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ON THE CHARACTERIZATION OF SELF-REGULARIZATION PROPERTIES OF A FULLY DISCRETE PROJECTION METHOD FOR SYMM'S INTEGRAL EQUATION

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Dedicated to Professor Erhard Meister
on the occasion of his retirement (Emeritierung)

ABSTRACT. The influence of small perturbations in the kernel and the righthand side of Symm's boundary integral equation, considered in an ill-posed setting, is analyzed. We propose a modification of a fully discrete projection method which is more economical in the sense of complexity and allows one to obtain the optimal order of accuracy in the power scale with respect to the level of the noise in the kernel or in the parametric representation of the boundary.

1. Introduction. In [**2**] the influence of small perturbations in the C^∞-smooth parametric representation of the boundary and the righthand side of Symm's boundary integral equation, discretized by collocation or quadrature methods, was analyzed recently. Our aim here is to extend the analysis of [**2**] by taking into account the infinite smoothness of the boundary curve and also to improve the order of accuracy of the approximate solution with respect to the level of the noise in the boundary parametrization. To do this we propose a slight modification of a fully discrete projection method. Our method uses the values of the kernel and free term of Symm's equation at equally-spaced points, and a trial space consisting of trigonometric polynomials, just as in [**1**], [**7**], [**2**].

Consider the numerical solution of Symm's integral equation

$$\int_\Gamma \log|x-y| v(y)\, ds_y = g(x), \quad x \in \Gamma, \tag{1.1}$$

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We regret to inform you of the second author's untimely death before the publication of this article.

with Γ being the boundary of a simply-connected planar domain Ω. This equation arises from solving the Dirichlet problem for Laplace's equation on Ω. As in [**1**], [**2**], [**4**], [**7**], we assume that Γ has a C^∞-smooth 1-periodic parametrization $\gamma : [0,1] \to \Gamma$ with $|\gamma'(t)| \neq 0$ for $t \in [0,1]$. Following the development in [**4**] or [**10**], rewrite (1.1) as

$$Au := A_0 u + Bu = f \tag{1.2}$$

with $u(t) = v(\gamma(t))|\gamma'(t)|$, $f(t) = g(\gamma(t))$,

(1.3)
$$(A_0 u)(t) = \int_0^1 \log|\sin\pi(t-s)|u(s)\,ds,$$

$$(Bu)(t) = \int_0^1 b(t,s)u(s)\,ds, \quad b(t,s) = \begin{cases} \log\dfrac{|\gamma(t)-\gamma(s)|}{|\sin\pi(t-s)|} & t \neq s \\ \log(|\gamma'(t)|/\pi) & t = s. \end{cases}$$

The operator A_0 arises from studying equation (1.1) on a circle. The eigenfunctions of A_0 are the trigonometric functions. Namely,

$$A_0 e^{2\pi ikt} = \begin{cases} -(2|k|)^{-1}e^{2\pi ikt} & k = \pm 1, \pm 2, \dots, \\ -\log 2 & k = 0. \end{cases} \tag{1.5}$$

The kernel $b(t,s)$ of the operator B is C^∞-smooth and 1-biperiodic. Now we would like to describe the smoothness properties of $b(t,s)$ more precisely. To do this we will use the scale of Gevrey classes of infinitely differentiable 1-periodic functions [**3**, p. 112]. Assume that the boundary parametrization $\gamma(t)$ is such that the kernel (1.4) belongs to the *Gevrey class* G_β of order β, $\beta \geq 1$, or Roumieu type in both variables or, more precisely (see Theorem 6.5 [**3**, pp. 112, 113]), there exists a constant $\mu > 0$ such that

$$[b]^2_{\beta,\mu} := \sum_{k,l=-\infty}^{\infty} |\hat{b}(k,l)|^2 \exp[2\mu(|k|^{1/\beta} + |l|^{1/\beta})] < \infty, \tag{1.6}$$

where

$$\hat{b}(k,l) = \int_0^1 \int_0^1 e^{-2\pi i(kt+ls)} b(t,s)\,dt\,ds$$

are the Fourier coefficients of $b(t,s)$. Note that for $\beta = 1$ from (1.6) it follows that the function $b(t,s)$ has in both variables analytic continuations into the strip $\{z : z = t + is, |s| < (\mu/(2\pi))\}$ of the complex plane.

In what follows we consider (1.2) in the Sobolev spaces H^λ, $\lambda \in (-\infty,\infty)$, of 1-periodic functions (distributions) $u(t)$ with the finite norm

$$\|u\|_\lambda = \left(\sum_{k=-\infty}^{\infty} [\max(1,|k|)]^{2\lambda} |\hat{u}(k)|^2\right)^{1/2},$$

where $\hat{u}(k)$ are the Fourier coefficients of $u(t)$, $H^0 = L_2(0,1)$. Due to (1.5), $A_0 : H^\lambda \to H^{\lambda+1}$ is an isomorphism for all $\lambda \in (-\infty,\infty)$. Since $B : H^\lambda \to H^{\lambda+1}$ is compact, the operator $A = A_0 + B : H^\lambda \to H^{\lambda+1}$ is also an isomorphism for all λ (we assume that $\operatorname{cap}\Gamma \neq 1$).

Introduce the n-dimensional space of trigonometric polynomials

$$\mathcal{T}_n = \left\{u_n : u_n = \sum_{k\in Z_n} c_k e^{2\pi ikt}\right\},$$
$$Z_n = \left\{k : -\frac{n}{2} < k \le \frac{n}{2},\ k = 0, \pm 1, \pm 2, \dots\right\}.$$

It is well known, see [**6**], that for any n and $v_n \in \mathcal{T}_n$

$$\|v_n\|_\lambda \le c_\lambda \|Av_n\|_{\lambda+1}. \tag{1.7}$$

Here and throughout the paper c_λ etc. denote generic constants. Moreover, in the sequel we shall often use the same symbol c for possibly different constants.

Let P_n and Q_n denote the corresponding orthogonal and interpolation projections, respectively,

$$P_n u = \sum_{k\in Z_n} \hat{u}(k) e^{2\pi ikt} \in \mathcal{T}_n,$$
$$Q_n u \in \mathcal{T}_n, \qquad (Q_n u)(jn^{-1}) = u(jn^{-1}), \quad j = 1,2,\dots,n.$$

It is known that (see [**6**], [**8**])

$$\|u - P_n u\|_\lambda \le \left(\frac{n}{2}\right)^{\lambda-\nu} \|u\|_\nu, \quad \lambda \le \nu, \quad u \in H^\nu, \tag{1.8}$$

$$\begin{gathered}\|u-Q_nu\|_\lambda \le c_{\lambda,\nu} n^{\lambda-\nu}\|u\|_\nu, \\ 0\le\lambda\le\nu,\quad u\in H^\nu,\quad \nu>\frac{1}{2}.\end{gathered} \tag{1.9}$$

Moreover, in our analysis we will refer to the following simple estimate

$$\|u-Q_nu\|_0 \le cn^{-1}\|u'\|_0,\quad u\in H^1. \tag{1.10}$$

We also need the Bernstein inverse estimates of the trigonometric polynomials

$$\|v_n\|_\nu \le 2^{\lambda-\nu}n^{\nu-\lambda}\|v_n\|_\lambda,\quad \lambda\le\nu,\quad v_n\in\mathcal{T}_n. \tag{1.11}$$

The most widespread method for approximate solution of Symm's equation (1.2) is the discrete collocation-Galerkin method consisting of an approximation of the equation (1.2) by the equation

$$\tilde{A}_nu_n := A_0u_n + Q_n\tilde{B}_nu_n = Q_nf,\quad u_n\in\mathcal{T}_n, \tag{1.12}$$

where

$$(\tilde{B}_nu)(t) = n^{-1}\sum_{j=1}^{n} b(t,jn^{-1})u(jn^{-1}).$$

This method was analyzed in [**1**], [**7**], [**2**]. It is clear that to obtain the approximate solution u_n from (1.12) it is necessary to have the following collection of values of $b(t,s)$ and $f(t)$ as information regarding equation (1.2)

$$b(in^{-1},jn^{-1}), f(in^{-1}),\quad i,j=1,2,\dots,n. \tag{1.13}$$

Information of such a type is called *collocation* information.

It is well known that Symm's integral equation (1.2), considered as equation in $H^0 = L_2(0,1)$, is ill-posed. Small perturbations of the data may cause dramatic changes in the solution of (1.2). These perturbations may be caused, e.g., by rounding errors preparing the problem to a discretization, measurement errors, and modelling errors. As a result, instead of $f(in^{-1})$, and $\gamma(jm^{-1})$ we have at our disposal some $f_\delta(in^{-1})$ and $\gamma_\varepsilon(jm^{-1})$ where the parameters $\delta>0$, $\varepsilon>0$, characterize the level of the noises in the data. As in [**2**], we accept the following model of perturbations of $f(t)$ and $\gamma(t)$:

$$\left(n^{-1}\sum_{j=1}^{n}|f_\delta(jn^{-1})-f(jn^{-1})|^2\right)^{1/2}\le \delta\|f\|_{\nu+1}, \tag{1.14}$$

$$\begin{gathered}|\gamma_\varepsilon(im^{-1})-\gamma(im^{-1})|\le\varepsilon,\qquad |\gamma'_\varepsilon(im^{-1})-\gamma'(im^{-1})|\le m\varepsilon,\\ i=1,2,\dots,m.\end{gathered} \tag{1.15}$$

Here we assume that $f\in H^{\nu+1}$. Let

$$b_\varepsilon(t,s)=\begin{cases}\log\dfrac{|\gamma_\varepsilon(t)-\gamma_\varepsilon(s)|}{|\sin\pi(t-s)|} & t\ne s,\\ \log(|\gamma'_\varepsilon(t)|/\pi) & t=s.\end{cases}$$

As has been shown in [**2**], from (1.15) it follows that

$$\begin{aligned}&|b_\varepsilon(km^{-1},lm^{-1})-b(km^{-1},lm^{-1})|\\ &\qquad\le\begin{cases}\dfrac{c\varepsilon}{|\sin(\pi(k-l)/m)|} & 1\le k,\, l\le m,\, k\ne l,\\ cm\varepsilon & k=l,\, 1\le l\le m.\end{cases}\end{aligned} \tag{1.16}$$

Let $u_{n,\varepsilon,\delta}$ be the solution of the perturbed problem $\tilde{A}_{n,\varepsilon}u=Q_nf_\delta$, where $\tilde{A}_{n,\varepsilon}$ corresponds to the perturbed data (cf. (1.4), (1.12) and (1.14)):

$$\begin{aligned}\tilde{A}_{n,\varepsilon}&=A_0+Q_n\tilde{B}_{n,\varepsilon},\\ (\tilde{B}_{n,\varepsilon}u)(t)&=n^{-1}\sum_{j=1}^{n}b_\varepsilon(t,jn^{-1})u(jn^{-1}).\end{aligned}$$

One of the main results of [**2**] yields the following theorem.

Theorem 1.1 ([2]). *Assume* $\operatorname{cap}\Gamma\ne 1$, $f\in H^{\nu+1}$ *and* $b(t,s)$ *satisfies the condition* (1.6) *for some* $\beta\ge 1$, $\mu\ge 0$. *Then for*

$$n\sim(\varepsilon+\delta)^{-1/(\nu+1)}, \tag{1.17}$$

$$\|u-u_{n,\varepsilon,\delta}\|_0\le c\left\{\delta^{\nu/(\nu+1)}+\varepsilon^{\nu/(\nu+1)}\log\frac{1}{\varepsilon+\delta}\right\}\|u\|_\nu, \tag{1.18}$$

where $u = A^{-1}f \in H^\nu$, $u_{n,\varepsilon,\delta} = \tilde{A}_{n,\varepsilon}^{-1} Q_n f_\delta$.

Note that in case of ε or δ-perturbations in the data of some well-posed problem we have the possibility to obtain the same order of accuracy of the approximate solution $O(\varepsilon)$ or $O(\delta)$. But in the ill-posed case we usually lose order of accuracy with respect to the level of the noise and obtain the accuracy of order $O(\delta^{\nu/(\nu+1)})$, for example.

The relationships (1.17) and (1.18) give an insight into how the discretization parameter n should be chosen to obtain a regularization effect for Symm's ill-posed problem (1.2); no special regularization of the problem is needed. This phenomenon is sometimes called the *self-regularization* of an ill-posed problem through its discretization. In some abstract settings, the self-regularization of ill-posed problems through projection methods has been analyzed in [**5**], [**9**], [**2**]. On the other hand, from estimate (1.18) one sees that caused by ill-posedness losses of accuracy, with respect to the level of the noise ε in the parametric representation of the boundary and with respect to the level of the noise δ in the righthand term, are more or less the same. As we shall see subsequently, this circumstance is connected only with the structure of the collocation-Galerkin method (1.12), where one discretization parameter n must attend to the noises of both types simultaneously. In the next section we propose another scheme of a fully discrete projection method which allows us to improve the order of accuracy with respect to ε up to $O(\varepsilon \log^q(1/\varepsilon))$.

2. Fully discrete projection method. Approximate the equation (1.2) by the equation

$$A_m u := A_0 u + B_m u = Q_n f, \quad n > m, \tag{2.1}$$

where

$$\begin{aligned} (B_m u)(t) &= \int_0^1 b_m(t,s) u(s)\, ds, \\ b_m(t,s) &= (Q_{m,t} \otimes Q_{m,s} b)(t,s) = \sum_{k,l \in Z_m} \hat{b}_m(k,l) e^{2\pi i (kt+ls)}, \\ \hat{b}_m(k,l) &= m^{-2} \sum_{p,q=1}^{m} e^{(-2\pi i/m)(kp+lq)} b(pm^{-1}, qm^{-1}). \end{aligned} \tag{2.2}$$

By definition $B_m : L_2(0,1) \to \mathcal{T}_m$,

$$b_m(km^{-1}, lm^{-1}) = b(km^{-1}, lm^{-1}), \quad k, l = 1, 2, \dots, m,$$

and for $n > m$

$$P_n B_m = B_m P_n = B_m. \tag{2.3}$$

Moreover, from (1.5) it follows that

$$P_n A_0 = A_0 P_n. \tag{2.4}$$

To obtain a finite linear system from which the solution $u_{n,m}$ of equation (2.1) can be calculated, note first that if (2.1) is solvable, then

$$A_0 u_{n,m} = Q_n f - B_m u_{n,m} \in \mathcal{T}_n.$$

This together with (1.5) implies that $u_{n,m}$ is a trigonometric polynomial of degree n. Thus

$$u_{n,m}(t) = \sum_{k \in Z_n} \hat{u}_{n,m}(k) e^{2\pi i k t},$$

where the unknown coefficients $\hat{u}_{n,m}(k)$ are determined from the following system of linear algebraic equations:

$$\begin{aligned} \lambda_k \hat{u}_{n,m}(k) + \sum_{l \in Z_m} \hat{b}_m(k, -l) \hat{u}_{n,m}(l) &= \hat{f}_n(k), \quad k \in Z_m, \\ \lambda_k \hat{u}_{n,m}(k) &= \hat{f}_n(k), \quad k \in Z_n \backslash Z_m. \end{aligned} \tag{2.5}$$

Here $\lambda_0 = -\log 2$, $\lambda_k = -(2|k|)^{-1}$,

$$\hat{f}_n(k) = n^{-1} \sum_{p=1}^{n} e^{-(2\pi i k p / n)} f(pn^{-1}). \tag{2.6}$$

It is interesting that, to determine an element $u_{n,m}$ belonging to the n-dimensional space of trigonometric polynomials $\mathcal{T}_n$, it suffices to solve the system of $m < n$ linear algebraic equations. Moreover, as in [**4**, eq. (4.3)], by introducing new unknowns, one can improve a condition number of the linear system (2.5).

In our analysis of the method (2.1), we will use some auxiliary approximation of the kernel $b(t,s)$ satisfying the condition (1.6). Let

$$b_{m,\beta}(t,s) = \sum_{k,l\in\Lambda_{m,\beta}} \hat{b}(k,l)e^{2\pi i(kt+ls)},$$

where $\Lambda_{m,\beta} = \{(k,l) : |k|^{1/\beta} + |l|^{1/\beta} < (m/2)^{1/\beta},\ k,l = 0, \pm 1, \pm 2, \dots\}$. Now we define the discretized operator $B_{m,\beta}$ by

$$(B_{m,\beta}u)(t) = \int_0^1 b_{m,\beta}(t,s)u(s)\,ds.$$

Lemma 2.1. *Assume that* $b(t,s)$ *satisfies the condition* (1.6). *Then, for* $m > 2(\beta\nu/\mu)^\beta$,

$$\|B - B_{m,\beta}\|_{H^0\to H^\nu} \le cm^\nu e^{-\chi m^{1/\beta}}[b]_{\beta,\mu},$$

where $\chi = \chi(\beta,\mu) = \mu/2^{1/\beta}$.

Proof. Using the Fourier representations, for any $v \in H^0$ we have

$$\begin{aligned}\|(B - B_{m,\beta})v\|_\nu^2 &= \Big|\sum_{|l|\ge(m/2)} \hat{b}(0,-l)\hat{v}(l)\Big|^2 \\ &\quad + \sum_{|k|>0} |k|^{2\nu}\Big|\sum_{l:(k,l)\notin\Lambda_{m,\beta}} \hat{b}(k,-l)\hat{v}(l)\Big|^2.\end{aligned} \tag{2.7}$$

We estimate only the second term in (2.7). The first term can be estimated in a similar manner. We obtain

$$\begin{aligned}\sum_{|k|>0} |k|^{2\nu}\Big|\sum_{l:(k,l)\notin\Lambda_{m,\beta}} \hat{b}(k,-l)\hat{v}(l)\Big|^2 &\le \|v\|_0^2 \sum_{|k|>0}|k|^{2\nu} \sum_{l:(k,l)\notin\Lambda_{m,\beta}} |\hat{b}(k,l)|^2 \\ &= \|v\|_0^2 \sum_{0<|k|\le(m/2)} |k|^{2\nu} \sum_{l:(k,l)\notin\Lambda_{m,\beta}} |\hat{b}(k,l)|^2 \\ &\quad + \|v\|_0^2 \sum_{|k|>(m/2)} |k|^{2\nu} \sum_{l:(k,l)\notin\Lambda_{m,\beta}} |\hat{b}(k,l)|^2 \\ &= S_1 + S_2;\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
S_1 &\le \|v\|_0^2 \left(\frac{m}{2}\right)^{2\nu} \\
&\cdot \sum_{k=-\infty}^{\infty} \sum_{l:(k,l)\notin\Lambda_{m,\beta}} e^{-2\mu(|k|^{1/\beta}+|l|^{1/\beta})} |\hat{b}(k,l)|^2 e^{2\mu(|k|^{1/\beta}+|l|^{1/\beta})} \\
&\le \|v\|_0^2 \left(\frac{m}{2}\right)^{2\nu} e^{-2\mu(m/2)^{1/\beta}} [b]^2_{\beta,\mu}.
\end{aligned} \tag{2.9}
$$

Note that $x = (\beta\nu/\mu)^\beta$ is the point at which the function $x^{2\nu}e^{-2\mu x^{1/\beta}}$ has a global maximum. Then for $|k| > (m/2) > (\beta\nu/\mu)^\beta$,

$$
|k|^{2\nu} e^{-2\mu|k|^{1/\beta}} < \left(\frac{m}{2}\right)^{2\nu} e^{-2\mu(m/2)^{1/\beta}}.
$$

Therefore,

$$
\begin{aligned}
S_2 &= \|v\|_0^2 \sum_{|k|>(m/2)} |k|^{2\nu} e^{-2\mu|k|^{1/\beta}} \sum_{l:(k,l)\notin\Lambda_{m,\beta}} e^{2\mu|k|^{1/\beta}} |\hat{b}(k,l)|^2 \\
&\le \|v\|_0^2 \left(\frac{m}{2}\right)^{2\nu} e^{-2\mu(m/2)^{1/\beta}} [b]^2_{\beta,\mu}.
\end{aligned} \tag{2.10}
$$

The assertion of the lemma follows from (2.7)–(2.10). □

Let

$$
\|\varphi\|^2_{\nu_1,\nu_2} := \sum_{k,l=-\infty}^{\infty} \max(1,|k|^{2\nu_1}) \max(1,|l|^{2\nu_2}) |\hat{\varphi}(k,l)|^2.
$$

Using an argument like that in the proof of Lemma 2.1 we get the following lemma.

Lemma 2.2. *Assume the conditions of Lemma* 2.1 *hold. Then*

$$
\|b-b_{m,\beta}\|_{0,0} \le ce^{-\chi m^{1/\beta}} [b]_{\beta,\mu}, \qquad \|b-b_{m,\beta}\|_{1,0} \le cme^{-\chi m^{1/\beta}} [b]_{\beta,\mu},
$$
$$
\|b-b_{m,\beta}\|_{0,1} \le cme^{-\chi m^{1/\beta}} [b]_{\beta,\mu}, \qquad \|b-b_{m,\beta}\|_{1,1} \le cm^2e^{-\chi m^{1/\beta}} [b]_{\beta,\mu}.
$$

Lemma 2.3. *Assume the conditions of Lemma* 2.1 *hold. Then*

$$\|b - b_m\|_{0,0} \le ce^{-\chi m^{1/\beta}}[b]_{\beta,\mu}.$$

Proof. From (1.10) it follows that, for $\varphi(t,s)$,

$$\|\varphi - Q_{m,t}\varphi\|_{0,0} \le cm^{-1}\left\|\frac{\partial\varphi}{\partial t}\right\|_{0,0} \le cm^{-1}\|\varphi\|_{1,0}.$$

Analogously

$$\begin{aligned}\|\varphi - Q_{m,s}\varphi\|_{0,0} &\le cm^{-1}\|\varphi\|_{0,1},\\ \|(I - Q_{m,t}) \otimes (I - Q_{m,s})\varphi\|_{0,0} &\le cm^{-2}\|\varphi\|_{1,1}.\end{aligned}$$

Then

$$\begin{aligned}&\|\varphi - Q_{m,t}\otimes Q_{m,s}\varphi\|_{0,0}\\ &\qquad \le c(m^{-1}\|\varphi\|_{1,0} + m^{-1}\|\varphi\|_{0,1} + m^{-2}\|\varphi\|_{1,1}).\end{aligned} \tag{2.11}$$

Now we note that, for $(k,l) \in \Lambda_{m,\beta} \subset Z_m \times Z_m$,

$$Q_{m,t} \otimes Q_{m,s}e^{2\pi i(kt+ls)} = Q_m e^{2\pi ikt} Q_m e^{2\pi ils} = e^{2\pi i(kt+ls)}. \tag{2.12}$$

Using (2.11), (2.12) and Lemma 2.2, we obtain the assertion of the lemma:

$$\begin{aligned}\|b-b_m\|_{0,0} &= \|(I-Q_{m,t} \otimes Q_{m,s})(b-b_{m,\beta})\|_{0,0}\\ &\le c(m^{-1}\|b-b_{m,\beta}\|_{1,0}+m^{-1}\|b-b_{m,\beta}\|_{0,1}+m^{-2}\|b-b_{m,\beta}\|_{1,1})\\ &\le ce^{-\chi m^{1/\beta}}[b]_{\beta,\mu}. \qquad \square\end{aligned}$$

Lemma 2.4. *Assume the conditions of Lemma* 2.1 *hold. Then*

$$\|B - B_m\|_{H^0\to H^\nu} \le cm^\nu e^{-\chi m^{1/\beta}}[b]_{\beta,\mu}.$$

Proof. Recalling Lemma 2.1, we have

$$\begin{aligned}\|B-B_m\|_{H^0\to H^\nu} &\le \|B-B_{m,\beta}\|_{H^0\to H^\nu} + \|B_{m,\beta}-B_m\|_{H^0\to H^\nu}\\ &\le cm^\nu e^{-\chi m^{1/\beta}}[b]_{\beta,\mu} + \|B_{m,\beta}-B_m\|_{H^0\to H^\nu}.\end{aligned} \tag{2.13}$$

Keeping in mind that $B_{m,\beta} - B_m : H^0 \to \mathcal{T}_m$ from (1.11) and Lemmas 2.2 and 2.3, we obtain the estimate

$$\begin{aligned}\|B_{m,\beta} - B_m\|_{H^0\to H^\nu} &\le 2^\nu m^\nu \|B_{m,\beta} - B_m\|_{H^0\to H^0}\\ &\le cm^\nu \|b_{m,\beta} - b_m\|_{0,0}\\ &\le cm^\nu (\|b - b_{m,\beta}\|_{0,0} + \|b - b_m\|_{0,0})\\ &\le cm^\nu e^{-\chi m^{1/\beta}} [b]_{\beta,\mu}.\end{aligned}$$

Summing up we get the assertion of the lemma. □

Now we are able to carry out the convergence analysis of our fully discrete projection method (2.1).

Theorem 2.1. *Let the assumptions of Theorem* 1.1 *be fulfilled. Then there is some* m_0 *such that, for* $m > m_0$,

$$\|u - u_{n,m}\|_0 \le c(n^{-\nu} + me^{-\chi m^{1/\beta}})[b]_{\beta,\mu}\|u\|_\nu. \tag{2.14}$$

Proof. First we show that, for any $v \in \mathcal{T}_n$ and $n > m > m_0$, the stability condition

$$\|v\|_0 \le \tilde{c}_0 \|A_m v\|_1 \tag{2.15}$$

holds with some constant $\tilde{c}_0$ which does not depend on n and m. Indeed, from (1.7) and Lemma 2.4 we have

$$\begin{aligned}\|v\|_0 &\le c_0\|Av\|_1\\ &\le c_0\|A_m v\|_1 + c_0\|(A - A_m)v\|_1\\ &= c_0\|A_m v\|_1 + c_0\|(B - B_m)v\|_1\\ &\le c_0\|A_m v\|_1 + cc_0 me^{-\chi m^{1/\beta}}[b]_{\beta,\mu}\|v\|_0.\end{aligned}$$

Consequently, for sufficiently large m,

$$\|v\|_0 \le \frac{c_0}{1 - cc_0 me^{-\chi m^{1/\beta}}[b]_{\beta,\mu}}\|A_m v\|_1 = \tilde{c}_0\|A_m v\|_1.$$

Now we pass to the estimation of the norm $\|u-u_{n,m}\|_0$. By (1.8) we have

$$\begin{aligned}\|u-u_{n,m}\|_0 &\le \|u-P_nu\|_0+\|P_nu-u_{n,m}\|_0\\ &\le \left(\frac{n}{2}\right)^{-\nu}\|u\|_\nu+\|P_nu-u_{n,m}\|_0.\end{aligned}\tag{2.16}$$

Since $P_nu-u_{n,m}\in\mathcal{T}_n$, from (2.15) we obtain

$$\begin{aligned}\|P_nu-u_{n,m}\|_0 &\le \tilde{c}_0\|A_m(P_nu-u_{n,m})\|_1\\ &= \tilde{c}_0\|A_mP_nu-Q_nf\|_1\\ &= \tilde{c}_0\|A_mP_nu-Q_nAu\|_1\\ &\le \tilde{c}_0\|P_nAu-Q_nAu\|_1+\tilde{c}_0\|P_nAu-A_mP_nu\|_1\\ &= \tilde{c}_0(T_1+T_2).\end{aligned}\tag{2.17}$$

Using (1.8) and (1.9) we find

$$\begin{aligned}T_1:=\|P_nAu-Q_nAu\|_1 &\le \|(I-P_n)Au\|_1+\|(I-Q_n)Au\|_1\\ &\le cn^{-\nu}\|Au\|_{\nu+1}\le cn^{-\nu}\|u\|_\nu.\end{aligned}\tag{2.18}$$

From Lemma 2.4 and (2.3), (2.4), it follows that, for $n>m$,

$$\begin{aligned}T_2:=\|P_nAu-A_mP_nu\|_1 &= \|P_n(A-A_m)u\|_1\\ &= \|P_n(B-B_m)u\|_1\\ &\le \|(B-B_m)u\|_1+\|(I-P_n)(B-B_m)u\|_1\\ &\le cme^{-\chi m^{1/\beta}}[b]_{\beta,\mu}\|u\|_0+cn^{-1}\|(B-B_m)u\|_2\\ &\le c(m+n^{-1}m^2)e^{-\chi m^{1/\beta}}[b]_{\beta,\mu}\|u\|_0\\ &\le cme^{-\chi m^{1/\beta}}[b]_{\beta,\mu}\|u\|_\nu.\end{aligned}\tag{2.19}$$

Now by virtue of (2.16)–(2.19), we get the assertion of the theorem. □

Remark. Using an argument like that in the proof of Theorem 2.1 we get the estimate

$$\|u-u_{n,m}\|_\lambda\le c(n^{-\nu+\lambda}+m^{\lambda+1}e^{-\chi m^{1/\beta}})[b]_{\beta,\mu}\|u\|_\nu,\quad 0\le\lambda<\nu.\qquad\square$$

Let us compare our result (2.14) with the convergence of the discrete collocation-Galerkin method (1.12). From [**6**], [**7**] it follows that, under the conditions of Theorem 1.1,

$$\|u - u_n\|_0 \le cn^{-\nu}\|u\|_\nu,$$

where u_n is the solution of (1.12). Keeping in mind the structure of (1.12) it is easy to see that to obtain the approximate solution of (1.2) with accuracy $O(n^{-\nu})$ one must solve a system of $O(n)$ linear algebraic equations and have a collection of $O(n^2)$ values (1.13). On the other hand, from (2.5) and Theorem 2.1, it follows that to guarantee an accuracy of order $O(n^{-\nu})$ within the framework of method (2.1), it suffices to take $m = ((\nu+1)/\chi)^\beta \log^\beta n$ to solve a system of $O(\log^\beta n)$ equations and to use $m^2 = O(\log^{2\beta} n)$ values of the kernel $b(t,s)$ and n values of the righthand side $f(t)$.

3. Characterization of self-regularization properties. In the above analysis we have assumed that $\gamma(t), b(t)$ and $f(t)$ have been determined exactly. Now we will discuss the influence of noises in the data. Assume that instead of γ, b, f we have at our disposal noisy data $\gamma_\varepsilon, b_\varepsilon, f_\delta$ satisfying (1.14)–(1.16).

Lemma 3.1. *Under the condition* (1.15)

$$\|B_m - B_{m,\varepsilon}\|_{H^0\to H^1} \le cm^{3/2}\varepsilon,$$

where

$$\begin{aligned}(B_{m,\varepsilon}u)(t) &= \int_0^1 b_{m,\varepsilon}(t,s)u(s)\,ds,\\ b_{m,\varepsilon}(t,s) &= (Q_{m,t}\otimes Q_{m,s}b_\varepsilon)(t,s).\end{aligned}$$

Proof. Since $B_m - B_{m,\varepsilon} : H^0 \to \mathcal{T}_m$, from (1.11) it follows that

$$\begin{aligned}\|B_m - B_{m,\varepsilon}\|_{H^0\to H^1} &\le cm\|B_m - B_{m,\varepsilon}\|_{H^0\to H^0}\\ &\le cm\|b_m - b_{m,\varepsilon}\|_{0,0}.\end{aligned} \tag{3.1}$$

Keeping in mind that in both variables the function $b_m(t,s)-b_{m,\varepsilon}(t,s)$ is a trigonometric polynomial from $\mathcal{T}_n$, we have

$$\begin{aligned}\|b_m - b_{m,\varepsilon}\|_{0,0}^2 &= \frac{1}{m^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\left|b_m\left(\frac{k}{m},\frac{l}{m}\right) - b_{m,\varepsilon}\left(\frac{k}{m},\frac{l}{m}\right)\right|^2 \\ &= I_{m,\varepsilon}.\end{aligned} \tag{3.2}$$

Due to (1.16) we can continue:

$$\begin{aligned} I_{m,\varepsilon} &= \frac{1}{m^2}\sum_{p=0}^{m-1}\sum_{\substack{1\le k,l\le m\\ |k-l|=p}}\left|b_m\left(\frac{k}{m},\frac{l}{m}\right) - b_{m,\varepsilon}\left(\frac{k}{m},\frac{l}{m}\right)\right|^2 \\ &\le \frac{c}{m^2}\sum_{\substack{1\le k,l\le m\\ k=l}} m^2\varepsilon^2 + \frac{c}{m^2}\sum_{p=1}^{m-1}\sum_{\substack{1\le k,l\le m\\ |k-l|=p}}\frac{\varepsilon^2}{\sin^2(\pi|k-l|/m)} \\ &= c\varepsilon^2 m + \frac{c}{m^2}\sum_{1\le p\le (m/2)}\sum_{\substack{1\le k,l\le m\\ |k-l|=p}}\frac{\varepsilon^2}{\sin^2(\pi|k-l|/m)} \\ &\quad + \frac{c}{m^2}\sum_{(m/2)<p<m}\sum_{\substack{1\le k,l\le m\\ |k-l|=p}}\frac{\varepsilon^2}{\sin^2(\pi-(\pi|k-l|/m))} \\ &= c(m\varepsilon^2 + I_{1,\varepsilon} + I_{2,\varepsilon}). \end{aligned} \tag{3.3}$$

Since $\sin x \ge (2x/\pi)$, $x\in[0,(\pi/2)]$, we obtain

$$\begin{aligned} I_{1,\varepsilon} &= \frac{1}{m^2}\sum_{1\le p\le (m/2)}\sum_{\substack{1\le k,l\le m\\ |k-l|=p}}\frac{\varepsilon^2}{\sin^2(\pi|k-l|/m)} \\ &\le \frac{c}{m^2}\sum_{1\le p\le (m/2)}\frac{(m-p)\varepsilon^2 m^2}{\pi p^2} \\ &\le c\varepsilon^2 m\sum_{1\le p\le (m/2)}\frac{1}{p^2} + c\varepsilon^2\sum_{1\le p\le (m/2)}\frac{1}{p} \\ &\le c(\varepsilon^2 m + \varepsilon^2\log m). \end{aligned} \tag{3.4}$$

Analogously, $I_{2,\varepsilon} \le c\varepsilon^2 \log m$ and the assertion of the lemma follows from (3.1)–(3.4). $\square$

Corollary 3.1. *Let the assumptions of Theorem* 1.1 *and Lemma* 3.1 *be fulfilled. Then for* $A_{m,\varepsilon} = A_0 + B_{m,\varepsilon}$ *and* $m \geq m_0$ *satisfying*

$$cm^{3/2}\varepsilon < q/\tilde{c}_0, \quad q \in (0,1),$$

the stability inequality

$$\|v\|_0 \leq c_0' \|A_{m,\varepsilon} v\|_1$$

holds for all $v \in \mathcal{T}_n$, $n \geq m$.

Proof. It follows from (2.15) and Lemma 3.1 that, for any $v \in \mathcal{T}_n$, $n \geq m$,

$$\begin{aligned}\|v\|_0 &\leq \tilde{c}_0 \|A_m v\|_1 \\ &\leq \tilde{c}_0 \|A_{m,\varepsilon} v\|_1 + \tilde{c}_0 \|(A_m - A_{m,\varepsilon})v\|_1 \\ &= \tilde{c}_0 \|A_{m,\varepsilon} v\|_1 + \tilde{c}_0 \|(B_m - B_{m,\varepsilon})v\|_1 \\ &\leq \tilde{c}_0 \|A_{m,\varepsilon} v\|_1 + \tilde{c}_0 c m^{3/2} \varepsilon \|v\|_0,\end{aligned}$$

which results in

$$\|v\|_0 \leq \frac{\tilde{c}_0}{1 - \tilde{c}_0 c m^{3/2}\varepsilon} \|A_{m,\varepsilon} v\|_1 = c_0' \|A_{m,\varepsilon} v\|_1$$

as claimed. ◻

Lemma 3.2. *Assume the conditions of Theorem* 1.1 *and* (1.14) *hold. Then*

$$\|u_{n,m} - u_{n,m,\delta}\|_0 \leq cn\delta \|u\|_\nu,$$

where $u_{n,m} = A_{n,m}^{-1} Q_n f$, $u_{n,m,\delta} = A_{n,m}^{-1} Q_n f_\delta$.

Proof. From Lemma 2.1 [**2**] it follows that under the condition (1.14)

$$\|Q_n f - Q_n f_\delta\|_0 \leq \delta \|f\|_{\nu+1}.$$

Moreover, it is easy to see that $u_{n,m} - u_{n,m,\delta} \in \mathcal{T}_n$. Then from (1.11) and (2.15) we have

$$\begin{aligned}\|u_{n,m} - u_{n,m,\delta}\|_0 &\leq \tilde{c}_0 \|A_{n,m}(u_{n,m} - u_{n,m,\delta})\|_1 \\ &= \tilde{c}_0 \|Q_n f - Q_n f_\delta\|_1 \\ &\leq 2n\tilde{c}_0 \|Q_n f - Q_n f_\delta\|_0 \\ &\leq cn\delta \|f\|_{\nu+1} \leq cn\delta \|u\|_\nu. \qquad ◻\end{aligned}$$

Within the framework of the fully discrete projection method (2.1) for solving Symm's integral equation (1.2), from the noisy data $\gamma_\varepsilon, b_\varepsilon, f_\delta$ one takes the solution $u_{n,m,\varepsilon,\delta}$ of the equation

$$A_{m,\varepsilon}u := A_0 u + B_{m,\varepsilon}u = Q_n f_\delta \tag{3.5}$$

as an approximate solution for (1.2).

Theorem 3.1. *Assume that the conditions of Theorem* 1.1 *and* (1.14), (1.15) *hold. Then, for*

$$n \sim \delta^{-1/(\nu+1)}, \quad m = \chi^{-\beta}\ln^\beta \frac{1}{\varepsilon} = \frac{2}{\mu^\beta}\log^\beta \frac{1}{\varepsilon} \sim \log^\beta \frac{1}{\varepsilon} \tag{3.6}$$

equation (3.5) *with perturbed data is uniquely solvable and*

$$\|u - u_{n,m,\varepsilon,\delta}\|_0 \le c\left(\delta^{\nu/(\nu+1)} + \varepsilon \log^{(3/2)\beta}\frac{1}{\varepsilon}\right)\|u\|_\nu. \tag{3.7}$$

Proof. It follows from Theorem 2.1 and Lemma 3.2 that for sufficiently large n, m,

$$\begin{aligned} \|u - u_{n,m,\varepsilon,\delta}\|_0 &\le \|u - u_{n,m}\|_0 + \|u_{n,m} - u_{n,m,\delta}\|_0 \\ &\quad + \|u_{n,m,\delta} - u_{n,m,\varepsilon,\delta}\|_0 \\ &\le c(n^{-\nu} + me^{-\chi m^{1/\beta}} + n\delta)\|u\|_\nu \\ &\quad + \|u_{n,m,\delta} - u_{n,m,\varepsilon,\delta}\|_0. \end{aligned} \tag{3.8}$$

Further, using Lemma 3.1 and Corollary 3.1 we find

$$\begin{aligned} \|u_{n,m,\delta} - u_{n,m,\varepsilon,\delta}\|_0 &\le c_0'\|A_{m,\varepsilon}(u_{n,m,\delta} - u_{n,m,\varepsilon,\delta})\|_1 \\ &= c_0'\|A_{m,\varepsilon}u_{n,m,\delta} - Q_n f_\delta\|_1 \\ &= c_0'\|A_{m,\varepsilon}u_{n,m,\delta} - A_m u_{n,m,\delta}\|_1 \\ &\le c_0'\|A_{m,\varepsilon} - A_m\|_{H^0\to H^1}\|u_{n,m,\delta}\|_0 \\ &= c_0'\|B_m - B_{m,\varepsilon}\|_{H^0\to H^1}\|u_{n,m,\delta}\|_0 \\ &\le c\varepsilon m^{3/2}\|u_{n,m,\delta}\|_0. \end{aligned} \tag{3.9}$$

Moreover, from Lemma 3.2 and Theorem 2.1 we have

$$\begin{aligned}\|u_{n,m,\delta}\|_0 &\le \|u_{n,m}\|_0 + \|u_{n,m,\delta} - u_{n,m}\|_0 \\ &\le \|u\|_0 + \|u - u_{n,m}\|_0 + cn\delta\|u\|_\nu \\ &\le \|u\|_0 + c(n^{-\nu} + me^{-\chi m^{1/\beta}})\|u\|_\nu + cn\delta\|u\|_\nu \\ &\le c\|u\|_\nu.\end{aligned} \tag{3.10}$$

Combining (3.8)–(3.10) with (3.6), we obtain the error estimate

$$\begin{aligned}\|u - u_{n,m,\varepsilon,\delta}\|_0 &\le c(n^{-\nu} + me^{-\chi m^{1/\beta}} + n\delta + \varepsilon m^{3/2})\|u\|_\nu \\ &\le c\left(\delta^{\nu/(\nu+1)} + \varepsilon \log^{(3/2)\beta}\frac{1}{\varepsilon}\right)\|u\|_\nu\end{aligned}$$

as claimed. $\square$

Estimates (1.17), (1.18) and (3.6), (3.7) characterize the self-regularization of the problem (1.2), considered in an ill-posed setting, through its discretizations

$$A_0 u + Q_n \tilde{B}_{n,\varepsilon} u = Q_n f_\delta \tag{3.11}$$

and (3.5), respectively. It is clear that, having the noises with levels ε and δ in the data of our problem (1.2), we cannot obtain an order of accuracy more than $O(\varepsilon)$ and $O(\delta)$. From Theorem 3.1 it follows that, unlike discretization (3.11), our fully discrete projection method (3.5) allows us to obtain the optimal order of accuracy in the power scale with respect to the level of the noise ε in the parametric representation of the boundary $\gamma(t)$.

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