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SZEGŐ KERNEL TRANSFORMATION LAW FOR PROPER HOLOMORPHIC MAPPINGS

MICHAEL BOLT

ABSTRACT. Let Ω_1, Ω_2 be smoothly bounded doubly connected regions in the complex plane. We establish a transformation law for the Szegő kernel under proper holomorphic mappings. This extends known results concerning biholomorphic mappings between multiply connected regions as well as proper holomorphic mappings from multiply connected regions to simply connected regions.

1. Introduction. In this manuscript, we establish a transformation law for the Szegő kernel under proper holomorphic mappings between doubly connected regions in the plane.

Let $\Omega \subset \mathbf{C}$ be a bounded region with C^∞ smooth boundary. The Szegő projection $\mathcal{S}$ is the orthogonal projection of $L^2(b\Omega)$ onto the subspace $H^2(b\Omega)$ of functions that extend holomorphically to Ω. It acts by integration against an Hermitian kernel $S_\Omega(\cdot,\cdot)$, called the Szegő kernel, according to

$$\mathcal{S}f(z) = \int_{w\in b\Omega} S_\Omega(z,w)\, f(w)\, ds_w \quad \text{for } z \in \Omega.$$

The integration is carried out with respect to arc length measure, ds.

The Szegő kernel is related to the Green's function and is considered to be one of the canonical functions associated with a bounded region in the plane. Its behavior is less understood than the Bergman kernel, its closest analytic relative.

The simplest transformation law for the Szegő kernel expresses the relationship between kernels for biholomorphically equivalent regions. The idea behind this result is the observation that arc length measure between biholomorphic regions is related by the derivative of the mapping. See Bell [**2,** page 44]. Using identities for the Bergman

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kernel, Jeong extended the result in her thesis to the case of a proper holomorphic mapping provided the target is simply connected. See **[5, 6]**.

Here we extend both results and establish a transformation law for proper holomorphic mappings between doubly connected regions. We prove:

Theorem 1.1. *Let Ω_1, Ω_2 be bounded, doubly connected regions with C^∞ boundaries, and let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping of order m. Let $f_1, \dots, f_m$ denote the m local inverses to f, and let $S_k(\cdot,\cdot)$ denote the Szegő kernel for Ω_k, $k = 1, 2$. Let $-(1/2\pi)\log r$ be the conformal modulus of Ω_1, and let ω_1 be the harmonic function on Ω_2 that takes value 1 on the inner boundary and value 0 on the outer boundary, respectively. Then the Szegő kernels transform according to*

$$\sum_{j=1}^{m} S_1(z, f_j(w))^2 \overline{f_j'(w)} = f'(z) S_2(f(z), w)^2 + \lambda_{r,m} \frac{\partial(\omega_1 \circ f)}{\partial z} \overline{\frac{\partial \omega_1}{\partial w}}$$

for $z \in \Omega_1$, $w \in \Omega_2$, where

$$\lambda_{r,m} = \frac{2(\log r^m)^2}{\pi^2} \left(\sum_{k>0} \frac{k(-1)^k r^{mk}}{1 - r^{2mk}} - \frac{1}{m} \sum_{k>0} \frac{k(-1)^k r^k}{1 - r^{2k}} \right).$$

A region $\Omega \subset\subset \mathbf{C}$ is doubly connected if its complement consists of two connected components, and a region is nondegenerate if these components contain more than one point. Every non-degenerate doubly connected region can be mapped biholomorphically to an annulus $\mathcal{A}_r = \{z : r < |z| < 1\}$ where $r \in (0, 1)$ is independent of the choice of the biholomorphism. The constant $-(1/2\pi)\log r$, called the conformal modulus of Ω, evidently is biholomorphically invariant, and coincides with Ahlfors's notion of extremal length [**1,** page 11].

The proof of Theorem 1.1 involves the corresponding transformation law for the Bergman kernel, the known relationship between the Szegő kernel and Bergman kernel, and the transformation law for the Szegő kernel under biholomorphic mappings. It also relies on a direct calculation of the Szegő kernel and Bergman kernel for an annulus.

The remainder of the manuscript is organized as follows. In the next two sections we recall what is known about the Szegő and Bergman kernels under proper holomorphic mappings. Theorem 1.1 extends the results from these sections, but these results also are needed for the proof of Theorem 1.1. Subsequently, we compute the Szegő and Bergman kernels for an annulus for the purpose of identifying the constant $\lambda_{r,m}$ that appears in Theorem 1.1. The last two sections contain the proof of Theorem 1.1 and some remarks about the case of regions with higher connectivity.

2. The Szegő kernel under proper holomorphic mappings. As noted, the Szegő kernel is the Hermitian kernel associated to the orthogonal projection from $L^2(b\Omega)$ to $H^2(b\Omega)$. The underlying inner product is given by

$$\langle f, g\rangle_{b\Omega} = \int_{b\Omega} f\,\overline{g}\,ds$$

where the arc length measure is denoted by ds and where the associated norm is given by $\|f\|^2_{b\Omega} = \langle f, f\rangle_{b\Omega}$ in both $L^2(b\Omega)$ and $H^2(b\Omega)$. The Szegő kernel $S(z, w)$ is known to be holomorphic in z and anti-holomorphic in w for $(z, w) \in \Omega \times \Omega$. The kernel extends to be C^∞ on the expanded set $\overline{\Omega} \times \overline{\Omega} \setminus \{(z, z) \in b\Omega \times b\Omega\}$.

One way to construct the Szegő kernel is to begin with an orthonormal basis $\{\psi_j\}$ for $H^2(b\Omega)$. Then, for given $z \in \Omega$, $w \in \overline{\Omega}$,

$$S(z, w) = \sum_j \psi_j(z)\overline{\psi_j(w)}.$$

Using this approach when $\Omega = \Delta$ is the unit disc, one finds that $S_\Delta(z, w) = (2\pi)^{-1}(1 - z\overline{w})^{-1}$.

For a biholomorphic mapping $f : \Omega_1 \to \Omega_2$ between bounded regions with C^∞ boundaries, it is known that f extends to be C^∞ on $\overline{\Omega}_1$, f' does not vanish on Ω_1, and f' is the square of a holomorphic function that extends to be C^∞ on $\overline{\Omega}_1$. See Bell [**2,** p42]. This leads to an isometry $L^2(b\Omega_2) \to L^2(b\Omega_1)$ given by $u \to (u \circ f)\sqrt{f'}$ that also maps $H^2(b\Omega_2) \to H^2(b\Omega_1)$. It, too, yields the following transformation law.

Theorem 2.1. *Let $f : \Omega_1 \to \Omega_2$ be a biholomorphic mapping between bounded regions with C^∞ boundaries, and let $S_k(\cdot, \cdot)$ denote the Szegő kernel for Ω_k, $k = 1, 2$. Then the Szegő kernels transform according to*

$$S_1(z, w) = f'(z)^{1/2}\, S_2(f(z), f(w))\, \overline{f'(w)^{1/2}}.$$

Jeong later extended Theorem 2.1 and proved the following. Her motivation was to establish a rationality criterion for proper maps to the unit disc. See [**5, 6**].

Theorem 2.2. *Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping of order m between bounded regions with C^∞ boundaries, and let Ω_2 be simply connected. Let $f_1, \dots, f_m$ denote the m local inverses to f, and let $S_k(z,w)$ denote the Szegő kernel for Ω_k, $k = 1, 2$. Then the Szegő kernels transform according to*

$$\sum_{j=1}^{m} S_1(z, f_j(w))^2 \, \overline{f_j'(w)} = f'(z) \, S_2(f(z), w)^2$$

for $z \in \Omega_1$, $w \in \Omega_2$.

The goal of the present work is to make a first step toward extending Theorem 2.2 to the case when the target region is multiply connected.

3. The Bergman kernel under proper holomorphic mappings. The closest analytic relative to the Szegő kernel is the Bergman kernel, also defined for a bounded region $\Omega \subset \mathbf{C}$. We still assume that Ω has C^∞ boundary.

Here, the Bergman projection $\mathcal{B}$ is the orthogonal projection from $L^2(\Omega)$ onto the subspace $B^2(\Omega)$ of holomorphic functions. It, too, acts by integration against an Hermitian kernel $K_\Omega(\cdot, \cdot)$, called the *Bergman kernel*, according to

$$\mathcal{B}f(z) = \iint_{w \in \Omega} K_\Omega(z, w) \, f(w) \, dA_w \quad \text{for } z \in \Omega.$$

The integration is carried out with respect to area measure, dA. In this case, the underlying inner product is given by

$$\langle f, g \rangle_\Omega = \iint_\Omega f \, \overline{g} \, dA,$$

and the associated norm is given by $\|f\|_\Omega^2 = \langle f, f \rangle_\Omega$ in both $L^2(\Omega)$ and $B^2(\Omega)$. The Bergman kernel $K(z, w)$ is known to be holomorphic in z and anti-holomorphic in w for $(z, w) \in \Omega \times \Omega$.

As for the Szegő kernel, one way to construct the Bergman kernel is to begin with an orthonormal basis $\{\psi_j\}$ for $B^2(\Omega)$. Then, for given $z \in \Omega$, $w \in \Omega$,

$$K(z,w) = \sum_j \psi_j(z)\overline{\psi_j(w)}.$$

For the unit disc, one finds that $K_\Delta(z,w) = \pi^{-1}(1 - z\overline{w})^{-2}$.

The behavior of the Bergman kernel under proper holomorphic mappings is understood completely. A proof of the following can be found in Bell [**2,** Section 16].

Theorem 3.1. *Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping of order m between bounded regions with C^∞ boundaries. Let $f_1, \dots, f_m$ denote the m local inverses to f, and let $K_k(z,w)$ denote the Bergman kernel for Ω_k, $k = 1, 2$. Then the Bergman kernels transform according to*

$$\sum_{j=1}^{m} K_1(z, f_j(w))\,\overline{f_j'(w)} = f'(z)\,K_2(f(z), w) \tag{1}$$

for $z \in \Omega_1$, $w \in \Omega_2$.

Before specializing to the case of a doubly connected region, we describe an additional result that expresses the relationship between the Szegő and Bergman kernels for a general region. Let $\Omega \subset \mathbf{C}$ be a bounded, n-connected region with C^∞ boundary. Let $\{\gamma_j\}_{j=1}^n$ denote the n boundary curves of Ω, and for convenience, let γ_n denote the outer boundary curve. The harmonic measure functions $\{\omega_j\}_{j=1}^n$ associated to Ω are defined as follows. The function ω_j is the harmonic function that solves the Dirichlet problem with boundary value 1 on γ_j and boundary value 0 on the remaining boundary curves. The associated holomorphic functions $\{F_j'\}$, defined on Ω, are then given by $F_j' = 2\partial\omega_j/\partial z$. The prime that is used in the notation is traditional.

The following result relates the Szegő and Bergman kernels. For a proof, see Bell [**2,** Section 23].

Theorem 3.2. *Let Ω be a bounded, n-connected region with C^∞*

boundary. Then the Szegő kernel and Bergman kernel are related via

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{j,k=1}^{n-1} c_{j,k} F_j'(z)\overline{F_k'(w)}$$

for constants $c_{j,k} \in \mathbf{C}$.

4. The Szegő kernel and Bergman kernel for an annulus. In this section we apply the constructions of the previous sections in order to compute the Szegő and Bergman kernels for an annulus and, for a given annulus, to identify the constant that appears in Theorem 3.2. Let $\mathcal{A}_r = \{z \in \mathbf{C} : r < |z| < 1\}$ for $0 < r < 1$.

For the Szegő kernel, one notes that an orthogonal basis for $H^2(b\mathcal{A}_r)$ is $\{z^j\}_{j\in\mathbf{Z}}$. Then, since

$$\|z^j\|_{b\mathcal{A}_r}^2 = \int_{b\mathcal{A}_r} |z|^{2j}\, ds = 2\pi(1 + r^{2j+1}),$$

an orthonormal basis for $H^2(b\mathcal{A}_r)$ is $\{\psi_j\}_{j\in\mathbf{Z}}$ where $\psi_j(z) = z^j/\sqrt{2\pi(1+r^{2j+1})}$. It follows that the Szegő kernel S_r for $\mathcal{A}_r$ is given by

$$S_r(z,w) = \frac{1}{2\pi}\sum_{j\in\mathbf{Z}} \frac{(z\overline{w})^j}{1+r^{2j+1}}. \tag{2}$$

For the Bergman kernel, one notes that an orthogonal basis for $B^2(\mathcal{A}_r)$ is again $\{z^j\}_{j\in\mathbf{Z}}$. Since

$$\|z^j\|_{\mathcal{A}_r}^2 = \iint_{\mathcal{A}_r} |z|^{2j}\, dA = \begin{cases} \pi(1-r^{2j+2})/(j+1) & \text{if } j \neq -1 \\ -2\pi \log r & \text{if } j = -1, \end{cases}$$

one finds after normalization that the Bergman kernel K_r for $\mathcal{A}_r$ is given by

$$K_r(z,w) = -\frac{1}{2\pi \log r}\frac{1}{z\overline{w}} + \frac{1}{\pi}\sum_{j\neq -1} \frac{(j+1)(z\overline{w})^j}{1-r^{2j+2}}. \tag{3}$$

To identify the constant in Theorem 3.2, one first identifies the harmonic measure functions for $\mathcal{A}_r$ as $\omega_1 = \log|z|/\log r$ and $\omega_2 = -\log|z|/\log r + 1$. The associated holomorphic functions are $F_1'(z) =$

$1/(z\log r)$ and $F_2'(z) = -1/(z\log r)$. Theorem 3.2 asserts for the annulus that there is a constant $c_r \in \mathbf{C}$ for which

$$K_r(z,w) = 4\pi S_r(z,w)^2 + c_r \cdot \frac{1}{z\overline{w}}. \tag{4}$$

To find c_r, substitute $z = -r/\overline{w} \in \mathcal{A}_r$ for fixed $w \in \mathcal{A}_r$. From (2),

$$\begin{aligned}
S_r(-r/\overline{w}, w) &= \frac{1}{2\pi}\sum_{j\in\mathbf{Z}} \frac{(-r)^j}{1+r^{2j+1}} \\
&= \frac{1}{2\pi}\sum_{j\geq 0} \frac{(-1)^j\, r^j}{1+r^{2j+1}} + \frac{1}{2\pi}\sum_{j<0} \frac{(-1)^j\, r^j}{1+r^{2j+1}} \\
&= \frac{1}{2\pi}\sum_{j\geq 0} \frac{(-1)^j\, r^j}{1+r^{2j+1}} + \frac{1}{2\pi}\sum_{k>0} \frac{(-1)^k\, r^{-k}}{1+r^{-2k+1}} \cdot \frac{r^{2k-1}}{r^{2k-1}} \\
&= \frac{1}{2\pi}\sum_{j\geq 0} \frac{(-1)^j\, r^j}{1+r^{2j+1}} + \frac{1}{2\pi}\sum_{k>0} \frac{(-1)^k\, r^{k-1}}{1+r^{2k-1}} \\
&= \frac{1}{2\pi}\sum_{j\geq 0} \frac{(-1)^j\, r^j}{1+r^{2j+1}} + \frac{1}{2\pi}\sum_{l\geq 0} \frac{(-1)^{l+1}\, r^l}{1+r^{2l+1}} \\
&= 0.
\end{aligned}$$

(In fact, this calculation can be used to show that the Ahlfors map f_a from $\mathcal{A}_r$ to the unit disc has zeros at exactly a and $-r/\overline{a}$. See Tegtmeyer and Thomas [**8**].) From (3),

$$K_r(-r/\overline{w}, w) = \frac{1}{2\pi}\frac{1}{r\log r} + \frac{1}{\pi}\sum_{j\neq -1} \frac{(j+1)(-r)^j}{1-r^{2j+2}}.$$

Substituting these in (4) and solving gives

$$\begin{aligned}
c_r &= -\frac{1}{2\pi}\frac{1}{\log r} + \frac{1}{\pi}\sum_{j\neq -1} \frac{(j+1)(-r)^{j+1}}{1-r^{2j+2}} \\
&= -\frac{1}{2\pi}\frac{1}{\log r} + \frac{1}{\pi}\sum_{k\neq 0} \frac{k(-1)^k r^k}{1-r^{2k}} \\
&= -\frac{1}{2\pi}\frac{1}{\log r} + \frac{1}{\pi}\left(\sum_{k>0} \frac{k(-1)^k r^k}{1-r^{2k}} + \sum_{l>0} \frac{-l(-1)^l r^{-l}}{1-r^{-2l}} \cdot \frac{r^{2l}}{r^{2l}}\right)
\end{aligned} \tag{5}$$

$$= -\frac{1}{2\pi}\frac{1}{\log r} + \frac{2}{\pi}\sum_{k>0}\frac{k(-1)^k\, r^k}{1-r^{2k}}.$$

5. Proof of Theorem 1.1. We begin by establishing Theorem 1.1 in the case that both regions are annuli. Subsequently, as any doubly connected region is biholomorphically equivalent to an annulus, we show that the result follows generally after using Theorem 2.1.

Suppose then that $\Omega_1^* = \mathcal{A}_r$ and $\Omega_2^* = \mathcal{A}_R$ for $0 < r, R < 1$, and let $f : \Omega_1^* \to \Omega_2^*$ be a proper holomorphic mapping of order m. It follows that $R = r^m$, and there exists $\theta \in [0, 2\pi)$ so that either $f(z) = e^{i\theta} z^m$ or $f(z) = e^{i\theta} r^m/z^m$. A proof of this fact can be found, for instance, in Narasimhan's book [**7,** subsection 7.1]. In both cases the relationship between kernels as found in the last section is expressed by

$$K_1^*(z, f_j(w)) = 4\pi S_1^*(z, f_j(w))^2 + c_r \frac{1}{\overline{z\overline{f_j(w)}}}$$
$$K_2^*(f(z), w) = 4\pi S_2^*(f(z), w)^2 + c_{r^m} \frac{1}{f(z)\overline{w}},$$

where S_k^* and K_k^* denote the Szegő and Bergman kernels for Ω_k^*, $k = 1, 2$, and where c_r (and c_{r^m}) are given by (5). Substituting these expressions and applying Theorem 3.1, which describes the behavior of the Bergman kernels under proper holomorphic mappings, yields

$$\sum_{j=1}^m S_1^*(z, f_j(w))^2 \overline{f_j'(w)} + \frac{c_r}{4\pi} \sum_{j=1}^m \frac{\overline{f_j'(w)}}{z\overline{f_j(w)}} = f'(z) S_2^*(f(z), w)^2 + \frac{c_{r^m}}{4\pi} \frac{f'(z)}{f(z)\overline{w}}. \tag{6}$$

(We also divided by 4π.) Considering the case $f(z) = e^{i\theta} z^m$, we use local inverses $f_j(w) = \eta_m^j (we^{-i\theta})^{1/m}$, where η_m is the principal mth root of unity, and find that $f'(z)/f(z) = m/z$ and $f_j'(w)/f_j(w) = 1/(mw)$. Considering the case $f(z) = e^{i\theta} r^m/z^m$, we use local inverses $f_j(w) = \eta_m^j\, r(e^{i\theta}/w)^{1/m}$ and find that $f'(z)/f(z) = -m/z$ and $f_j'(w)/f_j(w) = -1/(mw)$. In both cases, we find

$$\sum_{j=1}^m \frac{\overline{f_j'(w)}}{z\overline{f_j(w)}} = \frac{1}{m}\frac{f'(z)}{f(z)\overline{w}} = \frac{4(\log r^m)^2}{m} \frac{\partial(\omega_1^* \circ f)}{\partial z} \overline{\frac{\partial \omega_1^*}{\partial w}}$$

where $\omega_1^* = \log|w|/(\log r^m)$ is the relevant harmonic measure function for Ω_2^*. Then (6) yields

$$
\begin{aligned}
(7) \quad & \sum_{j=1}^{m} S_1^*(z, f_j(w))^2 \overline{f_j'(w)} \\
& \qquad = f'(z) S_2^*(f(z), w)^2 + \frac{c_{r^m} - c_r/m}{\pi} (\log r^m)^2 \, \frac{\partial(\omega_1^* \circ f)}{\partial z} \, \overline{\frac{\partial \omega_1^*}{\partial w}}.
\end{aligned}
$$

Since

$$
c_{r^m} - c_r/m = \frac{2}{\pi}\left(\sum_{k>0} \frac{k(-1)^k r^{mk}}{1 - r^{2mk}} - \frac{1}{m} \sum_{k>0} \frac{k(-1)^k r^k}{1 - r^{2k}} \right),
$$

this is the same as what is claimed by Theorem 1.1.

Now consider the general case of bounded, doubly connected regions Ω_1, Ω_2 with C^∞ boundaries. Let $g : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping of order m. Then there exist annuli $\mathcal{A}_r$, $\mathcal{A}_R$, and biholomorphic maps $\psi : \Omega_1 \to \mathcal{A}_r$, $\phi : \Omega_2 \to \mathcal{A}_R$, so that after composition, $f \stackrel{\text{def}}{=} \phi \circ g \circ \psi^{-1} : \mathcal{A}_r \to \mathcal{A}_R$ is a proper holomorphic mapping of order m, and necessarily, then $R = r^m$. If local inverses for g are denoted by g_j, then corresponding local inverses for f can be expressed as $f_j \stackrel{\text{def}}{=} \psi \circ g_j \circ \phi^{-1}$. The relationships between all these mappings is

$$
\begin{tikzcd}
\xi, \Omega_1 \arrow[r, shift left, "g"] \arrow[d, shift right, "\psi"'] & \eta, \Omega_2 \arrow[l, shift left, dashed, "g_j"] \arrow[d, shift right, "\phi"'] \\
z, \Omega_1^* = \mathcal{A}_r \arrow[r, shift left, "f"] \arrow[u, shift right, "\psi^{-1}"'] & w, \Omega_2^* = \mathcal{A}_{r^m} \arrow[l, shift left, dashed, "f_j"] \arrow[u, shift right, "\phi^{-1}"']
\end{tikzcd}
$$

FIGURE 1. Mappings used in the proof of Theorem 1.1.

illustrated in Figure 1.

The remainder of the proof involves the application of Theorem 2.1 to both sides of (7). To begin, an application of Theorem 2.1 using the biholomorphic map $\psi : \Omega_1 \to \mathcal{A}_r$ gives

$$S_1(\xi, g_j(\eta)) = \psi'(\xi)^{1/2}\, S_1^*(\psi(\xi), (\psi \circ g_j)(\eta))\, \overline{(\psi' \circ g_j)(\eta)^{1/2}}$$

for $\xi \in \Omega_1$, $\eta \in \Omega_2$. Using substitutions $z = \psi(\xi)$, $w = \phi(\eta)$ and rewriting gives

$$S_1^*(z, f_j(w)) = S_1(\xi, g_j(\eta))\, \psi'(\xi)^{-1/2}\, \overline{(\psi' \circ g_j)(\eta)^{-1/2}},$$

so that on the left-hand side of (7) there appears

(8)
$$\begin{aligned}\sum_{j=1}^{m} S_1^*(z, f_j(w))^2 \overline{f_j'(w)} &= \sum_{j=1}^{m} S_1(\xi, g_j(\eta))^2\, \psi'(\xi)^{-1}\, \overline{(\psi' \circ g_j)(\eta)^{-1}}\, \overline{f_j'(w)} \\ &= \sum_{j=1}^{m} S_1(\xi, g_j(\eta))^2\, \psi'(\xi)^{-1}\, \overline{g_j'(\eta)}\, \overline{((\phi^{-1})' \circ \phi)(\eta)} \\ &= \sum_{j=1}^{m} S_1(\xi, g_j(\eta))^2\, \overline{g_j'(\eta)} \cdot \psi'(\xi)^{-1}\, \overline{\phi'(\eta)^{-1}}.\end{aligned}$$

In the second step we applied the chain rule,

$$f_j'(w) = ((\psi \circ g_j \circ \phi^{-1})' \circ \phi)(\eta) = (\psi' \circ g_j)(\eta) \cdot g_j'(\eta) \cdot ((\phi^{-1})' \circ \phi)(\eta).$$

Likewise, an application of Theorem 2.1 to the biholomorphic map $\phi : \Omega_2 \to \mathcal{A}_{r^m}$ gives

$$S_2(g(\xi), \eta) = (\phi' \circ g)(\xi)^{1/2}\, S_2^*((\phi \circ g)(\xi), \phi(\eta))\, \overline{\phi'(\eta)^{1/2}},$$

so that the substitutions give

$$S_2^*(f(z), w) = S_2(g(\xi), \eta)\, (\phi' \circ g)(\xi)^{-1/2}\, \overline{\phi'(\eta)^{-1/2}}.$$

It follows that on the right-hand side of (7) there appears

$$\begin{aligned}f'(z) S_2^*(f(z), w)^2 &= f'(z)\, S_2(g(\xi), \eta)^2\, (\phi' \circ g)(\xi)^{-1}\, \overline{\phi'(\eta)^{-1}} \\ &= g'(\xi)\, ((\psi^{-1})' \circ \psi)(\xi)\, S_2(g(\xi), \eta)^2\, \overline{\phi'(\eta)^{-1}} \\ &= g'(\xi)\, S_2(g(\xi), \eta)^2 \cdot \psi'(\xi)^{-1}\, \overline{\phi'(\eta)^{-1}},\end{aligned} \tag{9}$$

where in the second step we again applied the chain rule,

$$f'(z) = ((\phi \circ g \circ \psi^{-1})' \circ \psi)(\xi) = (\phi' \circ g)(\xi) \cdot g'(\xi) \cdot ((\psi^{-1})' \circ \psi)(\xi).$$

To finish the proof, substitute (8) and (9) into (7), then multiply through by $\psi'(\xi)\,\overline{\phi'(\eta)}$. This gives

$$\sum_{j=1}^{m} S_1(\xi, g_j(\eta))^2 \overline{g_j'(\eta)} = g'(\xi) S_2(g(\xi), \eta)^2 + \lambda_{r,m} \frac{\partial(\omega_1^* \circ f)}{\partial z} \psi'(\xi)\, \overline{\frac{\partial \omega_1^*}{\partial w} \phi'(\eta)}.$$

The proof is complete upon realization that $\omega_1 = \omega_1^* \circ \phi$ is the relevant harmonic measure for Ω_2, and both

$$\frac{\partial \omega_1}{\partial \eta} = \frac{\partial \omega_1^*}{\partial w} \phi'(\eta) \quad \text{and} \quad \frac{\partial(\omega_1 \circ g)}{\partial \xi} = \frac{\partial(\omega_1^* \circ f \circ \psi)}{\partial \xi} = \frac{\partial(\omega_1^* \circ f)}{\partial z} \psi'(\xi).$$

6. Final remarks. In previous work, Chung and Jeong established a transformation law for the Szegő kernel under proper holomorphic mappings between (general) multiply connected regions [**4**]. That law, however, only expresses the relationship between kernels through composition with the Ahlfors maps $\Omega_k \to \Delta$, $k = 1, 2$, and their inverses. Indeed, the proof of that result is gotten through repeated application of Theorem 2.2. Although restricted to doubly connected regions, Theorem 1.1 demonstrates the contribution of the harmonic measure functions and expresses the relationship between kernels directly on the regions Ω_k, $k = 1, 2$.

We also point out recent work of Chung to document other relationships between the Szegő and Bergman kernels involving the harmonic measure functions and Ahlfors map [**3**].

To finish, we point out the difficulty in extending the proof of Theorem 1.1 to the case of regions with higher connectivity. The proof relied on the fact that proper holomorphic maps between annuli can be written explicitly. That there are so few of them (essentially z^{+m} and z^{-m}) can be seen via the Riemann-Hurwitz identity. In particular, by completing each doubly connected region to its double, namely a torus, and by extending the proper map to these tori, one sees that any proper holomorphic map must be unbranched. Already for the case of a map from a triply connected region to a doubly connected region, one needs to allow for branching.

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Department of Mathematics and Statistics, 1740 Knollcrest Circle SE, Calvin College, Grand Rapids, Michigan 49546
Email address: **mbolt@calvin.edu**