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EXISTENCE OF POSITIVE SOLUTIONS OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS ASYMPTOTIC TO ZERO

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ABSTRACT. The article contains some sufficient conditions for the existence of a positive solution of nonlinear neutral differential equations which converges to zero. The main results are presented in two theorems.

1. Introduction. This paper is concerned with the existence of positive solutions of neutral differential equations of the form

$$\frac{d^n}{dt^n}[x(t)-a(t)x(t-\tau)]=(-1)^{n+1}p(t)|x(t-\sigma)|^{\alpha-1}x(t-\sigma),\quad t\geq t_0, \tag{1}$$

where $n>0$ is an integer, $\tau>0$, $\sigma\geq 0$, $\alpha>0$, $a\in C([t_0,\infty),(0,\infty))$, $p\in C(R,(0,\infty))$.

By a solution of equation (1), we mean a function $x\in C([t_1-\tau,\infty),R)$ for some $t_1\geq t_0$, such that $x(t)-a(t)x(t-\tau)$ is n-times continuously differentiable on $[t_1,\infty)$ and such that equation (1) is satisfied for $t\geq t_1$.

The problem of the existence of solutions of neutral differential equations has been studied by many authors in recent years. We refer the reader to [**1–4**] and the references cited therein. However, in many of these papers, only the problem of the existence of solutions which do not converge to zero as $t\to\infty$ is treated. Theorems for the existence of solutions which tend to zero are absent because, using standard methods, difficulties sometimes arise. Due to new techniques, we establish two theorems which solve the problem.

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We also note that every positive solution of the ordinary differential equation

$$x'(t) = p(t)x^{\alpha}(t), \quad p(t) > 0,\ \alpha > 0,$$

has an increasing character. The situation for neutral equation (1) is different. In fact, it may happen that (1) has a positive solution which tends to zero. For example, the equation

$$[x(t) - (1 + e^{c\sigma})e^{-c\tau}x(t - \tau)]' = cx(t - \sigma), \quad t \geq t_0,$$

where $c > 0$, $\tau > 0$, $\sigma \geq 0$, has a solution $x(t) = \exp(-ct)$. So we are interested in the problem which includes the existence of such solutions.

The following fixed point lemma will be used to prove the main results in the next section.

Lemma 1.1 [**1, 2**] (Schauder's fixed point theorem). *Let Ω be a closed, convex and nonempty subset of Banach space X. Let $S : \Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is, there exists an $x \in \Omega$ such that $Sx = x$.*

2. Main results. The main results of this paper are the following.

Theorem 2.1. *Suppose that*

$$\int_{t_0}^{\infty} p(t)\, dt = \infty, \tag{2}$$

$\alpha > 0$, $0 < k_1 \leq k_2$,

$$\int_{t_0}^{\infty} t^{n-1} p(t) \exp\left(- \alpha k_1 \int_{t_0}^{t-\sigma} p(s)\, ds\right) dt < \infty \tag{3}$$

and there exists a $\gamma \geq 0$ such that

$$\frac{k_1}{k_2} \exp\left((k_2 - k_1) \int_{t_0-\gamma}^{t_0} p(t)\, dt\right) \geq 1, \tag{4}$$

$$\begin{aligned}
&\exp\Big(-k_2\int_{t-\tau}^{t}p(s)\,ds\Big)+\frac{1}{(n-1)!}\exp\Big(k_2\int_{t_0-\gamma}^{t-\tau}p(s)\,ds\Big)\\
&\quad\times\int_t^\infty (s-t)^{n-1}p(s)\exp\Big(-\alpha k_1\int_{t_0-\gamma}^{s-\sigma}p(\xi)\,d\xi\Big)\,ds\\
&\quad\le a(t)\\
&\quad\le \exp\Big(-k_1\int_{t-\tau}^{t}p(s)\,ds\Big)+\frac{1}{(n-1)!}\exp\Big(k_1\int_{t_0-\gamma}^{t-\tau}p(s)\,ds\Big)\\
&\qquad\times\int_t^\infty (s-t)^{n-1}p(s)\exp\Big(-\alpha k_2\int_{t_0-\gamma}^{s-\sigma}p(\xi)\,d\xi\Big)\,ds,\\
&\qquad\qquad t\ge t_0.
\end{aligned}$$

Then equation (1) *has a positive solution which tends to zero.*

Proof. We choose $t_1 \ge t_0 + \max\{\tau, \sigma\}$ and set

$$\begin{aligned}
u(t)&=\exp\Big(-k_2\int_{t_0-\gamma}^{t}p(s)\,ds\Big),\\
v(t)&=\exp\Big(-k_1\int_{t_0-\gamma}^{t}p(s)\,ds\Big),\quad t\ge t_0.
\end{aligned}$$

Let $C([t_0,\infty),R)$ be the set of all continuous bounded functions with the norm $||x|| = \sup_{t\ge t_0}|x(t)|$. Then $C([t_0,\infty),R)$ is a Banach space. We define a closed, bounded and convex subset Ω of $C([t_0,\infty),R)$ as follows:

$$\Omega=\{x=x(t)\in C([t_0,\infty),R): u(t)\le x(t)\le v(t),\ t\ge t_0\}.$$

Define the map $S:\Omega\to C([t_0,\infty),R)$ as follows:

$$(Sx)(t)=\begin{cases} a(t)x(t-\tau)-[1/(n-1)!] & \\ \quad \int_t^\infty (s-t)^{n-1}p(s)x^\alpha(s-\sigma)\,ds & t\ge t_1,\\ (Sx)(t_1)+v(t)-v(t_1) & t_0\le t\le t_1.\end{cases}$$

We shall show that, for any $x\in\Omega$, we have $Sx\in\Omega$. For every $x\in\Omega$

and $t \geq t_1$, we get

$$\begin{aligned}(Sx)(t) &\leq a(t)v(t-\tau) - \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}p(s)u^\alpha(s-\sigma)\,ds \\ &= a(t)\exp\left(-k_1\int_{t_0-\gamma}^{t-\tau} p(s)\,ds\right) \\ &\quad - \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}p(s)\exp\left(-\alpha k_2\int_{t_0-\gamma}^{s-\sigma} p(\xi)\,d\xi\right)ds \\ &\leq v(t).\end{aligned}$$

For $t \in [t_0, t_1]$, we have

$$(Sx)(t) = (Sx)(t_1) + v(t) - v(t_1) \leq v(t).$$

Furthermore, for $t \geq t_1$, we obtain

$$\begin{aligned}(Sx)(t) &\geq a(t)u(t-\tau) - \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}p(s)v^\alpha(s-\sigma)\,ds \\ &= a(t)\exp\left(-k_2\int_{t_0-\gamma}^{t-\tau} p(s)\,ds\right) \\ &\quad - \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}p(s)\exp\left(-\alpha k_1\int_{t_0-\gamma}^{s-\sigma} p(\xi)\,d\xi\right)ds \\ &\geq u(t).\end{aligned}$$

Finally, let $t \in [t_0, t_1]$, and set

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1).$$

Then, with regard to (4), it follows that

$$\begin{aligned}H'(t) &= v'(t) - u'(t) = -k_1p(t)v(t) + k_2p(t)u(t) \\ &= p(t)v(t)\left[-k_1 + k_2u(t)\exp\left(k_1\int_{t_0-\gamma}^{t} p(s)\,ds\right)\right] \\ &= p(t)v(t)\left[-k_1 + k_2\exp\left((k_1-k_2)\int_{t_0-\gamma}^{t} p(s)\,ds\right)\right] \\ &\leq p(t)v(t)\left[-k_1 + k_2\exp\left((k_1-k_2)\int_{t_0-\gamma}^{t_0} p(s)\,ds\right)\right] \leq 0, \\ &\qquad t_0 \leq t \leq t_1.\end{aligned}$$

Since $H(t_1) = 0$ and $H'(t) \leq 0$ for $t \in [t_0, t_1]$, this leads to the conclusion that

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1) \geq 0, \quad t_0 \leq t \leq t_1.$$

Then, for $t \in [t_0, t_1]$ and any $x \in \Omega$, we get

$$(Sx)(t) = (Sx)(t_1) + v(t) - v(t_1) \geq u(t_1) + v(t) - v(t_1) \geq u(t).$$

Thus, we have proved that $Sx \in \Omega$ for any $x \in \Omega$.

We now show that S is continuous. Let $x_i = x_i(t) \in \Omega$ be such that $x_i(t) \to x(t)$ as $i \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \geq t_1$, we have

$$\begin{aligned}
&|(Sx_i)(t) - (Sx)(t)| \\
&\qquad \leq a(t)|x_i(t-\tau) - x(t-\tau)| \\
&\qquad\quad + \frac{1}{(n-1)!}\left| \int_t^\infty (s-t)^{n-1} p(s)[x_i^\alpha(s-\sigma) - x^\alpha(s-\sigma)]\, ds \right| \\
&\qquad \leq a(t)|x_i(t-\tau) - x(t-\tau)| \\
&\qquad\quad + \frac{1}{(n-1)!} \int_{t_1}^\infty (s-t)^{n-1} p(s)|x_i^\alpha(s-\sigma) - x^\alpha(s-\sigma)|\, ds.
\end{aligned}$$

According to (3), we get $|x_i^\alpha(t-\sigma) - x^\alpha(t-\sigma)| \to 0$ as $i \to \infty$. By applying the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{i\to\infty} ||(Sx_i)(t) - (Sx)(t)|| = 0.$$

This means that S is continuous.

Next we show that $S\Omega$ is relatively compact. It is sufficient to show by Arzela's lemma that the family of functions $\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness follows from the definition of Ω. For the equicontinuity we only need to show that, for any given $\varepsilon > 0$, the interval $[t_0, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have a change of amplitude less than ε. Then, for $x \in \Omega$ and any $\varepsilon > 0$, we take $t^* \geq t_1$ large enough so that

$$(Sx)(t^*) \leq \frac{\varepsilon}{2}.$$

For $x \in \Omega, T_2 > T_1 \geq t^*$, we have

$$|(Sx)(T_2) - (Sx)(T_1)| \leq |(Sx)(T_2)| + |(Sx)(T_1)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For $x \in \Omega$ and $t_1 \leq T_1 < T_2 \leq t^*$, we get

$$\begin{aligned} |(Sx)(T_2) - (Sx)(T_1)| & \\ &\leq |a(T_2)x(T_2-\tau) - a(T_1)x(T_1-\tau)| \\ &\quad + \frac{1}{(n-1)!}\int_{T_1}^{T_2} s^{n-1}p(s)x^{\alpha}(s-\sigma)\,ds \\ &\leq |a(T_2)x(T_2-\tau) - a(T_1)x(T_1-\tau)| \\ &\quad + \max_{t_1 \leq s \leq t^*}\left\{\frac{1}{(n-1)!}s^{n-1}p(s)x^{\alpha}(s-\sigma)\right\}(T_2 - T_1). \end{aligned}$$

Thus, there exists a $\delta > 0$ such that

$$|(Sx)(T_2) - (Sx)(T_1)| < \varepsilon \quad \text{if} \quad 0 < T_2 - T_1 < \delta.$$

Finally, for any $x \in \Omega$, $t_0 \leq T_1 < T_2 \leq t_1$, there exists a $\delta > 0$ such that

$$|(Sx)(T_2) - (Sx)(T_1)| = |v(T_1) - v(T_2)| < \varepsilon \quad \text{if} \quad 0 < T_2 - T_1 < \delta.$$

Then $\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S\Omega$ is a relatively compact subset of $C([t_0, \infty), R)$. By Lemma 1.1, there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$. We see that $x_0(t)$ is a positive solution of equation (1)which tends to zero. The proof is complete. $\square$

Corollary 2.1. *Suppose that $\alpha > 0$, $k > 0$, (2) and (3) hold, and*

$$\begin{gathered} a(t) = \exp\left(-k\int_{t-\tau}^{t} p(s)\,ds\right) + \frac{1}{(n-1)!}\exp\left(k\int_{t_0}^{t-\tau} p(s)\,ds\right) \\ \times \int_t^{\infty} (s-t)^{n-1}p(s)\exp\left(-\alpha k\int_{t_0}^{s-\sigma} p(\xi)\,d\xi\right)ds, \\ t \geq t_0. \end{gathered}$$

Then equation (1) *has a solution*

$$x(t) = \exp\left(- k \int_{t_0}^{t} p(s)\,ds \right), \quad t \geq t_0.$$

Proof. We put $k_1 = k_2 = k$, $\gamma = 0$ and apply Theorem 2.1. $\square$

Theorem 2.2. *Suppose that* $\alpha > 0$, $0 < k_1 \leq k_2$, (2) *and* (3) *hold, there exists a* $\gamma \geq 0$ *such that* (4) *holds, and*

$$\begin{aligned}
&\exp\left(- k_1 \int_{t-\tau}^{t} p(s)\,ds \right) + \frac{1}{(n-1)!} \exp\left(k_1 \int_{t_0-\gamma}^{t-\tau} p(s)\,ds \right) \\
&\quad \times \int_t^{\infty} (s-t)^{n-1} p(s) \exp\left(- \alpha k_1 \int_{t_0-\gamma}^{s-\sigma} p(\xi)\,d\xi \right) ds \leq a(t) \\
&\quad \leq \exp\left(- k_2 \int_{t-\tau}^{t} p(s)\,ds \right) + \frac{1}{(n-1)!} \exp\left(k_2 \int_{t_0-\gamma}^{t-\tau} p(s)\,ds \right) \\
&\quad \times \int_t^{\infty} (s-t)^{n-1} p(s) \exp\left(- \alpha k_2 \int_{t_0-\gamma}^{s-\sigma} p(\xi)\,d\xi \right) ds, \\
&\qquad\qquad t \geq t_0.
\end{aligned}$$

Then equation (1) *has a positive solution which tends to zero.*

Proof. We choose $t_1 \geq t_0 + \max\{\tau, \sigma\}$, and set

$$\begin{aligned}
u(t) &= \exp\left(- k_2 \int_{t_0-\gamma}^{t} p(s)\,ds \right), \\
v(t) &= \exp\left(- k_1 \int_{t_0-\gamma}^{t} p(s)\,ds \right), \quad t \geq t_0.
\end{aligned}$$

Let $C([t_0, \infty), R)$ be the set as in the proof of Theorem 2.1. We define a closed, bounded and convex subset Ω of $C([t_0, \infty), R)$ as in the proof of Theorem 2.1. Define the map $S : \Omega \to C([t_0, \infty), R)$ as follows:

$$(Sx)(t) = \begin{cases} [1/a(t+\tau)]x(t+\tau) + [1/a(t+\tau)(n-1)!] & \\ \quad \times \int_{t+\tau}^{\infty} (s-t-\tau)^{n-1} p(s) x^{\alpha}(s-\sigma)\,ds & t \geq t_1, \\ (Sx)(t_1) + v(t) - v(t_1) & t_0 \leq t \leq t_1. \end{cases}$$

We now shall show that, for any $x \in \Omega$, $Sx \in \Omega$. For every $x \in \Omega$ and $t \geq t_1$, we have

$$\begin{aligned}
&(Sx)(t) \\
&\leq \frac{1}{a(t+\tau)} \bigg[v(t+\tau) + \frac{1}{(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^{n-1} p(s) v^{\alpha}(s-\sigma)\, ds \bigg] \\
&= \frac{1}{a(t+\tau)} \bigg[\exp\bigg(-k_1 \int_{t_0-\gamma}^{t+\tau} p(s)\, ds \bigg) \\
&\qquad + \frac{1}{(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^{n-1} p(s) \\
&\qquad\qquad \times \exp\bigg(-\alpha k_1 \int_{t_0-\gamma}^{s-\sigma} p(\xi)\, d\xi \bigg)\, ds \bigg] \leq v(t).
\end{aligned}$$

For $t \in [t_0, t_1]$, we obtain

$$(Sx)(t) = (Sx)(t_1) + v(t) - v(t_1) \leq v(t).$$

Furthermore, for $t \geq t_1$, we get

$$\begin{aligned}
&(Sx)(t) \\
&\geq \frac{1}{a(t+\tau)} \bigg[u(t+\tau) + \frac{1}{(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^{n-1} p(s) u^{\alpha}(s-\sigma)\, ds \bigg] \\
&= \frac{1}{a(t+\tau)} \bigg[\exp\bigg(-k_2 \int_{t_0-\gamma}^{t+\tau} p(s)\, ds \bigg) \\
&\qquad + \frac{1}{(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^{n-1} p(s) \\
&\qquad\qquad \times \exp\bigg(-\alpha k_2 \int_{t_0-\gamma}^{s-\sigma} p(\xi)\, d\xi \bigg)\, ds \bigg] \geq u(t).
\end{aligned}$$

Let $t \in [t_0, t_1]$, and set

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1).$$

Proceeding similarly as in the proof of Theorem 2.1, we obtain that for $t \in [t_0, t_1]$ and any $x \in \Omega$, $(Sx)(t) \geq u(t)$. Thus, we have proved that $Sx \in \Omega$ for any $x \in \Omega$.

In a similar way as in the proof of Theorem 2.1, we can show that $S\Omega$ is a relatively compact subset of $C([t_0,\infty),R)$. Thus, by Lemma 1.1, there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$. We see that $x_0(t)$ is a positive solution of equation (1). The proof of Theorem 2.2 is complete. $\square$

Example 2.1. Consider the nonlinear neutral differential equation

$$[x(t) - a(t)x(t-2)]' = p\,x^3(t-1), \quad t \geq t_0, \tag{5}$$

where $p \in (0,\infty)$. We will show that the conditions of Theorem 2.1 are satisfied. Condition (2) obviously holds. For condition (3), we have

$$p\exp(3pk_1t_0)\int_{t_0}^{\infty}\exp(-3pk_1t)\,dt < \infty,$$

$k_1 > 0$, and (4) has the form

$$\frac{k_1}{k_2}\exp\big((k_2-k_1)p\gamma\big) \geq 1, \tag{6}$$

$k_1 \leq k_2$, $\gamma \geq 0$. For function $a(t)$, we obtain

$$\begin{aligned}
&\exp(-2pk_2) + \frac{1}{3k_1}\\
&\qquad\times\exp\big(p[k_2(\gamma-t_0-2) - 3k_1(\gamma-t_0-1) + (k_2-3k_1)t]\big)\\
&\quad\leq a(t) \leq \exp(-2pk_1)\\
&\qquad+\frac{1}{3k_2}\exp\big(p[k_1(\gamma-t_0-2) - 3k_2(\gamma-t_0-1) + (k_1-3k_2)t]\big),\\
&\qquad\qquad t \geq t_0.
\end{aligned}$$

For $p=1$, $k_1=1$, $k_2=2$, $\gamma=1$, $t_0=1$, condition (6) is satisfied and

$$e^{-4} + \frac{1}{3e}e^{-t} \leq a(t) \leq e^{-2} + \frac{e^4}{6}e^{-5t}, \quad t \geq t_0 = 1. \tag{7}$$

If function $a(t)$ satisfies (7), then equation (5) has a solution which is bounded by functions $u(t) = \exp(-2t)$, $v(t) = \exp(-t)$, $t \geq 1$.

Example 2.2. We consider the nonlinear neutral differential equation

$$[x(t) - a(t)x(t-\tau)]^{(n)} = (-1)^{n+1}t\,|x(t-\sigma)|^{\alpha-1}x(t-\sigma), \quad t \geq t_0 \geq 0, \tag{8}$$

where $n > 0$ is an integer, $\alpha, \tau, \sigma \in (0, \infty)$, and

$$a(t) = \exp\bigg(- \frac{k\tau}{2}(2t-\tau)\bigg) + \frac{1}{(n-1)!}\exp\bigg(\frac{k}{2}\big((t-\tau)^2 - t_0^2\big)\bigg) \times \int_t^\infty s(s-t)^{n-1}\exp\bigg(- \frac{\alpha k}{2}\big((s-\sigma)^2 - t_0^2\big)\bigg)\,ds, \quad t \geq t_0,$$

where $k \in (0, \infty)$. Since all conditions of Corollary 2.1 are satisfied, then equation (8) has a solution

$$x(t) = \exp\bigg(- \frac{k}{2}\big(t^2 - t_0^2\big)\bigg), \quad t \geq t_0,$$

which converges to zero as $t \to \infty$.

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