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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS FOR INTEGRAL BOUNDARY PROBLEMS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN OPERATOR

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ABSTRACT. In this paper, we deal with the following integral boundary problem of nonlinear fractional differential equations with p-Laplacian operator

$$\begin{aligned}&D^{\gamma}_{0+}(\phi_p(D^{\alpha}_{0+}u(t))) + f(t,u(t)) = 0, \quad 0 < t < 1,\\ &u(0) = u'(0) = 0, \quad u'(1) = \int_0^{\eta} u(s)\, ds, \quad D^{\alpha}_{0+}u(t)|_{t=0} = 0,\end{aligned}$$

where $0 < \gamma < 1$, $2 < \alpha < 3$, D^{α}_{0+} is the standard Riemann-Liouville fractional derivative, $\phi_p(s) = |s|^{p-2}s, p > 1$, $(\phi_p)^{-1} = \phi_q$, $1/p + 1/q = 1$. By the properties of the Green function, the lower and upper solution method and fixed-point theorem in partially ordered sets, some new existence and uniqueness of positive solutions to the above boundary value problem are established. As applications, examples are presented to illustrate the main results.

1. Introduction. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex media, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so on, and involves derivatives of fractional order. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is

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the main advantage of fractional differential equations in comparison with classical integer-order models. For an extensive collection of such results, we refer the reader to the monographs by Samko et al. [**28**], Podlubny [**27**] and Kilbas et al. [**15**]. For the basic theory and recent development of the subject, we refer to a text by Lakshmikantham [**16**]. For more details and examples, see [**1, 2, 3, 4, 6, 7, 8, 17, 18, 21**] and the references therein. However, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages, and many aspects of this theory need to be explored; in particular, the existence and uniqueness of positive solutions for integral boundary problems of nonlinear fractional differential equations with p-Laplacian operators are relatively scarce.

In [**29**], Wang, Xiang and Liu considered the existence and multiplicity of positive solutions for the following boundary value problem of nonlinear fractional differential equations with p-Laplacian operators:

$$\begin{aligned}&D^{\gamma}_{0+}(\phi_p(D^{\gamma}_{0+}u(t)))u(t)+f(t,u(t),D^{\rho}_{0+}u(t))=0, \quad 0<t<1,\\&u(0)=u'(1)=u''(0)=0, \qquad D^{\gamma}_{0+}u(t)|_{t=0}=0,\end{aligned}$$

where $0<\gamma<1$, $2<\alpha<3$, $0<\rho\le 1$ and D^{γ}_{0+} denotes the Caputo derivative. By using the fixed point theorem, results for existence and multiplicity of positive solutions to the above boundary value problem are obtained. But the uniqueness is not treated.

Li, Luo and Zhou [**19**] considered the following three point boundary value problems of fractional order differential equations:

$$\begin{aligned}&D^{\alpha}_{0+}u(t)+f(t,u(t))=0, \quad 0<t<1,\ 1<\alpha\le 2,\\&u(0)=0, \qquad D^{\beta}_{0_+}u(1)=aD^{\beta}_{0_+}u(\xi),\end{aligned}$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative. The existence and multiplicity results of positive solutions are obtained by using some fixed-point theorems. But the uniqueness is not treated.

On the other hand, the study of the existence of solutions of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [**14**]. Then Gupta [**10**] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by several authors. We refer the reader to [**9, 11, 20, 22, 23**] and the references therein. However, all these papers are concerned with problems with three-point boundary condition restrictions on the slope of the solutions and the solutions themselves, for example,

$$\begin{aligned}
&u(0) = 0, && \alpha u(\eta) = u(1);\\
&u(0) = \beta u(\eta), && \alpha u(\eta) = u(1);\\
&u'(0) = 0, && \alpha u(\eta) = u(1);\\
&u(0) - \beta u'(0) = 0, && \alpha u(\eta) = u(1);\\
&\alpha u(0) - \beta u'(0) = 0, && u'(\eta) + u'(1) = 0;
\end{aligned}$$

etc. In this paper, we deal with the following three-point boundary value problem

$$D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}u(t))) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u'(0) = 0, \qquad u'(1) = \int_0^{\eta} u(s)ds, \qquad D_{0+}^{\alpha}u(t)|_{t=0} = 0, \tag{1.2}$$

where $0 < \gamma < 1$, $2 < \alpha \leq 3$ and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, $\phi_p(s) = |s|^{p-2}s, p > 1$, $(\phi_p)^{-1} = \phi_q$, $1/p + 1/q = 1$. η satisfies $0 < \eta^{\alpha} < \alpha(\alpha - 1)$. We will give some new existence and uniqueness criteria for boundary value problems (1.1) and (1.2) by using the lower and upper solution method and a fixed point theorem in partially ordered sets. Finally, we present some examples to demonstrate our results. The existence of fixed points in partially ordered sets has been considered recently in **[5, 12, 24, 25, 26]**. This work is motivated by papers **[5, 29]**. We also point out that problems (1.1) and (1.2) are motivated by the constitutive equation of viscoelastic fluid coming from rheology **[13]**.

2. Preliminaries. We need the following definitions and lemmas that will be used to prove our main results.

Definition 2.1. Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone, provided the following are satisfied:

(a) if $y \in P$ and $\lambda \geq 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leq$, that is, $x \leq y$ if and only if $y - x \in P$.

Definition 2.2 (**[27]**). The integral

$$I_{0+}^{s} f(x) = \frac{1}{\Gamma(s)} \int_0^x \frac{f(t)}{(x-t)^{1-s}}\, dt, \quad x > 0,$$

where $s > 0$, is called the Riemann-Liouville fractional integral of order s and $\Gamma(s)$ is the Euler gamma function defined by

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t}\, dt, \quad s > 0.$$

Definition 2.3 (**[15]**). For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{s} f(x) = \frac{1}{\Gamma(n-s)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{s-n+1}}\, dt,$$

where $n = [s] + 1, [s]$ denotes the integer part of the number s, called the Riemann-Liouville fractional derivative of order s.

The following two lemmas, found in **[3, 15]**, are crucial in finding an integral representation of fractional boundary value problems (1.1) and (1.2).

Lemma 2.1 (**[3, 15]**). *Let $\alpha > 0$ and $u \in C(0,1) \cap L(0,1)$. Then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

has

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$
$$c_i \in \mathbf{R}, \quad i = 0, 1, \ldots, n, \; n = [\alpha] + 1$$

as unique solutions.

Lemma 2.2 (**[3, 15]**). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbf{R}$, $i = 0, 1, \ldots, n$, $n = [\alpha] + 1$.

The following fixed-point theorems in partially ordered sets are fundamental and important to the proofs of our main results.

Theorem 2.1 ([**12**]). *Let $(E, \leq)$ be a partially ordered set, and suppose that there exists a metric d in E such that (E, d) is a complete metric space. Assume that E satisfies the following condition*:

(2.1) if $\{x_n\}$ is a nondecreasing sequence in E such that $x_n \to x$, then $x_n \leq x$, for all $n \in \mathbf{N}$.

Let $T : E \to E$ be a nondecreasing mapping such that

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \quad \text{for } x \geq y,$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that ψ is positive in $(0, +\infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If there exists $x_0 \in E$ with $x_0 \leq T(x_0)$, then T has a fixed point.

If we consider that $(E, \leq)$ satisfies the following condition

(2.2) for $x, y \in E$ there exists $z \in E$ which is comparable to x and y,

then we have the following result.

Theorem 2.2 ([**24**]). *Adding condition* (2.2) *to the hypotheses of Theorem* 2.1, *we obtain uniqueness of the fixed point.*

3. Related lemmas.

Lemma 3.1. *Let $0 < \eta^\alpha < \alpha(\alpha - 1)$. If $h \in C[0, 1]$, then the boundary value problems*

$$D^\alpha_{0+} u(t) + h(t) = 0, \quad 0 < t < 1, \ 2 < \alpha \leq 3, \tag{3.1}$$

$$u(0) = u'(0) = 0, \qquad u'(1) = \int_0^\eta u(s)\, ds, \tag{3.2}$$

have a unique solution

$$u(t) = \int_0^1 G(t, s) h(s)\, ds, \tag{3.3}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{3.4}$$

(3.5)

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1} & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-2} & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \frac{\alpha t^{\alpha-1}}{\alpha(\alpha-1) - \eta^\alpha} \int_0^\eta G_1(t,s)\, dt. \tag{3.6}$$

Proof. By Lemma 2.2, the solution of (3.1) can be written as

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds.$$

From (3.2), we know that $c_2 = c_3 = 0$ and

$$u'(t) = c_1(\alpha-1) - (\alpha-1) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} h(s)\, ds.$$

Thus, together with $u'(1) = \int_0^\eta u(s)\, ds$, we have

$$c_1 = \frac{1}{\alpha-1} \int_0^\eta u(s)\, ds + \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} h(s)\, ds.$$

Therefore, the unique solution of boundary value problem (3.1)–(3.2) is

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds + \frac{t^{\alpha-1}}{\alpha-1} \int_0^\eta u(s)\, ds \\ &\quad + t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} h(s)\, ds \\ &= \quad -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds + \frac{t^{\alpha-1}}{\alpha-1} \int_0^\eta u(s)\, dx \\ &\quad + \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} h(s)\, ds \\ &\quad + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} h(s)\, ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}) h(s)\, ds \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)} \int_t^1 t^{\alpha-1}(1-s)^{\alpha-2} h(s)\,ds + \frac{t^{\alpha-1}}{\alpha-1} \int_0^\eta u(s)\,ds \\
&= \int_0^1 G_1(t,s)h(s)\,ds + \frac{t^{\alpha-1}}{\alpha-1} \int_0^\eta u(s)\,ds, \qquad (3.7)
\end{aligned}
$$

where $G_1(t,s)$ is defined by (3.5).

From (3.7), we have

$$
\int_0^\eta u(t)\,dt = \int_0^\eta \int_0^1 G_1(t,s)h(s)\,ds\,dt + \frac{\eta^\alpha}{\alpha(\alpha-1)} \int_0^\eta u(s)\,ds.
$$

It follows that

$$
\int_0^\eta u(t)\,dt = \frac{\alpha(\alpha-1)}{\alpha(\alpha-1)-\eta^\alpha} \int_0^\eta \int_0^1 G_1(t,s)h(s)\,ds\,dt. \qquad (3.8)
$$

Substituting (3.8) into (3.7), we obtain

$$
\begin{aligned}
u(t) &= \int_0^1 G_1(t,s)h(s)\,ds + \frac{\alpha t^{\alpha-1}}{\alpha(\alpha-1)-\eta^\alpha} \int_0^\eta \int_0^1 G_1(t,s)h(s)\,ds\,dt \\
&= \int_0^1 G_1(t,s)h(s)\,ds + \int_0^1 G_2(t,s)h(s)\,ds \\
&= \int_0^1 G(t,s)h(s)\,ds,
\end{aligned}
$$

where $G(t,s)$, $G_1(t,s)$ and $G_2(t,s)$ are defined by (3.4), (3.5) and (3.6), respectively. The proof is complete. □

Lemma 3.2. *Let $0 < \eta^\alpha < \alpha(\alpha-1)$. If $f \in C([0,1]\times[0,+\infty),[0,+\infty))$, then the boundary value problem* (1.1)–(1.2) *is equivalent to the integral equation*

$$
u(t) = \int_0^1 G(t,s)\phi_q\left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} f(\tau,u(\tau))\,d\tau\right) ds, \qquad (3.9)
$$

where $G(t,s)$ is defined by (3.4).

Proof. By the boundary value problem (1.1)–(1.2) and Lemma 2.2, we have

$$
\phi_p(D_{0+}^\alpha u(t)) = ct^{\gamma-1} - \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} f(s,u(s))\,ds.
$$

By $D^{\alpha}_{0_+}u(t)|_{t=0} = 0$, there is $c = 0$, and then

$$D^{\alpha}_{0_+}u(t) = -\phi_q\left(\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} f(s, u(s))\, ds\right).$$

Therefore, boundary value problem (1.1)–(1.2) is equivalent to the following problem

$$D^{\alpha}_{0_+}u(t) + \phi_q\left(\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} f(s, u(s))\, ds\right) = 0,$$
$$0 < t < 1, \quad 2 < \alpha \le 3,$$
$$u(0) = u'(0) = 0, \qquad u'(1) = \int_0^\eta u(s)\, ds.$$

By Lemma 3.1, boundary value problem (1.1)–(1.2) is equivalent to the integral equation (3.9). The proof is complete. □

Lemma 3.3. *The function* $G_1(t, s)$ *defined by* (3.5) *satisfies*:

(i) G_1 *is a continuous function and* $G_1(t, s) \ge 0$ *for* $(t, s) \in [0, 1] \times [0, 1]$;

(ii)
$$\sup_{t\in[0,1]} \int_0^1 G_1(t, s)\, ds = \frac{1}{(\alpha-1)\Gamma(\alpha+1)}.$$

Proof.

(i) The continuity of G_1 is easily checked. On the other hand, for $0 \le t \le s \le 1$, it is obvious that

$$G_1(t, s) = \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \ge 0.$$

In the case $0 \le s \le t \le 1$ $(s \ne 1)$, we have

$$\begin{aligned} G_1(t,s) &= \frac{1}{\Gamma(\alpha)}\left[\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{1-s} - (t-s)^{\alpha-1}\right] \\ &\ge \frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] \\ &= \frac{1}{\Gamma(\alpha)}\left[(t-ts)^{\alpha-1} - (t-s)^{\alpha-1}\right] \\ &\ge 0. \end{aligned}$$

Moreover, as $G_1(t,1) = 0$, then we conclude that $G_1(t,s) \ge 0$ for all $(t,s) \in [0,1] \times [0,1]$.

(ii) Since

$$\begin{aligned} \int_0^1 G_1(t,s)\,ds &= \int_0^t G_1(t,s)\,ds + \int_t^1 G_1(t,s)\,ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1})\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-2}\,ds \\ &= \frac{1}{\Gamma(\alpha)}\left(\frac{t^{\alpha-1}}{\alpha-1} - \frac{1}{\alpha}t^{\alpha}\right). \end{aligned}$$

On the other hand, let

$$\sigma(t) = \int_0^1 G_1(t,s)\,ds = \frac{1}{\Gamma(\alpha)}\left(\frac{t^{\alpha-1}}{\alpha-1} - \frac{1}{\alpha}t^{\alpha}\right).$$

Then, as

$$\sigma'(t) = \frac{1}{\Gamma(\alpha)}\left(t^{\alpha-2} - t^{\alpha-1}\right) > 0 \quad \text{for } t > 0,$$

the function $\rho(t)$ is strictly increasing and, consequently,

$$\begin{aligned} \sup_{t\in[0,1]} \sigma(t) &= \sup_{t\in[0,1]} \int_0^1 G_1(t,s)\,ds = \sigma(1) = \frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha-1} - \frac{1}{\alpha}\right) \\ &= \frac{1}{\alpha(\alpha-1)\Gamma(\alpha)} = \frac{1}{(\alpha-1)\Gamma(\alpha+1)}. \end{aligned}$$

The proof is complete. $\square$

Remark 3.1. Obviously, by Lemmas 3.2 and 3.3, we have $u(t) \geq 0$ if $f(t, u(t)) \geq 0$ on $t \in [0, 1]$.

Lemma 3.4. *$G_1(t, s)$ is strictly increasing in the first variable.*

Proof. For s fixed, we let

$$g_1(t) = \frac{1}{\Gamma(\alpha)} \left(t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}\right) \quad \text{for } s \leq t,$$
$$g_2(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-s)^{\alpha-1} \quad \text{for } t \leq s.$$

It is easy to check that $g_1(t)$ is strictly increasing on $[s, 1]$ and $g_2(t)$ is strictly increasing on $[0, s]$. Then we have the following cases:

Case 1. $t_1, t_2 \leq s$ and $t_1 < t_2$. In this case, we have $g_2(t_1) < g_2(t_2)$, i.e. $G_1(t_1, s) < G_1(t_2, s)$.

Case 2. $s \leq t_1, t_2$ and $t_1 < t_2$. In this case, we have $g_1(t_1) < g_1(t_2)$, i.e. $G_1(t_1, s) < G_1(t_2, s)$.

Case 3. $t_1 \leq s \leq t_2$ and $t_1 < t_2$. In this case, we have $g_2(t_1) \leq g_2(s) = g_1(s) \leq g_1(t_2)$. We claim that $g_2(t_1) < g_1(t_2)$. In fact, if $g_2(t_1) = g_1(t_2)$, then $g_2(t_1) = g_2(s) = g_1(s) = g_1(t_2)$ and, from the monotone of g_1 and g_2, we have $t_1 = s = t_2$, which contradicts with $t_1 < t_2$. This fact implies that $G_1(t_1, s) < G_1(t_2, s)$. The proof is complete. □

Remark 3.2. Obviously, by Lemma 3.4, we have

$$\int_0^1 G_2(t, s)\, ds \leq \frac{\eta}{\Gamma(\alpha)\, [\alpha(\alpha-1) - \eta^\alpha]\, (\alpha - 1)}. \tag{3.10}$$

Proof. In fact, from Lemmas 3.4 and (3.6), we have

$$\begin{aligned} G_2(t, s) \leq G_2(1, s) &= \frac{\alpha\eta G_1(1, s)}{\alpha(\alpha-1) - \eta^\alpha} \\ &= \frac{\alpha\eta((1-s)^{\alpha-2} - (1-s)^{\alpha-1})}{\Gamma(\alpha)\, [\alpha(\alpha-1) - \eta^\alpha]}. \end{aligned}$$

Thus,

$$\int_0^1 G_2(t,s)ds \leq \frac{\alpha\eta \int_0^1 \left((1-s)^{\alpha-2} - (1-s)^{\alpha-1}\right) ds}{\Gamma(\alpha)\left[\alpha(\alpha-1) - \eta^\alpha\right]}$$
$$= \frac{\eta}{\Gamma(\alpha)\left[\alpha(\alpha-1) - \eta^\alpha\right](\alpha-1)},$$

for $s,t \in [0,1] \times [0,1]$. □

4. Uniqueness of a positive and nondecreasing solution for boundary value problems (1.1)–(1.2)**.** In this section, we establish the existence and uniqueness of a positive and nondecreasing solution for the boundary value problems (1.1)–(1.2) by using a fixed point theorem in partially ordered sets. The basic space used in this section is $E = C[0,1]$. Then E is a real Banach space with the norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Note that this space can be equipped with a partial order given by

$$x, y \in C[0,1], \qquad x \leq y \Leftrightarrow x(t) \leq y(t), \quad t \in [0,1].$$

In **[24]** it is proved that $(C[0,1], \leq)$ with the classic metric given by

$$d(x,y) = \sup_{0 \leq t \leq 1} \{|x(t) - y(t)|\}$$

satisfying condition (2.1) of Theorem 2.1. Moreover, for $x, y \in C[0,1]$ as the function $\max\{x,y\} \in C[0,1]$, $(C[0,1], \leq)$ satisfies condition (2.2).

For notational convenience, we denote

$$L := \frac{1}{\Gamma^{q-1}(\gamma+1)} \left(\frac{1}{(\alpha-1)\Gamma(\alpha+1)} + \frac{\eta}{\Gamma(\alpha)\left[\alpha(\alpha-1) - \eta^\alpha\right](\alpha-1)} \right)$$
$$> 0.$$

The main result of this section is the following.

Theorem 4.1. *The boundary value problem* (1.1)–(1.2) *has a unique positive and strictly increasing solution* $u(t)$ *if the following conditions are satisfied*:

(i) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and nondecreasing with respect to the second variable and* $f(t,u(t)) \not\equiv 0$ *for* $t \in Z \subset [0,1]$ *with* $\mu(Z) > 0$ (μ *denotes the Lebesgue measure*);

(ii) *there exists* $0 < \lambda + 1 < L^{-1}$ *such that, for* $u, v \in [0, +\infty)$ *with* $u \geq v$ *and* $t \in [0, 1]$

$$\phi_p(\ln(v+2)) \leq f(t,v) \leq f(t,u) \leq \phi_p(\ln(u+2)(u-v+1)^{\lambda}).$$

Proof. Consider the cone

$$K = \{u \in C[0,1] : u(t) \geq 0\}.$$

As K is a closed set of $C[0,1]$, K is a complete metric space with the distance given by $d(u,v) = \sup\limits_{t \in [0,1]} |u(t) - v(t)|$.

Now, we consider the operator T defined by

$$Tu(t) = \int_0^1 G(t,s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))\,d\tau\right) ds,$$

where $G(t,s)$ is defined by (3.4). By Lemma 3.3 and condition (i), we have that $T(K) \subset K$.

We now show that all the conditions of Theorems 2.1 and 2.2 are satisfied.

Firstly, by condition (i), for $u, v \in K$ and $u \geq v$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))\,d\tau\right) ds \\ &\geq \int_0^1 G(t,s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, v(\tau))\,d\tau\right) ds \\ &= Tv(t). \end{aligned}$$

This proves that T is a nondecreasing operator.

On the other hand, for $u \geq v$ and by condition (ii), we have

$$\begin{aligned} d(Tu,Tv) &= \sup_{0\leq t\leq 1} |(Tu)(t) - (Tv)(t)| \\ &= \sup_{0\leq t\leq 1} ((Tu)(t) - (Tv)(t)) \\ &\leq \int_0^1 G(t,s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))\,d\tau\right) ds \\ &\quad - \int_0^1 G(t,s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, v(\tau))\,d\tau\right) ds \end{aligned}$$

$$\begin{aligned}&\le \big(\ln(u+2)(u-v+1)^{\lambda} - \ln(v+2)\big)\\ &\quad \times \int_0^1 G(t,s)\phi_q\bigg(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}\,d\tau\bigg)\,ds\\ &\le \frac{1}{\Gamma^{q-1}(\gamma+1)} \ln\frac{(u+2)(u-v+1)^{\lambda}}{v+2}\int_0^1 G(t,s)\,ds\\ &\le (\lambda+1)\ln(u-v+1)\frac{1}{\Gamma^{q-1}(\gamma+1)}\int_0^1 G(t,s)\,ds.\end{aligned}$$

Since the function $h(x) = \ln(x+1)$ is nondecreasing, by Lemma 3.3 and condition (ii), then we have

$$\begin{aligned}d(Tu,Tv) &\le (\lambda+1)\ln(\|u-v\|+1)\\ &\qquad \frac{1}{\Gamma^{q-1}(\gamma+1)}\bigg(\sup_{0\le t\le 1}\int_0^1 G_1(t,s)\,ds + \sup_{0\le t\le 1}\int_0^1 G_2(t,s)\,ds\bigg)\\ &\le (\lambda+1)\ln(\|u-v\|+1)\cdot L\\ &\le \|u-v\| - (\|u-v\| - \ln(\|u-v\|+1)).\end{aligned}$$

Let $\psi(x) = x - \ln(x+1)$. Obviously $\psi : [0,+\infty) \to [0,+\infty)$ is continuous, nondecreasing, positive in $(0,+\infty)$, $\psi(0) = 0$ and $\lim_{x\to+\infty}\psi(x) = +\infty$. Thus, for $u \ge v$, we have

$$d(Tu,Tv) \le d(u,v) - \psi(d(u,v)).$$

As $G(t,s) \ge 0$ and $f \ge 0$, $(T0)(t) = \int_0^1 G(t,s)f(s,0)\,ds \ge 0$ and, by Theorem 2.1, we know that problem (1.1)–(1.2) has at least one nonnegative solution. As $(K,\le)$ satisfies condition (2.2), thus, Theorem 2.2 implies the uniqueness of the solution.

Finally, we will prove that this solution $u(t)$ is a strictly increasing function. As

$$u(0) = \int_0^1 G(0,s)\phi_q\bigg(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}f(\tau,u(\tau))\,d\tau\bigg)\,ds$$

and $G(0,s) = 0$, we have $u(0) = 0$.

Moreover, if we take $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we can consider the following cases.

Case 1. $t_1 = 0$. In this case, $u(t_1) = 0$ and, as $u(t) \geq 0$, suppose that $u(t_2) = 0$. Then

$$0 = u(t_2) = \int_0^1 G(t_2, s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))\, d\tau\right) ds.$$

This implies that

$$G(t_2, s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))\, d\tau\right) = 0,$$

almost everywhere (s) and, as $G(t_2, s) \neq 0$, almost everywhere (s), we get $f(s, u(s)) = 0$ almost everywhere (s).

On the other hand, f is nondecreasing with respect to the second variable. Then we have

$$f(s, 0) \leq f(s, u(s)) = 0, \quad \text{almost everywhere } (s),$$

which contradicts condition (i) $f(t, 0) \neq 0$ for $t \in Z \subset [0,1](\mu(Z) \neq 0)$. Thus, $u(t_1) = 0 < u(t_2)$.

Case 2. $0 < t_1$. In this case, let us take $t_2, t_1 \in [0,1]$ with $t_1 < t_2$. Then

$$\begin{aligned} u(t_2) - u(t_1) &= (Tu)(t_2) - (Tu)(t_1) \\ &= \int_0^1 (G(t_2, s) - G(t_1, s)) \\ &\quad \phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))\, d\tau\right) ds. \end{aligned}$$

Taking into account Lemma 3.4 and the fact that $f \geq 0$, we get $u(t_2) - u(t_1) \geq 0$.

Suppose that $u(t_2) = u(t_1)$. Then

$$\int_0^1 (G(t_2, s) - G(t_1, s))\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))\, d\tau\right) ds = 0,$$

and this implies

$$(G(t_2, s) - G(t_1, s))\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau))\, d\tau\right) = 0$$

almost everywhere (s). Again, Lemma 3.4 gives us

$$f(s, u(s)) = 0 \quad \text{almost everywhere } (s),$$

and, using the same reasoning as above, we have that this contradicts condition (i) $f(t, 0) \neq 0$ for $t \in Z \subset [0, 1](\mu(Z) \neq 0)$. Thus, $u(t_1) = 0 < u(t_2)$. The proof is complete. $\square$

5. Single positive solution of the boundary value problems (1.1)–(1.2). In this section, we establish the existence of a single positive solution for boundary value problems (1.1) and (1.2) by the lower and upper solution methods. We assume that $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous in this section.

Lemma 5.1. *If $u(t) \in C[0, 1]$ and is a positive solution of* (1.1) *and* (1.2)*, then $m\rho(t) \leq u(t) \leq M\rho(t)$, where*

$$\rho(t) = \int_0^1 G(t, s)\phi_q\bigg(\frac{1}{\Gamma(\gamma)} \int_0^s (s - \tau)^{\gamma-1}\, d\tau\bigg)\, ds$$

and m, M are two constants.

Proof. Since $u(t) \in C[0, 1]$, there exists $M' > 0$ so that $|u(t)| \leq M'$ for $t \in [0, 1]$. Taking

$$\phi_p(m) := \min_{(t,u)\in[0,1]\times[0,M']} f(t, u(t)),$$
$$\phi_p(M) := \max_{(t,u)\in[0,1]\times[0,M']} f(t, u(t)).$$

By view of Lemma 3.2, we have

$$\begin{aligned} m\rho(t) &\leq u(t) \\ &= \int_0^1 G(t, s)\phi_q\bigg(\frac{1}{\Gamma(\gamma)} \int_0^s (s - \tau)^{\gamma-1} f(\tau, u(\tau))\, d\tau\bigg)\, ds \leq M\rho(t). \end{aligned}$$

Thus, we have finished the proof of Lemma 5.1. $\square$

Now we introduce the following two definitions about the upper and lower solutions of fractional boundary value problems (1.1) and (1.2).

Definition 5.1. A function $\theta(t)$ is called a lower solution of fractional boundary value problems (1.1) and (1.2) if $\theta(t) \in C[0,1]$ and $\theta(t)$ satisfy

$$\begin{array}{llll} -D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}\theta(t))) \leq f(t,\theta(t)), & 0<t<1,\ 2<\alpha\leq 3, & & \\ \theta(0)\leq 0, & \theta'(0)\leq 0, & \theta'(1)\leq \int_0^{\eta}\theta(s)\,ds, & D_{0+}^{\alpha}\theta(t)|_{t=0}\leq 0. \end{array}$$

Definition 5.2. A function $\zeta(t)$ is called an upper solution of fractional boundary value problems (1.1) and (1.2) if $\zeta(t) \in C[0,1]$ and $\zeta(t)$ satisfy

$$\begin{array}{llll} -D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}\zeta(t))) \geq f(t,\zeta(t)), & 0<t<1,\ 2<\alpha\leq 3, & & \\ \zeta(0)\geq 0, & \zeta'(0)\geq 0, & \zeta'(1)\geq \int_0^{\eta}\zeta(s)\,ds, & D_{0+}^{\alpha}\zeta(t)|_{t=0}\geq 0. \end{array}$$

The main result of this section is the following.

Theorem 5.1. *The fractional boundary value problems* (1.1) *and* (1.2) *have a positive solution* $u(t)$ *if the following conditions are satisfied*:

(H_f) $f(t,u) \in C([0,1]\times[0,+\infty),\mathbf{R}^+)$ *is nondecreasing relative to* u, $f(t,\rho(t)) \not\equiv 0$ *for* $t \in (0,1)$ *and there exists a positive constant* $\mu < 1$ *such that*

$$k^{\mu} f(t,u) \leq f(t,ku), \quad \textit{for all } 0 \leq k \leq 1.$$

Proof. At first, we will prove that the functions $\theta(t) = k_1 g(t)$, $\zeta(t) = k_2 g(t)$ are lower and upper solutions of (1.1) and (1.2), respectively, where

$$0 < k_1 \leq \min\left\{\frac{1}{a_2}, a_1^{\mu/(1-\mu)}\right\}, \quad k_2 \geq \max\left\{\frac{1}{a_1}, a_2^{\mu/(1-\mu)}\right\}$$

and

$$\phi_p(a_1) = \min\left\{1, \inf_{t\in[0,1]} f(t,\rho(t))\right\} > 0,$$

$$\phi_p(a_2) = \max\left\{1, \sup_{t\in[0,1]} f(t,\rho(t))\right\}$$

and

$$g(t) = \int_0^1 G(t,s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1} f(\tau,\rho(\tau))\,d\tau\right) ds.$$

By view of Lemma 3.1, we know that $g(t)$ is a positive solution of the following equations
(5.1)

$$\begin{aligned}&D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}u(t)))+f(t,\rho(t))=0, \quad 0<t<1,\ 2<\alpha\le 3,\\ &u(0)=u'(0)=0, \qquad u'(1)=\textstyle\int_0^{\eta}u(s)\,ds, \qquad D_{0+}^{\alpha}u(t)|_{t=0}=0.\end{aligned}$$

From the conclusion of Lemma 5.1, we know that

$$a_1\rho(t)\le g(t)\le a_2\rho(t), \quad \text{for all } t\in[0,1]. \tag{5.2}$$

Thus, by virtue of the assumption of Theorem 5.1, this shows that

$$k_1a_1\le\frac{\theta(t)}{\rho(t)}\le k_1a_2\le 1, \qquad \frac{1}{k_2a_2}\le\frac{\rho(t)}{\gamma(t)}\le\frac{1}{k_2a_1}\le 1,$$
$$(k_1a_1)^{\mu}\ge k_1, \qquad (k_2a_2)^{\mu}\le k_2.$$

Therefore, we have

$$\begin{aligned}f(t,\theta(t))&=f\left(t,\frac{\theta(t)}{\rho(t)}\rho(t)\right)\ge\left(\frac{\theta(t)}{\rho(t)}\right)^{\mu}f(t,\rho(t))\\ &\ge(k_1a_1)^{\mu}f(t,\rho(t))\ge k_1f(t,\rho(t)),\\ k_2f(t,\rho(t))&=k_2f\left(t,\frac{\rho(t)}{\gamma(t)}\gamma(t)\right)\ge k_2\left(\frac{\rho(t)}{\gamma(t)}\right)^{\mu}f(t,\gamma(t))\\ &\ge k_2(k_2a_2)^{-\mu}f(t,\gamma(t))\ge f(t,\gamma(t)).\end{aligned}$$

This implies that

$$\begin{aligned}&-D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}\theta(t)))=k_1f(t,\rho(t))\le f(t,\theta(t)),\\ &0<t<1,\ 2<\alpha\le 3,\\ &-D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}\zeta(t)))=k_2f(t,\rho(t))\ge f(t,\zeta(t)),\\ &0<t<1,\ 2<\alpha\le 3.\end{aligned} \tag{5.3}$$

Obviously, $\theta(t)=k_1g(t)$ and $\zeta(t)=k_2g(t)$ satisfies the boundary conditions (1.2). So, $\theta(t)=k_1g(t)$ and $\zeta(t)=k_2g(t)$ are the lower and upper solutions of (1.1) and (1.2), respectively.

Next, we will prove the fractional boundary value problem
(5.4)

$$\begin{aligned}&D_{0+}^{\gamma}(\phi_p(D_{0+}^{\alpha}u(t)))+g(t,u(t))=0,\\ &0<t<1, \quad 0<\gamma\le 1, \quad 2<\alpha\le 3,\\ &u(0)=u'(0)=0, \qquad u'(1)=\textstyle\int_0^{\eta}u(s)\,ds, \qquad D_{0+}^{\alpha}u(t)|_{t=0}=0\end{aligned}$$

has a solution, where

$$g(t,u(t)) = \begin{cases} f(t,\theta(t)) & \text{if } u(t) \le \theta(t), \\ f(t,u(t)) & \text{if } \theta(t) \le u(t) \le \zeta(t), \\ f(t,\zeta(t)) & \text{if } \zeta(t) \le u(t). \end{cases}$$

Thus, we consider the operator $A : C[0,1] \to C[0,1]$ defined as follows

$$Au(t) = \int_0^1 G(t,s)\phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}g(\tau,u(\tau))\,d\tau\right)ds,$$

where $G(t,s)$ is defined by Lemma 3.1. It is clear that the operator A is continuous in $C[0,1]$. Since the function $f(t,u)$ in nondecreasing in u, this shows that, for any $u \in C[0,1]$,

$$f(t,\theta(t)) \le g(t,u(t)) \le f(t,\zeta(t)) \quad \text{for} \quad t \in [0,1].$$

The operator $A : C[0,1] \to C[0,1]$ is continuous in view of the continuity of $G(t,s)$ and $g(t,u(t))$. By means of the Arzela-Ascoli theorem, A is a compact operator. Therefore, from the Leray-Schauder fixed point theorem, the operator A has a fixed point, i.e., fractional boundary value problem (5.4) has a solution.

Finally, we will prove that fractional boundary value problems (1.1) and (1.2) have positive solutions.

Suppose $u^*(t)$ is a solution of fractional boundary value problem (5.4). Since the function $f(t,u)$ is nondecreasing in u, we know that

$$f(t,\theta(t)) \le g(t,u^*(t)) \le f(t,\zeta(t)) \quad \text{for } t \in [0,1].$$

Thus,

$$\begin{aligned} &-(D^\gamma_{0+}(\phi_p(D^\alpha_{0+}\zeta(t))) - D^\gamma_{0+}(\phi_p(D^\alpha_{0+}u^*(t)))) \\ &\qquad = f(t,\zeta(t)) - g(t,u^*(t)) \ge 0, \\ &(\zeta-u^*)(0) = (\zeta-u^*)'(0) = 0, \\ &(\zeta-u^*)'(1) = \int_0^\eta (\zeta-u^*)(s)\,ds, D^\alpha_{0+}(\zeta-u^*)|_{t=0} = 0, \end{aligned}$$

where $z(t) = \phi_p(D^\alpha_{0+}\zeta(t)) - \phi_p(D^\alpha_{0+}u^*(t))$. By virtue of Remark 3, $z(t) \ge 0$, i.e.,

$$\phi_p(D^\alpha_{0+}u^*(t)) \le \phi_p(D^\alpha_{0+}\zeta(t)) \quad \text{for } t \in [0,1].$$

Since ϕ_p is monotone increasing, we have $D^{\alpha}_{0+}u^*(t) \le D^{\alpha}_{0+}\zeta(t)$, that is, $D^{\alpha}_{0+}(\zeta(t) - u^*(t)) \ge 0$. By Lemma 3.1 and Remark 3, we have $\zeta(t) \ge u^*(t)$ for $t \in [0,1]$. Similarly, $\theta(t) \le u^*(t)$ for $t \in [0,1]$. Therefore, $u^*(t)$ is a positive solution of fractional boundary value problem (1.1) and (1.2). We have finished the proof of Theorem 5.1. $\square$

6. Examples.

Example 6.1. The fractional boundary value problem
(6.1)

$$\begin{cases} D^{1/2}_{0+}(\phi_2(D^{5/2}_{0+}u(t))) + (t^2+1)[\ln(2+u(t))]^2 = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u'(1) = \int_0^{1/2} u(s)\,ds, \quad D^{5/2}_{0+}u(t)|_{t=0} = 0 \end{cases}$$

has a unique and strictly increasing solution.

Proof. In this case, $\gamma = 1/2$, $\alpha = 5/2$, $p = 2$, $\eta = 1/2$, $f(t,u) = (t^2+1)[\ln(2+u)]^2$ for $(t,u) \in [0,1] \times [0,\infty)$. Note that f is a continuous function and $f(t,u) \ne 0$ for $t \in [0,1]$. Moreover, f is nondecreasing with respect to the second variable since $(\partial f)/(\partial u) = [2(t^2+1)/(u+2)]\ln(2+u(t)) > 0$ for $t \in [0,1]$. On the other hand, since $u \in C[0,1]$, without loss generality, we take $1 \ge u \ge v > 0$ with $(u-v)(t) \ge 1/3$ and, for $t \in [0,1]$, we have

$$\begin{aligned} \phi_2(\ln(v+2)) &\le (t^2+1)[\ln(v+2)]^2 \\ &= f(t,v) \le f(t,u) = (t^2+1)[\ln(u+2)]^2 \\ &\le 2[\ln(u+2)]^2 \\ &\le [\ln(u+2)(u-v+1)^2]^2 \\ &\le \phi_2(\ln(u+2)(u-v+1)^2). \end{aligned}$$

In this case, $\lambda = 2$. By simple computation, we have $0 < \lambda + 1 < 1/L$. Thus, Theorem 4.1 implies that boundary value problems (1.1)–(1.2) have a unique and strictly increasing solution. $\square$

Example 6.2. As an example we mention the following fractional boundary value problem:
(6.2)

$$\begin{cases} D^{1/2}_{0+}(\phi_2(D^{5/2}_{0+}u(t))) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u'(1) = \int_0^{1/2} u(s)\,ds, \quad D^{5/2}_{0+}u(t)|_{t=0} = 0, \end{cases}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, and

$$f(t,u) = t + u^{\mu}, \quad 0 < \mu < 1.$$

Proof. Since $k^{\mu} \le 1$, $0 < \mu < 1$ and $0 \le k \le 1$. It is easy to check that

$$k^{\mu} f(t,u) = k^{\mu} t + k^{\mu} u^{\mu} \le t + (ku)^{\mu} = f(t,ku).$$

Thus, by Theorem 5.1, we know that the boundary value problem (6.2) has a positive solution $u(t)$. $\square$

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