y(t) := t^{1+\alpha} \cos t, \quad t \in I$$

is the exact solution of (5.2). By a change of variables $y' = u$, equation (5.2) is transformed as a Volterra integral equation of the second kind with u being the new unknown.

We will present numerical results for the parameter α chosen as $0.1, 0.2, \ldots, 0.9$. Specifically, we present in Table 5.2A–5.10A the computed errors and orders of convergence of the collocation approximations u_h and y_h by using the space W_2 on the first subinterval corresponding to different parameters. For convenient presentation, we only list the spaces W_2, their dimensions l_2 and the corresponding collocation parameters in the first interval for the cases $\alpha = 0.1, 0.5, 0.9$. They are, for $\alpha = 0.1$,

$$W_2 = \mathrm{span}\,\{t^\sigma : \ \sigma = 0, 0.1, 0.2, \ldots, 1.9\}, \quad l_2 = 20$$

and the collocation parameters

$$\{0.1, 0.12, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45,\\ 0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.92, 0.95\};$$

for $\alpha = 0.5$,

$$W_2 = \text{ span}\,\{t^\sigma :\ \sigma = 0, 0.5, 1.0, 1.5\}, \quad l_2 = 4$$

and the collocation parameters

$$\{0.2, 0.4, 0.6, 0.8\};$$

and, for $\alpha = 0.9$,

$$W_2 = \text{ span}\,\{t^\sigma :\ \sigma = 0, 0.9, 1.0, 1.8, 1.9\}, \quad l_2 = 5$$

and collocation parameters

$$\{0.2, 0.4, 0.5, 0.6, 0.8\}.$$

Meanwhile, we use in Tables 5.2A–5.10A the collocation parameters

$$\tau_1 = 0.25, \quad \tau_2 = 0.8$$

on the other subintervals. Note that the theoretical optimal order of convergence is 2. The numerical results presented in these tables confirm the theoretical estimate in Theorem 4.2.

To investigate the global superconvergence property of the iterated collocation solution, we use

$$\tau_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad \tau_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}$$

as the collocation parameters and the corresponding piecewise linear polynomials for the subintervals except the first subinterval. We report the numerical results in Tables 5.2B–5.10B. The theoretical order of convergence for both $\hat{u}_h$ and $\hat{y}_h$ is 3. It can be seen from Tables 5.6B–5.10B that when $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9$, the numerical results demonstrate the superconvergence property of the iterated solution and thus confirm the theoretical estimates in these cases. It can be also observed from the numerical results that, when $\alpha < 0.5$, the improvement in convergence order by an iteration is not obvious. This interesting numerical phenomenon deserves further investigation.

TABLE 5.2A. Maximum errors of the collocation solution with $m=2$ and $\alpha=0.1$.

N	$\|u-u_h\|$	Order of conv.	$\|y-y_h\|$	Order of conv.
128	1.101e-0	—	1.272e-1	—
256	2.946e-1	1.90	3.399e-2	1.90
512	6.482e-2	2.18	7.477e-3	2.18
1024	1.541e-2	2.07	1.747e-3	2.09

TABLE 5.2B. Maximum errors of the iterated solution with $m=2$ and $\alpha=0.1$.

N	$\|u-\hat{u}_h\|$	Order of conv.	$\|y-y_h\|$	Order of conv.
128	1.211e-0	—	1.400e-1	—
256	3.172e-1	1.93	3.660e-2	1.94
512	7.054e-2	2.17	8.138e-3	2.17
1024	1.661e-2	2.09	1.916e-3	2.09

TABLE 5.3A. Maximum errors of the collocation solution with $m=2$ and $\alpha=0.2$.

N	$\|u-u_h\|$	Order of conv.	$\|y-y_h\|$	Order of conv.
32	7.902e-2	—	1.733e-2	—
64	2.124e-2	1.90	4.672e-3	1.89
128	3.914e-3	2.44	8.601e-4	2.44
256	9.062e-4	2.11	1.990e-4	2.11
512	1.783e-4	2.35	3.909e-5	2.35

TABLE 5.3B. Maximum errors of the iterated solution with $m=2$ and $\alpha=0.2$.

N	$\|u-\hat{u}_h\|$	Order of conv.	$\|y-y_h\|$	Order of conv.
32	9.196e-2	—	2.034e-2	—
64	2.423e-2	1.92	5.367e-3	1.92
128	4.688e-3	2.37	1.039e-3	2.37
256	1.101e-3	2.09	2.440e-4	2.09
512	2.277e-4	2.27	5.048e-5	2.27

TABLE 5.4A. Maximum errors of the collocation solution with $m = 2$ and $\alpha = 0.3$.

N	$\|u - u_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	5.454e-2	—	1.561e-2	—
32	1.484e-2	1.68	4.218e-3	1.61
64	2.327e-3	2.73	6.575e-4	2.76
128	3.759e-4	2.78	1.041e-4	2.88
256	7.233e-5	2.54	1.964e-5	2.67
512	1.258e-5	2.52	3.271e-6	2.59

TABLE 5.4B. Maximum errors of the iterated solution with $m = 2$ and $\alpha = 0.3$.

N	$\|u - \hat{u}_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	6.400e-2	—	1.841e-2	—
32	1.705e-2	1.91	4.975e-3	1.89
64	2.901e-3	2.56	8.523e-4	2.55
128	5.213e-4	2.48	1.533e-4	2.48
256	1.088e-4	2.26	3.193e-5	2.26
512	2.101e-5	2.37	6.173e-6	2.37

TABLE 5.5A. Maximum errors of the collocation solution with $m = 2$ and $\alpha = 0.4$.

N	$\|u - u_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	2.480e-2	—	7.689e-3	—
32	2.478e-3	2.30	7.515e-4	3.35
64	4.675e-4	2.41	1.363e-4	2.46
128	8.172e-5	2.52	2.206e-5	3.63
256	1.141e-5	2.84	2.463e-6	3.16
512	1.287e-6	3.15	1.408e-7	4.13

TABLE 5.5B. Maximum errors of the iterated solution with $m = 2$ and $\alpha = 0.4$.

N	$\|u - \hat{u}_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	2.818e-2	—	9.209e-3	—
32	3.398e-3	3.05	1.153e-3	3.00
64	6.941e-4	2.29	2.350e-4	2.29
128	1.381e-4	2.33	4.656e-5	2.34
256	2.555e-5	2.43	8.584e-6	2.44
512	4.772e-6	2.42	1.603e-6	2.42

TABLE 5.6A. Maximum errors of the collocation solution with $m = 2$ and $\alpha = 0.5$.

N	$\|u - u_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	2.402e-3	—	5.633e-4	—
32	5.916e-4	2.02	1.337e-4	2.07
64	5.902e-5	3.33	1.343e-5	3.32
128	9.027e-6	2.71	5.592e-6	1.26
256	4.065e-6	1.15	2.309e-6	1.28
512	1.271e-6	1.68	6.847e-7	1.75

TABLE 5.6B. Maximum errors of the iterated solution with $m = 2$ and $\alpha = 0.5$.

N	$\|u - \hat{u}_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	4.334e-3	—	1.582e-3	—
32	1.054e-3	2.04	3.773e-4	2.07
64	1.685e-4	2.65	6.052e-5	2.64
128	3.150e-5	2.42	1.115e-5	2.44
256	4.799e-6	2.71	1.691e-6	2.72
512	8.414e-7	2.51	2.948e-7	2.52

TABLE 5.7A. Maximum errors of the collocation solution with $m = 2$ and $\alpha = 0.6$.

N	$\|u - u_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	2.519e-3	—	7.281e-4	—
32	5.630e-4	2.16	1.519e-4	2.26
64	5.285e-5	3.41	6.675e-5	1.19
128	5.968e-6	3.15	5.086e-6	3.71
256	2.469e-6	1.27	1.707e-6	1.58

TABLE 5.7B. Maximum errors of the iterated solution with $m = 2$ and $\alpha = 0.6$.

N	$\|u - \hat{u}_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	3.739e-3	—	1.463e-3	—
32	8.570e-4	2.12	3.289e-4	2.17
64	1.264e-4	2.75	4.867e-5	2.75
128	1.805e-5	2.80	6.937e-6	2.81
256	3.047e-6	2.57	1.158e-6	2.60

TABLE 5.8A. Maximum errors and orders of convergence with $m = 2$ and $\alpha = 0.7$.

N	$\|u - u_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	1.912e-3	—	4.590e-4	—
32	1.150e-4	4.06	6.930e-5	2.73
64	3.264e-5	1.82	2.448e-5	1.50
128	9.449e-6	1.79	6.652e-6	1.88
256	2.720e-6	1.80	1.839e-6	1.85

TABLE 5.8B. Maximum errors of the iterated solution with $m = 2$ and $\alpha = 0.7$.

N	$\|u - \hat{u}_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	2.682e-3	—	9.876e-4	—
32	2.883e-4	3.22	1.140e-4	3.11
64	3.683e-5	2.97	1.481e-5	2.94
128	6.163e-6	2.58	2.403e-6	2.62
256	8.667e-7	2.83	3.347e-7	2.84

TABLE 5.9A. Maximum errors of the collocation solution with $m = 2$ and $\alpha = 0.8$.

N	$\|u - u_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	9.054e-4	—	1.664e-4	—
32	9.918e-5	3.19	8.433e-5	0.98
64	2.559e-5	1.95	2.051e-5	2.04
128	7.863e-6	1.70	5.864e-6	1.81
256	2.149e-6	1.87	1.558e-6	1.91

TABLE 5.9B. Maximum errors of the iterated solution with $m = 2$ and $\alpha = 0.8$.

N	$\|u - \hat{u}_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	1.494e-3	—	5.788e-4	—
32	1.361e-4	3.46	5.897e-5	3.30
64	2.603e-5	2.39	1.056e-5	2.48
128	3.222e-6	3.01	1.321e-6	3.00
256	4.091e-7	2.98	1.680e-7	2.98

TABLE 5.10A. Maximum errors of the collocation solution with $m = 2$ and $\alpha = 0.9$.

N	$\|u - u_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	3.758e-4	—	2.511e-4	—
32	8.348e-5	2.17	7.347e-5	1.77
64	2.448e-5	1.77	1.984e-5	1.89
128	6.471e-6	1.92	5.080e-6	1.97
256	1.691e-6	1.94	1.303e-6	1.96

TABLE 5.10B. Maximum errors of the iterated solution with $m = 2$ and $\alpha = 0.9$.

N	$\|u - \hat{u}_h\|$	Order of conv.	$\|y - y_h\|$	Order of conv.
16	8.023e-4	—	3.374e-4	—
32	9.643e-5	3.06	4.277e-5	2.98
64	1.205e-5	3.00	5.446e-6	2.97
128	1.760e-6	2.78	7.721e-7	2.82
256	2.355e-7	2.90	1.022e-7	2.92

REFERENCES

1. K.E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge Univ. Press, Cambridge, UK, 1997.

2. K.E. Atkinson, I. Graham and I. Sloan, *Piecewise continuous collocation for integral equations*, SIAM J. Numer. Anal. **20** (1983), 172–186.

3. J.G. Blom and H. Brunner, *The numerical solution of nonlinear Volterra integral equations of the second kind by collocation and iterated collocation methods*, SIAM J. Sci. Statist. Comput. **8** (1987), 806–830.

4. H. Brunner, *Nonpolynomial spline collocation for Volterra equations with weakly singular kernels*, SIAM J. Numer. Anal. **20** (1983), 1106–1119.

5. ———, *Iterated collocation methods and their discretizations for Volterra integral equations*, SIAM J. Numer. Anal. **21** (1984), 1132–1145.

6. ———, *The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes*, Math. Comp. **45** (1985), 417–437.

7. ———, *Polynomial spline collocation for Volterra integro-differential equations with weakly singular kernels*, IMA J. Numer. Anal. **6** (1986), 221–239.

8. ———, *Iterated collocation methods for Volterra integral equations with delay arguments*, Math. Comp. **62** (1994), 581–599.

9. H. Brunner, A. Pedas and G. Vainikko, *The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations*, Math. Comp. **68** (1999), 1079–1095.

10. ———, *Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels*, SIAM J. Numer. Anal. **39** (2001), 957–982.

11. H. Brunner and P.J. van der Houwen, *The numerical solution of Volterra equations*, North-Holland, Amsterdam, 1986.

12. Y. Cao, T. Herdman and Y. Xu, *A hybrid collocation method for Volterra integral equations with weakly singular kernels*, SIAM J. Numer. Anal. **41** (2003), 364–381.

13. T. Diogo, S. McKee and T. Tang, *Collocation methods for second-kind Volterra integral equations with weakly singular kernels*, Proc. Royal Soc. Edinburgh Sect. A **124** (1994), 199–210.

14. Q.-Y. Hu, *Stieltjes derivatives and β-polynomial spline collocation for Volterra integro-differential equations with singularities*, SIAM J. Numer. Anal. **33** (1996), 208–220.

15. ———, *Superconvergence of numerical solutions to Volterra integral equations with singularities*, SIAM J. Numer. Anal. **34** (1997), 1698–1707.

16. ———, *Geometric meshes and their application to Volterra integro-differential equations with singularities*, IMA J. Numer. Anal. **18** (1998), 151–164.

17. H. Kaneko and Y. Xu, *Gaussian-type quadratures for weakly singular integrals and their applications to the Fredholm integral equation of the second kind*, Math. Comp. **62** (1994), 739–753.

18. A. Pedas and G. Vainikko, *Numerical solution of weakly singular Volterra integral equations with change of variables*, Proc. Estonian Acad. Sci. Phys. Math. **53** (2004), 96–106.

19. ———, *Smoothing transformation and piecewise polynomial collocation for weakly singular Volterra integral equations*, Computing **73** (2004), 271–293.

20. H.J. Riele, *Collocation methods for weakly singular second-kind Volterra integral equations with non-smooth solution*, IMA J. Numer. Anal. **2** (1982), 437–449.

21. T. Tang, *Superconvergence of numerical solutions to weakly singular Volterra integral-differential equations*, Numer. Math. **61** (1992), 373–382.

22. ———, *A note on collocation methods for Volterra integro-differential equations with weakly singular kernels*, IMA J. Numer. Anal. **13** (1993), 93–99.

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