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A GEOMETRIC INTERPRETATION AND EXPLICIT FORM FOR HIGHER-ORDER HANKEL OPERATORS

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ABSTRACT. This paper deals with group-theoretic generalizations of classical Hankel operators called higher-order Hankel operators. We relate higher-order Hankel operators to the universal enveloping algebra of the Lie algebra of vector fields on the unit disk. From this novel perspective, higher-order Hankel operators are seen to be linear differential operators. An attractive combinatorial identity is used to find the exact form of these differential operators.

1. Introduction. A classical Hankel operator is a map between Hilbert spaces whose matrix representation is constant along antidiagonals. A survey of these operators and their applications appears in [**8**]. Hankel operators arise naturally in the study of holomorphic function spaces. Given $f(z) = \sum_{j\in\mathbf{Z}} f_j z^j \in L^2(S^1)$ and $x(z) = \sum_{j=1}^{\infty} x_j z^j$, define the operators

$$\mathcal{P}_+ f(z) = \sum_{j=0}^{\infty} f_j z^j, \quad M_x f(z) = x(z) f(z), \quad \mathcal{P}_- f(z) = \sum_{j=1}^{\infty} f_{-j} z^{-j}.$$

The projection $\mathcal{P}_+$ is known as the Cauchy-Szegő projection, and $\mathcal{P}_+ L^2(S^1)$ is the space of holomorphic functions on the open unit disk with square-integrable boundary values. The operator $B_1(x) = \mathcal{P}_+ M_x \mathcal{P}_-$ is a Hankel operator, whose matrix representation we will write

$$B_1(x) = \begin{pmatrix} \vdots & & & \cdot \\ x_3 & & \cdot & \\ x_2 & x_3 & \cdot & \\ x_1 & x_2 & x_3 & \dots \end{pmatrix}.$$

The function x is called the symbol of $B_1(x)$, and the map $x \mapsto B_1(x)$ is conformally equivariant in a sense explained in Section 2. Group-theoretic generalizations of this map, which are also conformally

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equivariant, were discovered in [**5**]. Here, we derive the following explicit expressions for these higher-order Hankel operators.

Theorem 1.1. *The higher-order Hankel operator of order* $s+1$ *with symbol* $x(z)(dz)^{-s}$, *where* $x(z) = \sum_{j=s+1}^{\infty} x_j z^{s+j}$, *has the formula*

$$B_{s+1}(x) = \mathcal{P}_+ L_s(x) \mathcal{P}_-,$$

where $L_s(x)$ *is the differential operator*

$$L_s(x) = \sum_{j=0}^{s} \frac{1}{s!} \binom{s}{j} \binom{s+j}{j} x^{(s-j)} \left(\frac{\partial}{\partial z} \right)^j .$$

In the case $s = 1$, this operator has the matrix representation

$$B_2(x) = \begin{pmatrix} \vdots & & & & \cdot & \\ 4x_5 & & & \cdot & & \\ 3x_4 & 2x_5 & \cdot & & & \\ 2x_3 & x_4 & 0 & & & \\ x_2 & 0 & -x_4 & -2x_5 & & \\ 0 & -x_2 & -2x_3 & -3x_4 & -4x_5 & \dots \end{pmatrix}.$$

The higher-order Hankel forms introduced in [**5**] are related to the transvectants $\tau_{k,l}^j$ of classical invariant theory [**3**] and to the Rankin-Cohen brackets appearing in the theory of modular forms [**1, 10**]. Our map B_{s+1} is the adjoint of the transvectant $\tau_{1/2,1/2}^s$. These objects and their relationships have been studied from many perspectives [**2, 6, 7, 9, 11**]. While the content of Theorem 1.1 is basically known, our method of proof, viewing $B_{s+1}(x)$ as an element of a universal enveloping algebra, appears to have some novelty.

The rest of the paper is as follows. In Section 2, we introduce spaces $H_{L^2}^m(\Delta)$ of sections of line bundles which are isomorphic to weighted Bergman spaces, and a group action on them. In Section 3, we outline the proof of Theorem 1.1, which involves first showing that B_{s+1} is a linear differential operator of order $\leq s$, and then determining its coefficients. In Section 4, we find embeddings of the spaces $H_{L^2}^{-s}(\Delta)$

into the universal enveloping algebra of the Lie algebra of vector fields on the unit disk. These embeddings determine the form of B_{s+1}. In Section 5, we find the image under B_{s+1} of a key element of $H^{-s}_{L^2}(\Delta)$. In Section 6, we prove Theorem 1.1, using several identities for binomial coefficients. In Section 7, we present a pair of binomial coefficient identities resulting from Theorem 1.1, and in Section 8 we relate the maps B_{s+1} to transvectants and higher-order Hankel forms.

2. Notation. The group

$$G = PSU(1,1) = \left\{ g = \pm \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}$$

acts on the Riemann sphere $\widehat{\mathbf{C}}$ by linear fractional transformations,

$$g : z \longmapsto \frac{\overline{b} + \overline{a}z}{a + bz}.$$

Let Δ and Δ^* denote the open unit disks around 0 and infinity. The decomposition

$$\widehat{\mathbf{C}} = \Delta^* \sqcup S^1 \sqcup \Delta, \tag{1}$$

is stable under the action of G. Restricting its action to Δ identifies G with the group of conformal automorphisms of Δ.

For each half-integer m, the action of G on $\widehat{\mathbf{C}}$ lifts to an action of $SU(1,1)$ on κ^m, the mth (tensor) power of the canonical bundle. This induces an action of $SU(1,1)$ on the space of holomorphic sections of $\kappa^m|_\Delta$, the holomorphic differentials of degree m on Δ. We denote such a differential by $f(z)(dz)^m$, and the space of all such differentials by $H^m(\Delta)$. The action of $g \in SU(1,1)$ on $H^m(\Delta)$ is

$$g : f(z)(dz)^m \longmapsto f\left(\frac{-\overline{b} + az}{\overline{a} - bz}\right)(\overline{a} - bz)^{-2m}(dz)^m. \tag{2}$$

The action of $SU(1,1)$ on $H^m(\Delta)$ is essentially unitary for $m > 0$; the dense Hilbert subspace of $H^m(\Delta)$ with Hermitian inner product

$$\langle f(z)(dz)^m, g(z)(dz)^m \rangle_m = \begin{cases} \int_\Delta f(z)\overline{g}(z)(1-|z|^2)^{2m-2}\, dz\, d\overline{z} & m = 1, 3/2, 2, \ldots, \\ \int_{S^1} f(z)\overline{g}(z)\, dz, & m = 1/2 \end{cases} \tag{3}$$

will be denoted $H^m_{L^2}(\Delta)$. Thus, $f(z)(dz)^m \in H^m_{L^2}(\Delta)$ if and only if $f(z)$ belongs to the weighted Bergman space $A^2_{2m-2}(\Delta)$.

The space of sections of $\kappa^{1/2}|_{S^1}$ will be denoted $\Omega^{1/2}(S^1)$. The actions of $SU(1,1)$ and the $SU(1,1)$-invariant Hermitian inner product on this space are also given by (2) and (3), with $m = 1/2$. We will denote the Hilbert subspace by $\Omega^{1/2}_{L^2}(S^1)$, so that $f(z)(dz)^{1/2} \in \Omega^{1/2}_{L^2}(S^1)$ if and only if $f(z) \in L^2(S^1)$.

The decomposition (1) corresponds to an $SU(1,1)$-stable decomposition

$$\Omega^{1/2}_{L^2}(S^1) = H^{1/2}_{L^2}(\Delta) \oplus H^{1/2}_{L^2}(\Delta^*). \tag{4}$$

We abuse notation slightly by using $\mathcal{P}_+$ to denote the projection onto $H^{1/2}_{L^2}(\Delta)$, and $\mathcal{P}_-$ to denote the projection onto $H^{1/2}_{L^2}(\Delta^*)$. As noted, $\mathcal{P}_+ f(z)(dz)^{1/2} = f_+(z)(dz)^{1/2}$, where $f_+(z)$ is the Cauchy-Szegő projection (Cauchy transform) of $f(z)$.

If $\theta = f(z)(dz)^{1/2} \in H^{1/2}_{L^2}$ and $\eta = g(z)(dz)^{1/2} \in H^{1/2}_{L^2}(\Delta^*)$, then $\theta\eta = fg\,dz$ is a one density on S^1 that can be integrated to a nontrivial constant, so $(H^{1/2}_{L^2}(\Delta))^* = H^{1/2}_{L^2}(\Delta^*)$. The decomposition (4) induces a diagonal action of $SU(1,1)$ on Hilbert-Schmidt operators sending $H^{1/2}_{L^2}(\Delta)$ to $H^{1/2}_{L^2}(\Delta)$, via the identifications

$$\begin{aligned}\mathcal{L}_2\left(H^{1/2}_{L^2}(\Delta^*), H^{1/2}_{L^2}(\Delta)\right) &= H^{1/2}_{L^2}(\Delta) \otimes \left(H^{1/2}_{L^2}(\Delta)\right)^* \\ &= H^{1/2}_{L^2}(\Delta) \otimes H^{1/2}_{L^2}(\Delta).\end{aligned}$$

Calculating the character of the rotation group $S^1 \subset PSU(1,1)$, one sees that

$$H^{1/2}_{L^2}(\Delta) \otimes H^{1/2}_{L^2}(\Delta) = H^1_{L^2}(\Delta) \oplus H^2_{L^2}(\Delta) \oplus H^3_{L^2}(\Delta) \oplus \cdots$$

so for each $m \in \mathbf{N}$ there is an intertwining map

$$H^m_{L^2}(\Delta) \to \mathcal{L}_2(H^{1/2}_{L^2}(\Delta^*), H^{1/2}_{L^2}(\Delta)).$$

Let $x \in H^0(\Delta)/\mathbf{C}$. Then x acts on $\Omega^{1/2}(S^1)$ by multiplication. With respect to the decomposition (4), the multiplication operator M_x can be written

$$M_x = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

$B = B_1(x) = \mathcal{P}_+ M_x \mathcal{P}_- \in \mathcal{L}_2(H^{1/2}_{L^2}(\Delta^*), H^{1/2}_{L^2}(\Delta))$ is the Hankel operator associated to x. The action of G on $H^0(\Delta)/\mathbf{C}$ intertwines with the diagonal action of $SU(1,1)$ on $\mathcal{L}_2(H^{1/2}_{L^2}(\Delta^*), H^{1/2}_{L^2}(\Delta))$, and $B_1(x)$ is Hilbert-Schmidt precisely when $\theta = x'dz \in H^1_{L^2}(\Delta)$. Thus B_1 is an $SU(1,1)$-equivariant map from $H^1_{L^2}(\Delta)$ to $\mathcal{L}_2(H^{1/2}_{L^2}(\Delta^*), H^{1/2}_{L^2}(\Delta))$. This motivates the following.

Definition 2.1. The Hankel operator of order m with symbol $\theta \in H^m_{L^2}(\Delta)$ is the image of θ under the intertwining map

$$H^m_{L^2}(\Delta) \longrightarrow \mathcal{L}_2(H^{1/2}_{L^2}(\Delta^*), H^{1/2}_{L^2}(\Delta)) = H^{1/2}_{L^2}(\Delta) \otimes H^{1/2}_{L^2}(\Delta).$$

The space of Hankel operators of order m is the image of $H^m_{L^2}(\Delta)$ under this map.

3. An outline of the proof. Our geometric interpretation of higher-order Hankel operators relies upon two facts. First, for $m > 0$, the action of $SU(1,1)$ on $H^m(\Delta)$ is essentially irreducible. But if $m = -s$ for $s \in \mathbf{N}$, there is a short exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbf{C}^{2s+1} &= \text{span}\left\{(dz)^{-s}, \dots, z^{2s}(dz)^{-s}\right\} \\ &\longrightarrow H^{-s}(\Delta) \xrightarrow{I_s} H^{s+1}(\Delta) \longrightarrow 0. \end{aligned}$$

$H^{-s}(\Delta)/\mathbf{C}^{2s+1}$ can be given a Hilbert space structure via the intertwining map I_s, and we refer to this Hilbert space as $H^{-s}_{L^2}(\Delta)$. We study the maps $B_{s+1} : H^{-s}_{L^2}(\Delta) \to H^{1/2}_{L^2}(\Delta) \otimes H^{1/2}_{L^2}(\Delta)$.

Second, in the case $s = 1$, $H^{-1}(\Delta)$ is the Lie algebra of complex vector fields on the unit disk (note $(dz)^{-1} = \partial/(\partial z)$). These vector fields act on $\Omega^{1/2}(S^1)$ as differential operators of order 1 via the Lie derivative. Composing this action with $\mathcal{P}_+, \mathcal{P}_-$ gives us a Hankel operator of order two. For $\theta = f(z)(dz)^{1/2} \in H^{1/2}_{L^2}(\Delta^*)$ and $x(z)(\partial/\partial z) \in H^{-1}_{L^2}(\Delta)$, a formal calculation leads to

$$B_2(x)\theta := \mathcal{P}_+ \frac{1}{2} L_{x(z)(\partial/\partial z)} f(z)(dz)^{1/2} = \mathcal{P}_+ \left(\frac{1}{2}x'f + xf'\right)(dz)^{1/2}.$$

The key to the higher-order Hankel operators is that the above action of $H^{-1}(\Delta)$ determines an action of $H^{-s}(\Delta)$ on $\Omega^{1/2}(S^1)$, as differential operators of order $\leq s$. Each representation of $H^{-1}(\Delta)$ corresponds to a representation of the universal enveloping algebra $\mathcal{U}(H^{-1}(\Delta))$. The elements of $\mathcal{U}(H^{-1}(\Delta))$ act as linear differential operators on $\Omega^{1/2}(S^1)$. Let $S^s(H^{-1}(\Delta))$ be the space of symmetric tensors of order $\leq s$ over $H^{-1}(\Delta)$. $S(H^{-1}(\Delta)) = \oplus_{s=0}^{\infty} S^s(H^{-1}(\Delta))$ is isomorphic as a filtered vector space to $\mathcal{U}(H^{-1}(\Delta))$, so elements of $S^s(H^{-1}(\Delta))$ can be mapped to linear differential operators of order $\leq s$ on $\Omega^{1/2}(S^1)$. In Section 3, we show the existence of a G-equivariant embedding of $H^{-s}(\Delta)$ into $S^s(H^{-1}(\Delta))$. Thus, the image of $H^{-s}(\Delta)$ under B_{s+1} contains linear differential operators of order $\leq s$.

B_{s+1} is unique up to multiplication by a constant and must map the lowest-weight vector in $H^{-s}(\Delta)/\mathcal{C}^{2s+1}$, namely, $z^{2s+1}(\partial/\partial z)^s$, to the lowest-weight vector l_s of weight $2(s+1)$ in $H^{1/2}_{L^2}(\Delta) \otimes H^{1/2}_{L^2}(\Delta)$. We find the form of l_s in Section 4 and use it to find B_{s+1} in Section 5.

Finally, a technical point. The elements of $H^{-1}(\Delta)$ may not extend to holomorphic vector fields on S^1, so neither $H^{-1}(\Delta)$ nor $\mathcal{U}(H^{-1}(\Delta))$ act naturally on $\Omega^{1/2}(S^1)$ a priori. However, the polynomial sections of $\kappa^{-1}|_{S^1}$, denoted by $H^{-1}_{poly}(\Delta)$, do extend to S^1, and the space of polynomial sections in $H^{-s}(\Delta)$, $H^{-s}_{poly}(\Delta)$, is mapped into $\mathcal{U}(H^{-1}_{poly}(\Delta))$ by the embedding from Section 3. Since $H^{-s}_{poly}(\Delta)$ is dense in $H^{-s}_{L^2}(\Delta)$, the action of $H^{-s}_{poly}(\Delta)$ on $\Omega^{1/2}(S^1)$ extends to an action of $H^{-s}_{L^2}(\Delta)$.

4. The equivariant cross-section of $S^s(H^{-1}(\Delta)) \to H^{-s}(\Delta)$. $S^s(H^{-1}(\Delta))$ sits inside $T^s(H^{-1}(\Delta))$, the space of tensors of order s. We will write a monomial in $S^s(H^{-1}(\Delta))$ as

$$\bigodot_{i=1}^{s} f_i(z)\frac{\partial}{\partial z} = \frac{1}{s!}\sum_{\sigma\in S_s}\left(f_{\sigma(1)}(z)\frac{\partial}{\partial z}\otimes\cdots\otimes f_{\sigma(s)}(z)\frac{\partial}{\partial z}\right),$$

where S_s is the symmetric group on s elements. For each s, $S^s(H^{-1}(\Delta))$ projects onto $H^{-s}(\Delta)$ via the map

$$\mathcal{P}_s : \bigodot_{i=1}^{s} f_i(z)\frac{\partial}{\partial z} \longmapsto \prod_{i=1}^{s} f_i(z)\left(\frac{\partial}{\partial z}\right)^s.$$

Let $d_p = z^p(\partial/\partial z)$. We refer to p as the *power* of d_p. The vectors

$$\left\{ \bigodot_{i=1}^{s} d_{p_i} \;\middle|\; p_1 \geq p_2 \geq \cdots \geq p_s, \quad p_i \in \mathbf{N}\right\}$$

are a basis for $S^s(H^{-1}(\Delta))$. We refer to $\sum_{i=1}^s p_i$ as the *total power* of such a basis vector.

The infinitesimal action of $\mathfrak{sl}(2,\mathbf{C})$ on $H^m(\Delta)$, in terms of the coordinate f, is

$$\begin{aligned} A^- &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \longmapsto -\frac{\partial}{\partial z}, \\ A^+ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \longmapsto z^2 \frac{\partial}{\partial z} + 2mz, \\ E &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longmapsto 2z\frac{\partial}{\partial z} + 2m. \end{aligned}$$

The action of $\mathfrak{sl}(2,\mathbf{C})$ on $T^s(H^{-1}(\Delta))$ is by a Liebniz rule and preserves the subspace of symmetric tensors. We denote this action by $\pi_\odot(X)$. The following lemma is key.

Lemma 4.1. $(\pi_\odot(A^+))^{2s}(d_0)^{\odot s} = C_s(d_2)^{\odot s}$, *for some constant* $C_s \in \mathbf{R}$.

Proof. The raising operator $\pi_\odot(A^+)$ maps a monomial of total power p into a linear combination of monomials having total power $p+1$. Thus,

$$\left(\pi_\odot(A^+)\right)^{2s}(d_0)^{\odot s} = \sum_{p\in\mathcal{P}_s^{2s}} a_p \bigodot_{i=1}^{s} d_{p(i)},$$

where $\mathcal{P}_s^{2s}$ is the set of partitions of $2s$ into s or fewer parts, arranged so that $p(i) \geq p(i+1)$. At the same time, $A^+d_2 = 0$, so none of the parts $p(i)$ can be greater than 2. But there is only one partition of $2s$ into s or fewer parts less than or equal to 2, namely $p(i) = 2$ for all i. $\square$

Proposition 4.2. *For each* s, *there is a* $PSU(1,1)$*-equivariant cross-section of* $S^s(H^{-1}(\Delta)) \to H^{-s}(\Delta)$.

Proof. Our strategy is to map the basis elements of $H^{-s}(\Delta)$, namely, the set $\{z^p(\partial/\partial z)^s\}_{p=0}^\infty$, into $S^s(H^{-1}(\Delta))$, in such a way that the resulting cross-section is equivariant.

It is clear from the action of $\pi_\odot(E)$ on $S^s(H^{-1}(\Delta))$ that $z^p(\partial/\partial z)^s$ must be mapped to an element having total power p. Thus, we map $(\partial/\partial z)^s$ to the only monomial of total power zero, the vector $(d_0)^{\odot s}$. We map $z^{2s+1}(\partial/\partial z)^s$ into a vector v_{2s+1} of total power $2s+1$. The images of all other basis vectors are obtained by applying the raising operator. The resulting cross-section will be equivariant, provided that

$$\pi_\odot(A^-)v_{2s+1} = \left(\pi_\odot(A^+)\right)^{2s}(d_0)^{\odot s} = C_s\,(d_2)^{\odot s}\,. \tag{5}$$

In fact, we take

$$\begin{aligned} v_{2s+1} = &-\frac{C_s}{3}d_3 \odot (d_2)^{\odot(s-1)} \\ &- \sum_{4\le p\le 2s+1} d_p \odot \sum_{\substack{m+n\le s-1\\ 2m+n=2s+1-p}} A_{pn}(d_2)^{\odot m}\odot(d_1)^{\odot n}\odot(d_0)^{\odot(s-m-n-1)}. \end{aligned}$$

For example, for $s=2$, one has

$$-\frac{3}{C_2}v_5 = d_3\odot d_2 - \frac{1}{2}d_4\odot d_1 + \frac{1}{10}d_5\odot d_0.$$

The monomials in v_{2s+1} all have total power $2s+1$. The largest power p ranges from 3 to $2s+1$. Among basis elements with highest power p, the sum includes all those where the remaining powers are no greater than two. Thus, the inner sum is really over partitions of $2s+1-p$ into $s-1$ or fewer parts, each part being 1 or 2. The coefficients A_{pn} are indexed by the highest power and n, the (symmetric tensor) exponent of d_1. If p is even, n is odd, and vice-versa. Modulo parity, n ranges from 0 to the minimum of $p-3$ and $2s+1-p$. Thus, the range of n increases until $p=s+2$, and then decreases to $n=0$ when $p=2s+1$. For all other n,p, take $A_{pn}=0$.

Because the monomials in v_{2s+1} are all products of d_p, d_2, d_1 and d_0, applying $\pi_\odot(A^-)$ to each monomial results in at most three terms. In fact,

$$\begin{aligned} \pi_\odot(A^-)v_{2s+1} = C_s\,(d_2)^{\odot(s)} + \left(\frac{2C_s(s-1)}{3} + 4A_{41}\right)d_3 \\ \odot (d_2)^{\odot(s-2)}\odot d_1 \end{aligned}$$

$$+\sum_{4\le q\le 2s} d_q \odot \sum_{\substack{l+k\le s-1\\ 2l+k=2s-q}} B_{qk}(d_2)^{\odot l}\odot(d_1)^{\odot k}\odot(d_0)^{\odot(s-l-k-1)},$$

where

$$B_{qk}=(q+1)A_{(q+1)k}+(2s-q-k+2)A_{q(k-1)}+(k+1)A_{q(k+1)}.$$

Thus, choosing $A_{41}=-[C_s(s-1)]/6$, and defining

$$A_{(p+1)n}=-\frac{1}{p+1}\left[(2s-p-n+2)A_{p(n-1)}+(n+1)A_{p(n+1)}\right]$$

for all other n and p, one obtains (5). This recursion relation is linear and terminates at $p=2s+1$. Thus, it can be solved, and v_{2s+1} exists and satisfies (5). This proves the proposition. □

5. Lowest weight vectors in $\mathcal{L}_2(H^{1/2}_{L^2}(\Delta^*),H^{1/2}_{L^2}(\Delta))$. The action of $\mathfrak{sl}(2,\mathbf{C})$ on $\mathcal{L}_2(H^{1/2}_{L^2}(\Delta^*),H^{1/2}_{L^2}(\Delta))=H^{1/2}_{L^2}(\Delta)\otimes H^{1/2}_{L^2}(\Delta)$ is again by a Liebniz rule and will be denoted $\pi_\otimes(X)$. The irreducible subspaces of $H^{1/2}_{L^2}(\Delta)\otimes H^{1/2}_{L^2}(\Delta)$ are lowest-weight representations of $SU(1,1)$. We now identify the vectors in $H^{1/2}_{L^2}(\Delta)\otimes H^{1/2}_{L^2}(\Delta)$ annihilated by $\pi_\otimes(A^-)$.

Proposition 5.1. *The set of vectors*

$$\left\{l_s:=\sum_{i=0}^{s}(-1)^i\binom{s}{i}z^{s-i}(dz)^{1/2}\otimes z^i(dz)^{1/2}\right\}_{s=0}^{\infty}$$

are annihilated by the operator $\pi_\otimes(A^-)$. The vector l_s has weight $2(s+1)$.

Proof. Let $b_p=z^p(dz)^{1/2}$. Applying $\pi_\otimes(A^-)$ to $b_{s-i}\otimes b_i$ results in two terms unless $i=0$ or $i=s$, so we pull these cases out of the sum l_s. Thus,

$$\begin{aligned}-\pi_\otimes(A^-)\,[l_s]&=sb_{s-1}\otimes b_0\\&\quad+\sum_{i=1}^{s-1}(-1)^i\binom{s}{i}\pi_\otimes(A^-)\,[b_{s-i}\otimes b_i]\\&\quad+(-1)^s sb_0\otimes b_{s-1}.\end{aligned}$$

The middle term is

$$\begin{aligned}
\sum_{i=1}^{s-1}(-1)^i\binom{s}{i}&\left[(s-i)b_{s-i-1}\otimes b_i + ib_{s-i}\otimes b_{i-1}\right]\\
&= \sum_{i=1}^{s-1}(-1)^i\frac{s\cdots(s-i)}{i!}b_{s-i-1}\otimes b_i\\
&\quad + \sum_{i=1}^{s-1}(-1)^i\frac{s\cdots(s-i+1)}{(i-1)!}b_{s-i}\otimes b_{i-1}\\
&= \sum_{i=1}^{s-1}(-1)^i\frac{s\cdots(s-i)}{i!}b_{s-i-1}\otimes b_i\\
&\quad + \sum_{j=0}^{s-2}(-1)^{j+1}\frac{s\cdots(s-j)}{j!}b_{s-j-1}\otimes b_j\\
&= -sb_{s-1}\otimes b_0\\
&\quad + \sum_{j=1}^{s-2}\left[(-1)^j+(-1)^{j+1}\right]\frac{s\cdots(s-j)}{j!}b_{s-j-1}\otimes b_j\\
&\quad + (-1)^{s-1}sb_0\otimes b_{s-1}\\
&= -sb_{s-1}\otimes b_0 + (-1)^{s-1}sb_0\otimes b_{s-1}.
\end{aligned}$$

Thus,

$$A^-\left[l_s\right] = sb_{s-1}\otimes b_0 - sb_{s-1}\otimes b_0 + (-1)^{s-1}sb_0\otimes b_{s-1} + (-1)^s sb_0\otimes b_{s-1} = 0.$$

To prove the second claim, simply notice that

$$E\left[b_{s-i}\otimes b_i\right] = 2(s-i+i+1)b_{s-i}\otimes b_i = 2(s+1)b_{s-i}\otimes b_i. \qquad \square$$

6. An explicit formula for B_{s+1}. We will use the following lemma to prove Theorem 1.1. For $x(z)(\partial/\partial z)^s \in H^{-s}_{L^2}(\Delta)$, let

$$\mathcal{O}_j\left(x\right) = \mathcal{P}_+ x^{(s-j)}\left(\frac{\partial}{\partial z}\right)^j \in \mathcal{L}_2(H^{1/2}_{L^2}(\Delta^*), H^{1/2}_{L^2}(\Delta)).$$

Lemma 6.1. *Let $k > s - j$. As an element of $H^{1/2}_{L^2}(\Delta) \otimes H^{1/2}_{L^2}(\Delta)$,*

$$\mathcal{O}_j\left(z^k\right) = \sum_{i=0}^{k-s-1} (-1)^j \frac{(i+j)!}{i!} \frac{k!}{(k-s+j)!} z^{(k-s-1)-i} (dz)^{1/2} \otimes z^i (dz)^{1/2}.$$

Proof. Let $f(z)(dz)^{1/2} \in H^{1/2}_{poly}(\Delta^*)$, with $f(z) = \sum_{n=1}^{N} f_{-n} z^{-n}$ for $N > k - s$. Then,

$$\left(\frac{\partial}{\partial z}\right)^j f(z) = \sum_{n=1}^{N} (-1)^j \frac{(n+j-1)!}{(n-1)!} f_{-n} z^{-(n+j)}.$$

Also,

$$\left(\frac{\partial}{\partial z}\right)^{s-j} z^k = \frac{k!}{(k-s+j)!} z^{k-s+j}.$$

Thus,

$$\left[\left(\frac{\partial}{\partial z}\right)^{s-j} z^k\right]\left[\left(\frac{\partial}{\partial z}\right)^j f(z)\right] = \sum_{n=1}^{N} (-1)^j \frac{(n+j-1)!}{(n-1)!} \frac{k!}{(k-s+j)!} f_{-n} z^{k-s-n},$$

and so

$$\mathcal{O}_j\left(z^k\right) f(z)(dz)^{1/2} = \sum_{n=1}^{k-s} (-1)^j \frac{(n+j-1)!}{(n-1)!} \frac{k!}{(k-s+j)!} f_{-n} z^{k-s-n} (dz)^{1/2}.$$

Since

$$f_{-n} z^{k-s-n} (dz)^{1/2} = \left(z^{k-s-n}(dz)^{1/2} \otimes z^{n-1}(dz)^{1/2}\right) f(z)(dz)^{1/2},$$

the required formula is obtained after reindexing. But this formula depends only upon the first $k - n$ coefficients of f, so it applies to all of $H^{1/2}_{L^2}(\Delta^*)$. $\square$

Proof of Theorem 1.1. By Proposition 4.2, we know $B_{s+1}(v) = \mathcal{P}_+ L_s(v)$, where

$$L_s(v) = \sum_{j=0}^{s} c_j(v) \left(\frac{\partial}{\partial z} \right)^j .$$

The cases $s = 0$ and $s = 1$ suggest that $c_j(v) = a_j v^{(s-j)}$. Since B_{s+1} is unique, we need only find coefficients a_j which satisfy

$$\mathcal{P}_+ L_s \left(z^{2s+1} \right) = l_s. \tag{6}$$

By Proposition 5.1 and Lemma 6.1, (6) is equivalent to

$$\sum_{j=0}^{s} a_j \sum_{i=0}^{s} (-1)^j \frac{(i+j)!}{i!} \frac{(2s+1)!}{(s+j+1)!} z^{s-i} (dz)^{1/2} \otimes z^i (dz)^{1/2} = \sum_{i=0}^{s} (-1)^i \binom{s}{i} z^{s-i} (dz)^{1/2} \otimes z^i (dz)^{1/2}.$$

In other words, the a_j's must solve the linear system

$$M_s \vec{a}_s := \left((-1)^j \frac{(i+j)!}{i!} \frac{(2s+1)!}{(s+j+1)!} \right)_{i,j=0}^{s} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_s \end{pmatrix} = \begin{bmatrix} 1 \\ -s \\ \vdots \\ (-1)^s \end{bmatrix} =: \vec{l}_s.$$

Pleasantly, the matrix M_s is easily factored. First, we rewrite M_s as

$$N_s D_s := \left(\frac{(i+j)!}{i!j!} \right)_{i,j=0}^{s} \operatorname{diag} \left(\left((-1)^j j! \frac{(2s+1)!}{(s+j+1)!} \right)_{j=0}^{s} \right)_0 .$$

Now, a beautiful combinatorial identity comes into play, and one has

$$N_s = \left(\binom{i+j}{i} \right)_{i,j=0}^{s} = \left(\binom{i}{j} \right)_{i,j=0}^{s} \left(\binom{j}{i} \right)_{i,j=0}^{s} =: L_s U_s,$$

using (5.23) from [**4**]. Let

$$\widetilde{L}_s = \left((-1)^{i+j} \binom{i}{j} \right)_{i,j=0}^{s},$$

and let $P = L_s\widetilde{L}_s$. Then the (i,j)th entry of P is

$$\begin{aligned} p_{ij} &= \sum_{k=j}^{i}(-1)^{k+j}\binom{i}{k}\binom{k}{j} \\ &= \binom{i}{j}\sum_{k=j}^{i}(-1)^{k+j}\binom{i-j}{i-k} \\ &= \binom{i}{j}\sum_{k=0}^{i-j}(-1)^{k}\binom{i-j}{k} \\ &= \delta_{ij}, \end{aligned}$$

since $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = (1-1)^n = \delta_{n0}$. Thus, $\widetilde{L}_s = L_s^{-1}$. Analogously,

$$U_s^{-1} = \left((-1)^{i+j}\binom{j}{i}\right)_{i,j=0}^{s}.$$

Now,

$$\begin{aligned} L_s^{-1}\vec{l}_s &= \left(\sum_{k=0}^{s}(-1)^{j+k}(-1)^k\binom{j}{k}\binom{s}{k}\right)_{j=0}^{s} \\ &= \left((-1)^j\binom{s+j}{j}\right)_{j=0}^{s}, \end{aligned}$$

again using (5.23) from [**4**]. Next,

$$\begin{aligned} U_s^{-1}L_s^{-1}\vec{l}_s &= \left((-1)^j\sum_{k=0}^{s}\binom{k}{j}\binom{s+k}{k}\right)_{j=0}^{s} \\ &= \left((-1)^j\binom{s+j}{j}\sum_{k=j}^{s}\binom{s+k}{s+j}\right)_{j=0}^{s} \\ &= \left((-1)^j\binom{s+j}{j}\binom{2s+1}{s+j+1}\right)_{j=0}^{s}, \end{aligned}$$

using upper summation ([**4**, (5.10)]), and finally,

$$\begin{aligned} \vec{a}_s = M_s^{-1}l_s &= \left(\frac{(s+j+1)!}{j!(2s+1)!}\binom{s+j}{j}\binom{2s+1}{s+j+1}\right)_{j=0}^{s} \\ &= \left(\frac{1}{s!}\binom{s}{j}\binom{s+j}{j}\right)_{j=0}^{s}. \qquad \square \end{aligned}$$

7. Binomial coefficient identities. The equivariance of B_{s+1} implies a pair of identities relating sums of products of binomial coefficients.

Proposition 7.1. *For $s \in \mathbf{N}$, $k \geq 2s+1$, and $l = 0, \dots, k-s$,*

$$\sum_{j=0}^{s}(-1)^j \binom{s+j}{j}\binom{k}{s-j}\left[\binom{l+j}{j}(k-s) - \binom{l+j-1}{j-1}l\right] = \sum_{j=0}^{s}(-1)^j \binom{s+j}{j}\binom{k+1}{s-j}\binom{l+j}{j}(k-2s),$$

and, for $s \in \mathbf{N}$, $i+j \geq s$,

$$\sum_{l=0}^{s}(-1)^l \binom{s+l}{l}\binom{j+l}{l}\binom{i+j+s+1}{s-l} = \sum_{l=0}^{s}(-1)^l \binom{s}{j}\binom{j}{l}\binom{i}{s-l}.$$

Proof. To prove the first identity, expand the equation

$$B_{s+1}\left(A^+ z^k\right) = \pi_\otimes\left(A^+\right) B_{s+1}\left(z^k\right)$$

in the basis $\{z^i(dz)^{1/2} \otimes z^j(dz)^{1/2}\}$; to prove the second, expand the equation

$$B_{s+1}\left(\left(A^+\right)^{(i+j-s)} z^{2s+1}\right) = \left(\pi_\otimes\left(A^+\right)\right)^{(i+j-s)} B_{s+1}\left(z^{2s+1}\right). \qquad \square$$

8. Connections with prior work. The *transvectant* $\tau^s_{1/2,1/2} = \tau_{s+1}$ is the essentially unique equivariant map $\tau_{s+1} : H^{1/2}_{L^2}(\Delta) \otimes H^{1/2}_{L^2}(\Delta) \to H^{s+1}_{L^2}(\Delta)$. Let $\theta = f(z)(dz)^{1/2}, \eta = g(z)(dz)^{1/2} \in H^{1/2}_{L^2}(\Delta)$. Then

$$\tau_{s+1}(\theta\otimes\eta) = \tau_{s+1}(f,g)(dz)^s{+}1 = \left(\sum_{j=1}^{s}(-1)^j\binom{s}{j}^2 f^{(s-j)}g^{(j)}\right)(dz)^s{+}1.$$

In [**5**], this map is used to construct higher-order Hankel bilinear forms on the space $H^{1/2}_{L^2}(\Delta) \otimes H^{1/2}_{L^2}(\Delta)$. If $\nu = x(z)(\partial/\partial z)^s \in H^{-s}_{L^2}(\Delta)$ and

$\mu = y(z)(dz)^{s+1} \in H^{s+1}_{L^2}(\Delta)$, then $\overline{\nu}\mu = \overline{x}y\,dz$ is a one-density on S^1, and so $(H^{s+1}_{L^2}(\Delta))^* = H^{-s}_{L^2}(\Delta)$. As in [**5**], define the Hankel form of order $s+1$ with symbol x to be

$$K_{s+1}(x)[f,g] = \int_{S^1} \overline{\nu}\tau_{s+1}(\theta\otimes\eta) = \int_{S^1} \overline{x}\tau_{s+1}(f,g)\,dz,$$

where $\nu = x(z)(\partial/\partial z)^s$. Since $B_{s+1}(x)\overline{\theta} \in H^{1/2}_{L^2}(\Delta)$, another bilinear form is defined by

$$\widetilde{K}_{s+1}(x)[f,g] = \langle B_{s+1}(x)\overline{\theta},\eta\rangle_{1/2} = \int_{S^1} \overline{B_{s+1}(x)\overline{f}}g\,d\theta,$$

and $K_{s+1}(x)$ and $\widetilde{K}_{s+1}(x)$ are easily seen to be equivalent. Another expression for $\widetilde{K}_{s+1}(x)$ is

$$\widetilde{K}_{s+1}(x)[f,g] = \langle B_{s+1}(x),\theta\otimes\eta\rangle_{\otimes} = \mathrm{tr}\,\left(B_{s+1}(x)(\theta\otimes\eta)^*\right),$$

where $\langle\ ,\ \rangle_{\otimes}$ is the inner product on $H^{1/2}_{L^2}(\Delta)\otimes H^{1/2}_{L^2}(\Delta)$. Thus, B_{s+1} is the adjoint of the map τ_{s+1}, and we have the diagram

$$\begin{tikzcd}
H^{-s}_{L^2}(\Delta) \arrow[d, "I_s"'] \arrow[dr, "B_{s+1}"] & \\
H^{s+1}_{L^2}(\Delta) & \mathcal{L}_2(H^{1/2}_{L^2}(\Delta^*), H^{1/2}_{L^2}(\Delta)). \arrow[l, "\tau_{s+1}"']
\end{tikzcd}$$

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