

HOMOGENEITY PROPERTIES WITH ISOMETRIES AND LIPSCHITZ FUNCTIONS

JAN J. DIJKSTRA

ABSTRACT. We consider metric variants of homogeneity, countable dense homogeneity (CDH) and strong local homogeneity (SLH) by requiring that the homeomorphisms that witness the homogeneity be isometries, respectively bi-Lipschitz maps that are almost isometries: iso-homogeneity, iso-CDH, iso-SLH, L-homogeneity, LCDH, and LSLH. We prove metric versions of Bennett's theorem that SLH implies CDH for complete spaces, and we show that every separable Banach space is LCDH. As applications, we investigate how a number of standard examples of CDH spaces fare with respect to metric homogeneity.

1. Introduction. Every topological space in this paper is assumed to be separable metric. A space X is *countable dense homogeneous* (CDH) if, given any two countable dense subsets $A, B \subset X$, there is a homeomorphism h of X such that $h(A) = B$. Well-known examples of CDH spaces are \mathbf{R} , the Cantor set, the Hilbert cube and Hilbert space. The standard method for proving that a space is CDH uses a theorem of Bennett [1] which states that complete spaces are CDH whenever they are *strongly locally homogeneous* (SLH), that is, the space has a basis such that for every basis element B and $x, y \in B$ there exists an autohomeomorphism of the space that is supported on B and that maps x onto y .

In the context of separable metric spaces the notions of topological homogeneity allow a natural strengthening, by requiring that the homeomorphism h be of a particularly nice form. When the homeomorphisms are required to be isometries we obtain the notions iso-homogeneous, iso-CDH and iso-SLH. When the homeomorphisms are required to be bi-Lipschitz maps that are close to isometric we obtain L-homogeneous, LCDH and LSLH. Precise definitions can be found in Section 2.

2010 AMS *Mathematics subject classification.* Primary 54E40, 46B04.

Keywords and phrases. Countable dense homogeneous, strongly locally homogeneous, isometry, bi-Lipschitz map.

Received by the editors on August 24, 2006, and in revised form on April 22, 2008.

DOI:10.1216/RMJ-2010-40-5-1505 Copyright ©2010 Rocky Mountain Mathematics Consortium

In Section 3 we show that Bennett's theorem has metric extensions and that Banach spaces are LCDH. Of particular interest is that it turns out that our metric extension of Bennett's theorem can also be used to prove countable dense homogeneity in certain cases where the space in question is not strongly locally homogeneous; see Remark 10 and Proposition 31. In Section 4 we consider a number of examples to which our theorems are applied.

Related research was carried out by Väisälä [12], Hohti [4] and Hohti and Junnila [5] who investigated homogeneity with bi-Lipschitz maps. Kojman and Shelah [7] investigated almost isometric imbedding between metric spaces. Among other results they showed that the Urysohn space is LCDH.

2. Definitions and preliminaries.

Definition 1. Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \rightarrow Y$ be a one-to-one function. Then the *norm* of f is defined by

$$\|f\| = \sup \left\{ \left| \log \frac{\rho(f(x), f(y))}{d(x, y)} \right| : x, y \in X \text{ with } x \neq y \right\} \in [0, \infty].$$

Remark 2. Note that if $\|f\| < \infty$, then f is, in particular, an imbedding of X in Y . We have $\|f\| = 0$ if and only if f is an isometric imbedding. If $g : Y \rightarrow Z$ is also one-to-one, then $\|g \circ f\| \leq \|f\| + \|g\|$. If f is a bijection, then $\|f^{-1}\| = \|f\|$.

The following lemma is standard.

Lemma 3. Let (X, d) and (Y, ρ) be metric spaces such that ρ is complete. Let D be a dense subset of X , and let $f : D \rightarrow Y$ be such that $\|f\| < \infty$. Then there exists a unique continuous extension $\bar{f} : X \rightarrow Y$. The map \bar{f} has the same norm as f , and \bar{f} is closed if and only if d is complete.

Recall that a topological space X is called *homogeneous* if for any two points $x, y \in X$ there exists a homeomorphism $h : X \rightarrow X$ such that $h(x) = y$.

Definition 4. Let (X, d) be a separable metric space. We say that (X, d) is *iso-homogeneous* if for any two points $x, y \in X$ there exists an isometric bijection $h : X \rightarrow X$ such that $h(x) = y$.

The space is *L-homogeneous* if for any two points $x, y \in X$ and any $\varepsilon > 0$ there is a bijection $h : X \rightarrow X$ such that $h(x) = y$ and $\|h\| \leq \varepsilon$.

Hohti and Junnila [5] have shown that for every homogeneous compact space X and $\varepsilon > 0$ there is a compatible metric d on X such that for any two points $x, y \in (X, d)$ there is a bijection $h : X \rightarrow X$ such that $h(x) = y$ and $\|h\| \leq \varepsilon$.

Definition 5. Let (X, d) be a separable metric space. We say that (X, d) is *iso-CDH* if for any two countable dense subsets A, B there exists an isometric bijection $h : X \rightarrow X$ such that $h(A) = B$.

The space is *LCDH* if for any two countable dense subsets A, B and any $\varepsilon > 0$ there is a bijection $h : X \rightarrow X$ such that $h(A) = B$ and $\|h\| \leq \varepsilon$.

While the usual notion of countable dense homogeneity is topological, the newly introduced notions depend on the geometry of the metric space.

A space X is called *strongly locally homogeneous (SLH)* if there is a basis \mathcal{B} for the topology such that for every $B \in \mathcal{B}$ and $x, y \in B$ there exists a homeomorphism $h : X \rightarrow X$ that is supported on B and that maps x to y . (A function $f : X \rightarrow X$ is said to be *supported on* a subset A of X if the restriction $f|(X \setminus A)$ is the identity.) Bennett [1] has shown that for complete spaces SLH implies CDH.

Definition 6. A metric space (X, d) is called *iso-SLH* if for every $x \in X$ and every neighborhood U of x there is a neighborhood V of x such that for each $y \in V$ there exists an isometric bijection $h : X \rightarrow X$ that h is supported on U and that maps x to y .

A metric space (X, d) is called *LSLH* if for every $x \in X$, every $\varepsilon > 0$ and every neighborhood U of x , there is a neighborhood V of x such that for each $y \in V$ there exists a homeomorphism $h : X \rightarrow X$ such that h is supported on U , $h(x) = y$ and $\|h\| \leq \varepsilon$.

Example 7. It is easily seen that the real line \mathbf{R} with the Euclidean metric is LSLH, LCDH and iso-homogeneous but not iso-SLH or iso-CDH.

If $x \in X$, then $\tau(x) = \{h(x) : h \text{ is an autohomeomorphism of } X\}$ is the topological type of x . The collection $\{\tau(x) : x \in X\}$ forms a partition of X . If X is SLH or CDH, then every $\tau(x)$ is clopen, see [1]. So, for connected spaces, both SLH and CDH imply homogeneity. Similarly, if a connected space is iso-SLH or iso-CDH, then it is iso-homogeneous. The interval $(0, 1)$ with the regular metric and ellipses in the plane (that are not circles) are easily seen to be connected spaces that are LSLH. These spaces are LCDH by Theorem 9 and Remark 12 but not L-homogeneous by Proposition 21.

Proposition 8. *A space X is SLH if and only if for every $x \in X$ and every neighborhood U of x there is a neighborhood V of x such that for each $y \in V$ there exists a homeomorphism $h : X \rightarrow X$ that is supported on U and that maps x to y . Thus, LSLH implies SLH.*

Proof. The ‘only if’ part is trivial so consider a space that satisfies the second condition. Let $x \in X$, and let U be an open neighborhood of x . Consider the following equivalence relation on U , $a \sim b$ if there is a homeomorphism $h : X \rightarrow X$ that is supported on U and that maps a to b . Clearly, every equivalence class of \sim is open and hence clopen in U . Let B be the class that contains x , and let $y \in B$. Then there exists a homeomorphism $h : X \rightarrow X$ that is supported on U and that maps x to y . Note that $h(B) = B$. Since B is clopen in U , it is clear that the map that restricts to $h|_B$ on B and that is supported on B is a homeomorphism of X that maps x to y . Thus, X is SLH. \square

3. Theorems. We have the following metric analogues to Bennett’s theorem:

Theorem 9. *Let (X, d) be complete. If the space is LSLH, then it is LCDH. If the space is iso-SLH, then it is iso-CDH.*

Proof. Let (X, d) be complete and LSLH, let $\varepsilon > 0$, and let $A = \{a_i : i \in \omega\}$ and $B = \{b_i : i \in \omega\}$ be dense subsets of X . We construct by induction a sequence h_0, h_1, \dots of permutations of X such that for each $n \in \omega$,

- (1) $h_n(a_i) \in B$ for $0 \leq i < n$,
- (2) if $n \geq 1$ then $h_n(a_i) = h_{n-1}(a_i)$ for $0 \leq i < n-1$,
- (3) $h_n^{-1}(b_i) \in A$ for $0 \leq i < n$,
- (4) if $n \geq 1$ then $h_n^{-1}(b_i) = h_{n-1}^{-1}(b_i)$ for $0 \leq i < n-1$, and
- (5) $\|h_n\| \leq \varepsilon(1 - 2^{-n})$.

Assume that the construction has been performed. Define $h : A \rightarrow B$ by $h(a_i) = h_{i+1}(a_i)$ for $i \in \omega$ and note that the hypotheses imply that h is well defined, that $h(A) = B$, and that $\|h\| \leq \varepsilon$. By Lemma 3 there exists an extension $\bar{h} : X \rightarrow X$ with $\|\bar{h}\| \leq \varepsilon$ and $\bar{h}(X) = X$ because B is dense.

It remains to perform the construction of the h_n . Let h_0 be the identity. Assume that h_n has been found. Consider the point a_n . If $h_n(a_n) \in F = \{h_n(a_0), \dots, h_n(a_{n-1}), b_0, \dots, b_{n-1}\}$, then let α be the identity map. If $h_n(a_n) \notin F$, then we select with the LSLH property a permutation α of X such that $\|\alpha\| \leq \varepsilon 2^{-n-2}$, $\alpha(a_n) \in B$, and α fixes all points of $h_n(F)$. In the same way we can find a permutation β of X such that $\|\beta\| \leq \varepsilon 2^{-n-2}$, $\beta(b_n) \in \alpha \circ h_n(A)$, and β fixes all points of $\alpha(F \cup \{h_n(a_n)\})$. We put $h_{n+1} = \beta^{-1} \circ \alpha \circ h_n$ and note that it clearly satisfies the induction hypotheses (1)–(4). For hypothesis (5) note that by Remark 2 we have

$$\begin{aligned} \|h_{n+1}\| &\leq \|\beta^{-1}\| + \|\alpha\| + \|h_n\| = \|\beta\| + \|\alpha\| + \|h_n\| \\ &\leq \varepsilon 2^{-n-2} + \varepsilon 2^{-n-2} + \varepsilon(1 - 2^{-n}) \\ &= \varepsilon(1 - 2^{-n-1}). \end{aligned}$$

For the isometry part of the theorem, just substitute $\varepsilon = 0$ in the above proof. \square

Remark 10. Inspection of the proof gives that the LSLH and iso-SLH premises of Theorem 9 can be weakened somewhat. We may replace LSLH by the property $LSLH^-$ which means that, for any $x \in X$, any $\varepsilon > 0$ and any finite subset F of $X \setminus \{x\}$, there is a neighborhood V

of x such that for each $y \in V$ there is a permutation f of X such that points of F are fixed, $\|f\| \leq \varepsilon$ and $f(x) = y$. The property iso-SLH can be weakened to *iso-SLH*⁻, that is, for any $x \in X$ and any finite subset F of $X \setminus \{x\}$, there is a neighborhood V of x such that for each $y \in V$ there is an isometry of X that fixes F and that maps x to y . See Proposition 31 for an application that shows that this version of Theorem 9 may be used to prove the CDH property for spaces that are not SLH.

Remark 11. There is a slightly stronger version of Bennett's theorem: if X is complete and SLH, O is open in X and A, B are countable dense subsets of O , then there is a homeomorphism $h : X \rightarrow X$ that is supported on O and that maps A onto B . Theorem 9 can be strengthened in the same way.

Remark 12. The advantage in using Lemma 3 in the proof of Theorem 9 is that we do not have to be concerned with the question whether the sequence of homeomorphisms converges which leads to Remark 10. However, if we use the inductive convergence criterion, cf. [10, Exercises 1.6.1 and 1.6.2], instead of Lemma 3, then we can weaken the premise that d is complete to the requirement that X is merely topologically complete, that is, the theorem remains valid if d is incomplete as long as there is some other admissible metric that is complete. For instance, $\mathbf{R} \setminus \mathbf{Q}$ is LSLH in the standard metric; see Proposition 19. Thus, this space is also LCDH.

Theorem 13. *Every normed vector space is LSLH and every Banach space is LCDH.*

Proof. Let $(X, |\cdot|)$ be a normed vector space. Let $z \in X$, and let $\varepsilon > 0$. Since the metric $|x - y|$ is invariant, it suffices to consider the case $z = 0$. Let $\delta = \varepsilon(1 - e^{-\varepsilon})$ and note that $0 < \delta < \varepsilon$. Put $U = \{x \in X : |x| < \varepsilon\}$ and $V = \{x \in X : |x| < \delta\}$. Let $a \in V$, and define the function $f : X \rightarrow X$ by

$$f(x) = \begin{cases} x + (\varepsilon - |x|/\varepsilon)a & \text{if } |x| < \varepsilon; \\ x & \text{if } |x| \geq \varepsilon. \end{cases}$$

Note that f is supported on U and that $f(0) = a$. If $|x| < \varepsilon$, then

$|f(x)| \leq |x| + (\varepsilon - |x|)|a|/\varepsilon < \varepsilon$ so $f(U) \subset U$. If $x, y \in U$ with $f(x) = f(y)$, then $|x - y| = ||x| - |y|||a|/\varepsilon \leq |x - y||a|/\varepsilon$. Since $|a| < \delta < \varepsilon$ we have that $x = y$ so f is one-to-one. Now let $y \in U \setminus \{a\}$ be arbitrary, and put $\alpha_t = a + t(y - a)$ for $t \geq 1$. Note that $|\alpha_1| < \varepsilon$ and $\lim_{t \rightarrow \infty} |\alpha_t| = \infty$ so there is an $r > 1$ with $|\alpha_r| = \varepsilon$. Observe that $f((1/r)\alpha_r) = y$; thus, f is a bijection.

We now obtain an estimate for $\|f\|$. Let $x, y \in X$ be such that $x \neq y$. Since the case $x, y \notin U$ is of no interest, we may assume that $x \in U$. If $y \in U$, then

$$\begin{aligned} |x - y| \left(1 - \frac{|a|}{\varepsilon}\right) &\leq |f(x) - f(y)| = \left| (x - y) + (|y| - |x|) \frac{|a|}{\varepsilon} \right| \\ &\leq |x - y| \left(1 + \frac{|a|}{\varepsilon}\right). \end{aligned}$$

If $y \notin U$, then

$$\begin{aligned} |x - y| \left(1 - \frac{|a|}{\varepsilon}\right) &\leq |f(x) - f(y)| = \left| (x - y) + (\varepsilon - |x|) \frac{|a|}{\varepsilon} \right| \\ &\leq |x - y| \left(1 + \frac{|a|}{\varepsilon}\right). \end{aligned}$$

Since $-\log(1 - r) \geq \log(1 + r)$, we have that $\|f\| \leq -\log(1 - (|a|/\varepsilon)) < -\log(1 - (\delta/\varepsilon)) = \varepsilon$. We may conclude that X is LSLH and that Banach spaces are LCDH by Theorem 9.

Note that $|\log t| < -\log(1 - r)$ whenever $|t - 1| < r < 1$, a fact that we will use several times. \square

Remark 14. Zamora Aviles [13] proved that \mathbf{R}^n is LCDH by a different method. (We always assume that \mathbf{R}^n is equipped with the standard Euclidean metric.)

Any topological group admits a (left-)invariant metric; thus, it is iso-homogeneous with respect to that metric (see [3, Theorem 8.3]).

Proposition 15. *If (X, d) is iso-CDH, then $d(X \times X)$ is countable and hence X is zero-dimensional. If (X, d) is iso-SLH⁻, then $d(\{x\} \times X)$ is countable for each $x \in X$ and hence X is zero-dimensional.*

Proof. If (X, d) is iso-CDH and D is a countable dense subset, then $d(D \times D) = d(X \times X)$. Let (X, d) be iso-SLH⁻, and let $x \in X$. If $y \neq x$, then there is a neighborhood V_y of y such that for every $z \in V$ there is an isometry h of X with $h(x) = x$ and $h(y) = z$. Note that $d(x, y) = d(x, z)$ for every $z \in V_y$. Since X is separable metric we can cover $X \setminus \{x\}$ by countably many V_y s and hence $d(\{x\} \times X)$ is countable. \square

Remark 16. Unlike CDH the concept SLH is also meaningful for nonseparable spaces. The proof of Proposition 15 uses separability. For nonseparable iso-SLH⁻ spaces the set $d(\{x\} \times X)$ may be uncountable but we still have that the set $\{y \in X : d(x, y) = r\}$ is clopen for each $r > 0$ and $x \in X$. Thus, iso-SLH⁻ implies $\text{ind} \leq 0$ also for general, not necessarily separable, metric spaces.

Theorem 17. *For a subspace A of \mathbf{R}^n , the following statements are equivalent:*

- (1) *A is iso-CDH.*
- (2) *A is iso-SLH.*
- (3) *A is iso-SLH⁻.*
- (4) *A is discrete.*

Proof. The implications (4) \Rightarrow (1), (4) \Rightarrow (2), and (2) \Rightarrow (3) are trivial.

(1) \Rightarrow (4). Let A be an iso-CDH subspace of \mathbf{R}^n . Select a finite subset F of A such that F and A have the same affine hull V . Let R be the countable set $d(A \times A)$. Define $f : A \rightarrow R^F$ by $(f(a))(b) = d(a, b)$ for $a \in A$ and $b \in F$. We show that f is one-to-one. Let $x, y \in A$ be such that $x \neq y$ and $f(x) = f(y)$. Then for each $b \in F$, $d(x, b) = d(y, b)$; thus, F is a subset of the hyperplane H that is equidistant from x and y . Then $A \subset V \subset H$ which contradicts the fact $x, y \notin H$. Since R^F is countable, we may conclude that A is countable. Every countable

(iso-)CDH space is obviously discrete because if $a \in A$ is not isolated; then $A \setminus \{a\}$ is a countable dense subset of A that cannot be mapped onto A with a permutation of A .

(3) \Rightarrow (4). Let A be iso-SLH $^-$, and let $a \in A$. If a is not in the affine hull of $A \setminus \{a\}$, then a is isolated in A . So we may choose a finite subset F of $A \setminus \{a\}$ such that F and A have the same affine hull. Let U be a neighborhood of a in A such that for each $x \in U$ there exists an isometry of A that fixes the points in F and that maps a to x . Let $x \in U \setminus \{a\}$ and consider an isometry f of A that fixes F and that maps a to x . As in the argument given above F , and hence A , are contained in the hyperplane H that is equidistant from a and x . Since $a, x \notin H$, we may conclude that $U = \{a\}$ and thus a is isolated in A . \square

Proposition 18. *A compact metric space is iso-homogeneous if and only if it is L-homogeneous.*

Proof. Let (X, d) be a compact L-homogeneous space. Let $a, b \in X$ and select for each $n \in \omega$ a permutation f_n of X such that $\|f_n\| \leq 2^{-n}$ and $f_n(a) = b$. Thus, we have $\lim_{n \rightarrow \infty} d(f_n(x), f_n(y))/d(x, y) = 1$ for $x \neq y$. Select a countable dense subset $A = \{a_k : k \in \omega\}$ in X such that $a_0 = a$. By compactness we find using recursion a sequence $\omega = I_0 \supset I_1 \supset \dots$ of infinite sets such that for each $k \in \omega$ there is a $b_k \in X$ with $\lim_{n \in I_k} f_n(a_k) = b_k$. We define $f : A \rightarrow X$ by $f(a_k) = b_k$ for each $k \in \omega$. Note that $f(a) = b$. If $a_k \neq a_m$ and $k < m$, then $d(f(a_k), f(a_m)) = \lim_{n \in I_m} d(f_n(a_k), f_n(a_m)) = d(a_k, a_m) \lim_{n \in I_m} d(f_n(a_k), f_n(a_m))/d(a_k, a_m) = d(a_k, a_m)$. Thus, $\|f\| = 0$ and by Lemma 3 there is an extension $\bar{f} : X \rightarrow X$ such that $\|\bar{f}\| = 0$. By [11, page 181] and compactness we have that \bar{f} is also surjective and hence X is iso-homogeneous. \square

4. Examples. We consider several examples. Let $\mathbf{P} = \mathbf{R} \setminus \mathbf{Q}$ be the space of irrational numbers with the Euclidean metric.

Proposition 19. *The space \mathbf{P} is LSLH, LCDH and L-homogeneous but not iso-SLH $^-$, iso-CDH, or iso-homogeneous.*

Proof. The set \mathcal{I} of isometric permutations of \mathbf{P} consists of all linear functions of the form $\pm x + q$ where $q \in \mathbf{Q}$. Obviously, every function like that generates an isometric permutation of \mathbf{P} . If f is an isometric bijection of \mathbf{P} , then by Lemma 3 it extends to an isometry \bar{f} of \mathbf{R} . Clearly, we have $\bar{f}(x) = \pm x + q$ with $q = \bar{f}(0) \in \mathbf{Q}$. Since \mathcal{I} is countable, we have that \mathbf{P} is neither iso-SLH⁻, nor iso-CDH, nor iso-homogeneous.

We now verify that \mathbf{P} is LSLH. Let $a \in \mathbf{P} \cap (p, q)$ where $p, q \in \mathbf{Q}$, and let $\varepsilon > 0$ be such that $\varepsilon < \min\{q - a, a - p\}$. Put $\delta = 1 - e^{-\varepsilon} \in (0, 1)$, and select sequences of rational points $p = p_0 < p_1 < p_2 < \dots$ and $q = q_0 > q_1 > q_2 > \dots$ such that $\lim_{i \rightarrow \infty} p_i = \lim_{i \rightarrow \infty} q_i = a$. Pick an arbitrary point $b \in \mathbf{P}$ such that $|b - a| < \varepsilon\delta$. The slope m of the secant line through the points (p, p) and (a, b) in the plane satisfies $|m - 1| = |(b - p)/(a - p) - 1| = |(b - a)/(a - p)| < \delta$. We can then choose a rational point p'_1 close enough to p_1 such that both the slope from (p, p) to (p_1, p'_1) and the slope from (p_1, p'_1) to (a, b) have distance less than δ towards 1. Continuing this process, we find a sequence $p = p'_0 < p'_1 < p'_2 < \dots$ such that $\lim_{i \rightarrow \infty} p'_i = b$ and the slope $m_i = (p'_{i+1} - p'_i)/(p_{i+1} - p_i)$ from (p_i, p'_i) to (p_{i+1}, p'_{i+1}) has distance less than δ towards 1 for every $i \in \omega$. We can find a similar sequence $q = q'_0 > q'_1 > q'_2 > \dots$ on the other side of b with slopes $n_i = (q'_{i+1} - q'_i)/(q_{i+1} - q_i)$ such that $|n_i - 1| < \delta$. Now define $h : \mathbf{R} \rightarrow \mathbf{R}$ by

$$h(x) = \begin{cases} x & \text{if } x < p \text{ or } x > q; \\ b & \text{if } x = a; \\ p'_i + m_i(x - p_i) & \text{if } x \in [p_i, p_{i+1}] \text{ for } i \in \omega; \\ q'_i + n_i(x - q_i) & \text{if } x \in [q_{i+1}, q_i] \text{ for } i \in \omega. \end{cases}$$

Thus, h is supported on (p, q) and $h(a) = b$. The derivative h' of the continuous function h exists in all but countably many points and assumes only the values 1, m_i or n_i ; thus, for all $x, y \in \mathbf{R}$ with $x < y$, we have

$$\left| \frac{h(y) - h(x)}{y - x} - 1 \right| = \left| \frac{\int_x^y (h'(t) - 1) dt}{y - x} \right| \leq \delta.$$

Hence, we have $\|h\| \leq -\log(1 - \delta) = \varepsilon$. Apart from $h(a) = b$ the function h is piecewise linear with rational coefficients and thus $h(\mathbf{Q}) = \mathbf{Q}$ and $h(\mathbf{P}) = \mathbf{P}$. We have shown that \mathbf{P} is LSLH.

Since \mathbf{P} is LSLH and topologically complete, we have that the space is LCDH by Remark 12. Let $a, b \in \mathbf{P}$ and $\varepsilon > 0$. Since \mathbf{P} is LSLH, there is a neighborhood U of b such that for every $x \in U$ there is a permutation f of \mathbf{P} with $\|f\| < \varepsilon$ and $f(b) = x$. Select a $q \in \mathbf{Q}$ such that $a + q \in U$, and let g be a permutation of \mathbf{P} such that $\|g\| < \varepsilon$ and $g(b) = a + q$. Then $h(x) = g^{-1}(x + q)$ is a permutation of \mathbf{P} such that $\|h\| = \|g\|$ and $h(a) = b$, proving that \mathbf{P} is L-homogeneous. \square

Thus, there exist topologically complete spaces that are L-homogeneous but not iso-homogeneous. In view of Proposition 18 this suggests

Question 20. *Is there a complete metric space (X, d) that is L-homogeneous but not iso-homogeneous?*

Let $Q = [0, 1]^\omega$ be the Hilbert cube, and let the Cantor set C be represented by the subset $\{0, 1\}^\omega$ of Q . It is well known that Q and C are homogeneous, SLH and CDH; see [9, subsection 6.1]. We consider the following standard product metrics on Q . Let $s = (s_0, s_1, \dots)$ be a monotone sequence of positive real numbers that converges to zero. The metric ρ_s on Q is defined by

$$\rho_s(x, y) = \max\{s_i | x_i - y_i| : i \in \omega\} \quad \text{for } x, y \in Q.$$

According to Väisälä [12] and Hohti [4] we have that for every pair $x, y \in Q$ there is a permutation f of (Q, ρ_s) with $f(x) = y$ and $\|f\| < \infty$ if and only if $\sup\{s_i/s_{i+1} : i \in \omega\} < \infty$.

If (X, d) is bounded, then a point $x \in X$ is called *extremal* if $\sup\{d(x, y) : y \in X\} = \text{diam } X$.

Proposition 21. *If a space (X, d) is bounded and L-homogeneous, then every point of X is extremal. Consequently, the space (Q, ρ_s) is not L-homogeneous.*

Proof. Let $x \in X$ be such that $\varepsilon = \text{diam } X - \sup\{d(x, y) : y \in X\} > 0$. Select $y, z \in X$ such that $d(y, z) \geq \text{diam } X - \varepsilon/2$. Let f be a permutation of X such that $f(y) = x$. Then $d(f(y), f(z)) = d(x, f(z)) \leq \text{diam } X - \varepsilon$. Consequently, $\|f\| \geq \log((\text{diam } X - \varepsilon/2)/(\text{diam } X - \varepsilon))$. \square

Question 22. *Is (Q, ρ_s) LCDH or LSLH? Is there a compatible metric on Q that makes the space LCDH, LSLH, or iso-homogeneous?*

The following result was communicated to the author by Michael Hrušák [6].

Theorem 23. *The Cantor set $C = \{0, 1\}^\omega$, seen as a subspace of (Q, ρ_s) , is iso-homogeneous, iso-SLH and iso-CDH.*

Proof. Consider the standard boolean group structure on C : $(x \triangle y)_i = x_i + y_i \bmod 2$. Clearly, for $a \in C$, the function $x \mapsto x \triangle a$ is an isometry so C is iso-homogeneous. Let $x \in C$, and let $n \in \omega$. Consider the clopen neighborhood

$$U_n = \{y \in C : y_i = x_i \text{ for every } i \text{ with } s_i > 2^{-n}\}$$

of x . Let $y \in U_n$, and define the function

$$h(z) = \begin{cases} z \triangle x \triangle y & \text{if } z \in U_n; \\ z & \text{if } z \in C \setminus U_n. \end{cases}$$

It is easily verified that h is an isometry that maps x to y . Since the U_n s form a neighborhood basis at x we have that C is iso-SLH (and iso-CDH by Theorem 9). \square

Note that we have shown that C is *iso-SLH*⁺: there exists a basis \mathcal{B} for the topology such that for each $B \in \mathcal{B}$ and $x, y \in B$ there exists an isometry of the space that is supported on B and that maps x to y . We can of course add *iso-SLH*⁺ to the list in Theorem 17.

Question 24. *Are *iso-SLH*[−], *iso-SLH*, and *iso-SLH*⁺ equivalent?*

Analogously, we can call a space (X, d) *LSLH*⁺ if there exists for every $x \in X$ and $\varepsilon > 0$ a neighborhood U of x with $\text{diam } U \leq \varepsilon$ such that for each $y \in U$ there exists a permutation h of X such that h is supported on U , $h(x) = y$, and $\|h\| \leq \varepsilon$. This concept is very different from LSLH in that it applies only to zero-dimensional spaces (note that the

neighborhoods U must be clopen). Also note that the zero-dimensional space \mathbf{P} is LSLH by Proposition 19 but clearly not LSLH^+ . LSLH^- is also not equivalent to LSLH; see Proposition 31.

Remark 25. Similarly to the Cantor set we can equip \mathbf{P} with an admissible metric that makes the space iso-homogeneous, iso- SLH^+ and iso-CDH. Represent \mathbf{P} by \mathbf{N}^ω and use the metric

$$\rho_s(x, y) = \max\{s_i \min\{1, |x_i - y_i|\} : i \in \omega\} \quad \text{for } x, y \in \mathbf{N}^\omega.$$

Proposition 26. *Some (C, ρ_s) admits an isometric imbedding in Hilbert space and hence in Theorem 17 we cannot substitute Hilbert space for \mathbf{R}^n .*

Proof. Let the Hilbert space be represented by ℓ^2 , the vector space of square summable sequences $x = (x_0, x_1, \dots)$ of real numbers. We have the usual inner product $x \cdot y = \sum_{i=0}^\infty x_i y_i$, norm $\|x\| = \sqrt{x \cdot x}$, and metric $d(x, y) = \|x - y\|$ for $x, y \in \ell^2$. We identify \mathbf{R}^n for $n \in \omega$ with the subspace $\{x \in \ell^2 : x_i = 0 \text{ for } i \geq n\}$ of ℓ^2 , and we put $\mathbf{R}^{-1} = \emptyset$. Let e^0, e^1, \dots be the standard orthonormal basis for ℓ^2 , that is, $e_i^n = 1$ if $i = n$ and $e_i^n = 0$ if $i \neq n$. The affine hull of a subset A of ℓ^2 is denoted as $\text{aff } A$.

If $n \in \omega$ we put $C_n = \{x \in C : x_i = 0 \text{ for } i \geq n\}$ and if positive real numbers s_0, \dots, s_{n-1} have been chosen, then we let ρ_n be the metric on C_n that is determined by $d_n(x, y) = \max_{i=0}^{n-1} s_i |x_i - y_i|$. Let p_0, p_1, \dots enumerate $D = \bigcup_{n=0}^\infty C_n$ in the canonical way, that is, $p_k = x$ if and only if $k = \sum_{i \in \omega} x_i 2^i$.

By recursion we will find for $n \in \mathbf{N}$ a positive real number s_{n-1} and a function $\alpha_n : C_n \rightarrow \ell^2$ such that

- (1) α_n is an isometric imbedding of (C_n, ρ_n) in ℓ^2 ,
- (2) $\alpha_n \upharpoonright C_{n-1} = \alpha_{n-1}$ for $n \geq 2$,
- (3) $\alpha_n(p_k) \in \mathbf{R}^k \setminus \mathbf{R}^{k-1}$ for $0 \leq k < 2^n$, and
- (4) $s_{n-1} < (1/2)s_{n-2}$ for $n \geq 2$.

If this construction has been performed, then $\alpha = \bigcup_{n=1}^\infty \alpha_n$ is an isometric imbedding of (D, ρ_s) in ℓ^2 , where $s = (s_0, s_1, \dots)$. Since

D is dense, we have by Lemma 3 that there is an isometric imbedding $\bar{\alpha}$ of (C, ρ_s) in ℓ^2 .

As the basis step we put $s_0 = 1$, $\alpha_1(p_0) = 0$ and $\alpha_1(p_1) = e^0$. Assume now that s_0, \dots, s_{n-1} and $\alpha_1, \dots, \alpha_n$ have been found. Define

$$r = \min\{d(\alpha_n(x), \text{aff } \alpha_n(C_n \setminus \{x\})) : x \in C_n\}.$$

By hypothesis (3) we have that the elements of $\alpha_n(C_n)$ are geometrically independent; thus, we have $r > 0$. Choose $s_n > 0$ such that $s_n < (1/2)s_{n-1}$ and $s_n < 2r2^{-n/2}$.

We will define $\alpha_{n+1}(p_{2^n}), \dots, \alpha_{n+1}(p_{2^{n+1}-1})$ recursively such that α_{n+1} extends α_n and, for each k with $0 \leq k \leq 2^n$, we have

- (a) $\alpha_{n+1}(p_i) \in \mathbf{R}^i \setminus \mathbf{R}^{i-1}$ for $0 \leq i < k + 2^n$,
- (b) $\alpha_{n+1}|_{\{p_i : i < k + 2^n\}}$ is an isometric imbedding, and
- (c) for each j with $k \leq j < 2^n$,

$$d(\alpha_n(p_j), \text{aff } \{\alpha_{n+1}(p_i) : i \neq j, i < k + 2^n\}) \geq \sqrt{r^2 - ks_n^2/4}.$$

Clearly, the hypotheses are satisfied for $k = 0$. Let $0 \leq k \leq 2^n - 1$, and consider the point p_k . Note that p_{k+2^n} is obtained from p_k by changing the coordinate with index n from 0 to 1 and hence $\rho_{n+1}(p_k, p_{k+2^n}) = s_n$. We put

$$V = \text{aff } \{\alpha_{n+1}(p_i) : i \neq k, i < k + 2^n\}$$

and $t = d(\alpha_n(p_k), V)$. Let N be the unit normal vector to the hyperplane V in \mathbf{R}^{k+2^n-1} such that $t = N \cdot (\alpha_n(p_k) - x)$ for each $x \in V$. By hypothesis (c) and the property $s_n < 2r2^{-n/2}$, we have

$$t \geq \sqrt{r^2 - ks_n^2/4} > \sqrt{r^2 - (2^n - 1)r^2 2^{-n}} = r2^{-n/2} > s_n/2.$$

Thus, $a = (s_n/2t) < 1$ and we may define the vector

$$z = s_n(-aN + \sqrt{1 - a^2} e^{k+2^n}).$$

We put $\alpha_{n+1}(p_{k+2^n}) = \alpha_n(p_k) + z$. For hypothesis (a), it is clear that $\alpha_{n+1}(p_{k+2^n}) \in \mathbf{R}^{k+2^n} \setminus \mathbf{R}^{k+2^n-1}$. Note that $\|z\| = s_n$ and hence $d(\alpha_{n+1}(p_{k+2^n}), \alpha_n(p_k)) = \rho_{n+1}(p_{k+2^n}, p_k)$. Now let i be such

that $i \neq k$ and $i < k + 2^n$. Then we have $\alpha_{n+1}(p_i) \in V$; thus, $z \cdot (\alpha_n(p_k) - \alpha_{n+1}(p_i)) = -s_n at = -s_n^2/2$, and hence

$$\begin{aligned} & \|\alpha_{n+1}(p_{k+2^n}) - \alpha_{n+1}(p_i)\|^2 \\ &= \|z\|^2 + 2z \cdot (\alpha_n(p_k) - \alpha_{n+1}(p_i)) + \|\alpha_n(p_k) - \alpha_{n+1}(p_i)\|^2 \\ &= \rho_{n+1}(p_k, p_i)^2 = \rho_{n+1}(p_{k+2^n}, p_i)^2. \end{aligned}$$

So we have that $\alpha_{n+1}|_{\{p_i : i \leq k + 2^n\}}$ is an isometric imbedding.

For hypothesis (c), let $k + 1 \leq j < 2^n$, and let

$$W = \text{aff} \{ \alpha_{n+1}(p_i) : i \neq j, i < k + 2^n \}$$

and

$$W' = \text{aff} \{ \alpha_{n+1}(p_i) : i \neq j, i \leq k + 2^n \} = W + \mathbf{R}z.$$

Let M be the unit normal vector to W in \mathbf{R}^{k+2^n-1} such that $d(\alpha_n(p_j), W) = (\alpha_n(p_j) - x) \cdot M$ for each $x \in W$. Consider the vector

$$y = \sqrt{1 - a^2} M + a(M \cdot N) e^{k+2^n}.$$

Since M and e^{k+2^n} are both perpendicular to W and since $y \cdot z = 0$, we have that y is perpendicular to the hyperplane W' in \mathbf{R}^{k+2^n} . Since, by hypothesis (c), $d(\alpha_n(p_j), W) \geq \sqrt{r^2 - ks_n^2/4}$ we have

$$\begin{aligned} d(\alpha_n(p_j), W') &= \frac{|(\alpha_n(p_j) - \alpha_n(p_k)) \cdot y|}{\|y\|} \\ &= d(\alpha_n(p_j), W) \sqrt{\frac{1 - a^2}{1 - a^2 + a^2(M \cdot N)^2}} \\ &\geq \sqrt{r^2 - ks_n^2/4} \sqrt{1 - a^2} \\ &\geq \sqrt{(r^2 - ks_n^2/4) \left(1 - \frac{s_n^2/4}{r^2 - ks_n^2/4}\right)} \\ &= \sqrt{r^2 - (k+1)s_n^2/4}, \end{aligned}$$

where we used $a = s_n/(2t)$ and $(*)$. Thus, hypothesis (c) has been verified for $k + 1$. We note that for $k = 2^n$ the map α_{n+1} is defined on C_{n+1} and that it satisfies hypotheses (1)–(3), so the proof is complete. \square

Cantor sets in the real line are not as well behaved as (C, ρ_s) :

Proposition 27. *Let T be the middle third Cantor set in \mathbf{R} . If $f : T \rightarrow T$ is a surjection such that $|f(x) - f(y)| < (5/3)|x - y|$ for $x, y \in T$ with $x \neq y$, then $f(x) \equiv x$ or $f(x) \equiv 1 - x$. Consequently, T is neither LSLH, nor LCDH, nor L -homogeneous.*

Proof. We use the standard construction of T . Let $T_0 = [0, 1]$, and let every T_{n+1} be a union of 2^{n+1} disjoint closed intervals that is obtained from T_n by removing the open middle third interval from each component of T_n . Let $f : T \rightarrow T$ be a surjection such that $|f(x) - f(y)| < (5/3)|x - y|$ for $x, y \in T$ with $x \neq y$. By possibly replacing $f(x)$ by $1 - f(x)$ we can arrange that $f(0) \leq 1/3$.

Let $n \in \mathbf{N}$ and consider a component A of T_n . If $f(A \cap T)$ intersects more than one component of T_n , then we select two points $a, b \in A \cap T$ of minimal distance such that $f(a)$ and $f(b)$ are in different components of T_n . Since the largest gap in $A \cap T$ has length 3^{-n-1} we have $|a - b| \leq 3^{-n-1}$. The smallest gap in T_n has length 3^{-n} so $|f(a) - f(b)|/|a - b| \geq 3^{-n}/3^{-n-1} = 3$, contradicting the assumption about f . Thus, $f(A)$ is contained in some component of T_n which means that f generates a permutation P_n of the components of T_n because f is onto.

Note that since $f(0) \leq 1/3$ we have that P_1 is the identity. If f is not the identity, then there is a P_k that is not the identity, and we let $n \geq 1$ be the highest index such that P_n is the identity permutation. Let A_1 and A_2 be two distinct components of T_{n+1} such that $P_{n+1}(A_1) = A_2$. Then there exist components B_1 and B_2 of T_n , and a component D of T_{n-1} such that $A_1 \cup A_2 = B_1 \cap T_{n+1}$ and $B_1 \cup B_2 = D \cap T_n$. Let a and b be the endpoints of B_1 and B_2 , respectively, such that $|a - b| = 3^{-n}$. Since $P_n(B_1) = B_1$, we have $P_{n+1}(A_2) = A_1$ and we may assume by symmetry that $a \in A_1$ and $f(a) \in A_2$. Since a is an endpoint of B_1 this means that $|f(a) - a| \geq 2 \cdot 3^{-n-1}$. Since $P_n(B_2) = B_2$ we have $|a - f(b)| \geq |a - b| = 3^{-n}$ thus $|f(a) - f(b)| \geq 5 \cdot 3^{-n-1} = (5/3)|a - b|$,

a contradiction. We may conclude that f is the identity which proves the proposition. \square

Definition 28. A metric space (X, d) is called *L-rigid* if the identity is the only permutation of X with finite norm.

Proposition 29. *There exist L-rigid Cantor sets in the real line.*

Proof. We use a standard construction for our example: K as an intersection $\bigcap_{n=0}^{\infty} K_n$. Let $K_0 = [0, 1]$, and let every K_n for $n \geq 1$ be a union of 2^n disjoint closed intervals that are obtained from K_{n-1} by replacing every component of K_{n-1} by two subintervals such that

- (a) the boundary of K_{n-1} (in \mathbf{R}) is contained in the boundary of K_n ,
- (b) for each component A of K_n we have $\log(g_n/\text{diam } A) > n$, where g_n is the length of the smallest gap in K_n , and
- (c) for each pair of distinct components A and B of K_n we have $|\log(\text{diam } B/\text{diam } A)| > n$.

By property (a) we have that for each component A of K_n , $\text{diam}(K \cap A) = \text{diam } A$.

Let F be a permutation of K such that $\|f\| < \infty$. Consider K_n with $n > \|f\|$. Let A be a component of K_n such that $f(A \cap K)$ meets more than one component of K_n . Then $\|f\| \geq \log(g_n/\text{diam } A) > n$, a contradiction. Since f is onto we now have that f generates a permutation P_n of the components of K_n . Let A be a component of K_n such that $P_n(A) \neq A$. Since f is surjective we have

$$\|f\| \geq \left| \log \frac{\text{diam}(P_n(A) \cap K)}{\text{diam}(A \cap K)} \right| = \left| \log \frac{\text{diam } P_n(A)}{\text{diam } A} \right| > n.$$

Thus, we have that P_n is the identity permutation for each $n > \|f\|$ and hence f is the identity. \square

Clearly, a bounded subset of \mathbf{R} is L-homogeneous if and only if it contains at most two points, see Proposition 21.

Proposition 30. *There exist Cantor sets in \mathbf{R} that are $LSLH^+$ and $LCDH$.*

Proof. Consider the Cantor set C with metric ρ_s for $s_i = 3^{-i^2}$, $i \in \omega$. We define the map $\alpha : C \rightarrow \mathbf{R}$ by

$$\alpha(x) = \sum_{n=0}^{\infty} x_n 3^{-n^2}.$$

Since $\sum_{i=n+1}^{\infty} 3^{-i^2} \leq \sum_{i=1}^{\infty} 3^{-n^2-2n-i} = (1/2)3^{-n^2-2n} \leq (1/2)3^{-n^2}$, we have that α is an imbedding and $(1/2)\rho_s(x, y) \leq d(x, y) \leq (3/2)\rho_s(x, y)$ for all $x, y \in C$. So $d(x, y) = |\alpha(x) - \alpha(y)|$ is a compatible metric on C that makes (C, d) isometrically imbeddable in \mathbf{R} . Let x and y be distinct points in C so $\rho_s(x, y) = 3^{-n^2}$ for some $n \in \omega$ with $|x_n - y_n| = 1$ and $x_i = y_i$ for $i < n$. Thus, $d(x, y) = |(x_n - y_n)3^{-n^2} + \sum_{i=n+1}^{\infty} (x_i - y_i)3^{-i^2}|$, and hence

$$|d(x, y) - \rho_s(x, y)| \leq \sum_{i=1}^{\infty} 3^{-n^2-2n-i} = \frac{1}{2}3^{-n^2-2n} < \rho_s(x, y)3^{-2n}.$$

Consequently, we have

$$\left| \log \frac{d(x, y)}{\rho_s(x, y)} \right| < -\log(1 - 3^{-2n})$$

whenever $0 < \rho_s(x, y) \leq 3^{-n^2}$ with $n \geq 1$.

Let $a \in X$, let $n \geq 2$ and consider the clopen neighborhood $U_n = \{x \in C : \rho_s(x, a) \leq 3^{-n^2}\}$ of a . Let b be an arbitrary element of U_n and define (just as in the proof of Theorem 23) the ρ_s -isometry

$$h(x) = \begin{cases} x \triangle a \triangle b & \text{if } x \in U_n; \\ x & \text{if } x \in C \setminus U_n; \end{cases}$$

that maps a to b .

It remains to find an estimate for $\|f\|$, the norm with respect to d . We can ignore the case $x, y \in C \setminus U_n$ so let $x \in U_n$. First, let $y \in U_n \setminus \{x\}$ and note that $\rho_s(x, y) = \rho_s(h(x), h(y)) \leq 3^{-n^2}$. Then we have

$$\begin{aligned} \left| \log \frac{d(h(x), h(y))}{d(x, y)} \right| &= \left| \log \frac{d(h(x), h(y))}{\rho_s(h(x), h(y))} - \log \frac{d(x, y)}{\rho_s(x, y)} \right| \\ &\leq -2 \log(1 - 3^{-2n}). \end{aligned}$$

If $y \in C \setminus U_n$, then $\rho_s(x, y) \geq 3^{-(n-1)^2}$. Since $h(y) = y$ we have that

$$\left| \frac{d(h(x), h(y))}{d(x, y)} - 1 \right| \leq \frac{d(h(x), x)}{d(x, y)} \leq \frac{\frac{3}{2}\rho_s(h(x), x)}{\frac{1}{2}\rho_s(x, y)} \leq \frac{3^{-n^2+1}}{3^{-(n-1)^2}} = 3^{-2n+2}.$$

Combining these estimates we find $\|f\| \leq -2 \log(1 - 3^{-2n+2})$ which proves that (C, d) is LSLH^+ and hence LCDH by Theorem 9. \square

The following proposition shows that Theorem 9 with Remark 10 can also be used in certain cases where Bennett's theorem does not apply.

Proposition 31. *Complete Erdős space \mathfrak{E}_c is not SLH but it admits a complete metric that makes the space iso-homogeneous, LSLH^- and LCDH .*

Proof. We begin by presenting a particularly elegant model of \mathfrak{E}_c that is featured in Dijkstra [2] and called *harmonic Erdős space*. Consider the Cantor set C with its boolean group structure \triangle . We define the following 'norm' from C to $[0, \infty]$:

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{x_n}{n+1}.$$

Note that $\varphi(x \triangle y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in C$ and hence $\mathfrak{E}_c = \{x \in C : \varphi(x) < \infty\}$ is a subgroup of C . Moreover, it follows that $d(x, y) = \varphi(x \triangle y)$ defines an invariant metric on \mathfrak{E}_c that makes \mathfrak{E}_c into a topological group and an iso-homogeneous space. It is shown in [2] that that (\mathfrak{E}_c, d) is homeomorphic to complete Erdős space, that d is complete, and that the empty set is the only bounded clopen subset of \mathfrak{E}_c .

It is known that \mathfrak{E}_c is not SLH and this fact is easily shown as follows. Let $f : \mathfrak{E}_c \rightarrow \mathfrak{E}_c$ be a continuous function such that $f(a) \neq a$ for some $a \in \mathfrak{E}_c$. Let n be such that $f(a)_n \neq a_n$. Then $O = \{x \in C : x_n = a_n \text{ and } f(x)_n = f(a)_n\}$ is a clopen set that contains a and hence $\text{diam } O = \infty$. Note that no point of O is fixed because $f(O) \cap O = \emptyset$. So we have that the identity is the only map from \mathfrak{E}_c to itself with bounded support.

To show that (\mathfrak{E}_c, d) is LSLH^- and hence LCDH by Remark 10 let $a \in \mathfrak{E}_c$, let $F \subset \mathfrak{E}_c \setminus \{a\}$ be finite, and let $\varepsilon > 0$. Select an $n \in \mathbf{N}$ such that for each $b \in F$ there is an $i < n$ with $b_i \neq a_i$. Put $\delta = (1 - e^{-\varepsilon})/n$ and define the clopen neighborhood $U = \{x \in \mathfrak{E}_c : x_i = a_i \text{ for } i < n\}$ of a . Let $b \in U$ be such that $d(a, b) < \delta$ and define the function $h : \mathfrak{E}_c \rightarrow \mathfrak{E}_c$ by

$$h(x) = \begin{cases} x \triangle a \triangle b & \text{if } x \in U; \\ x & \text{if } x \in \mathfrak{E}_c \setminus U. \end{cases}$$

Note that h is supported on U ; thus, h fixes the points of F . Also $h(a) = b$ and $h \upharpoonright U$ is an isometry of U . So to estimate $\|h\|$ we only have to consider points $x \in U$ and $y \in \mathfrak{E}_c \setminus U$. Since $h(y) = y$ and $d(x, y) \geq 1/n$, we have that

$$\left| \frac{d(h(x), h(y))}{d(x, y)} - 1 \right| \leq \frac{d(h(x), x)}{d(x, y)} \leq \frac{\varphi(x \triangle a \triangle b \triangle x)}{1/n} = n\varphi(a \triangle b) < n\delta.$$

Thus, $\|h\| \leq -\log(1 - n\delta) = \varepsilon$. \square

In particular, we have that \mathfrak{E}_c is CDH , a fact that is already contained in Kawamura, Oversteegen, and Tymchatyn [7]. Of course, \mathfrak{E}_c is neither iso-CDH nor iso-SLH^- because $\dim \mathfrak{E}_c = 1$, see Proposition 15.

Acknowledgments. The author wishes to thank Michael Hrušák [6] not only for his valuable comments during the preparation of this paper but especially for asking the questions that led to this investigation.

REFERENCES

1. R. Bennett, *Countable dense homogeneous spaces*, Fund. Math. **74** (1972), 189–194.
2. J.J. Dijkstra, *A criterion for Erdős spaces*, Proc. Edinburgh Math. Soc. **48** (2005), 595–601.
3. E. Hewitt and K.A. Ross, *Abstract harmonic analysis*, Vol. 1, Springer-Verlag, Berlin, 1963.
4. A. Hohti, *On Lipschitz homogeneity of the Hilbert cube*, Trans. Amer. Math. Soc. **291** (1985), 75–86.
5. A. Hohti and H. Junnila, *A note on homogeneous metrizable spaces*, Houston J. Math. **13** (1987), 231–234.

6. M. Hrušák, personal communication.
7. K. Kawamura, L.G. Oversteegen and E.D. Tymchatyn, *On homogeneous totally disconnected 1-dimensional spaces*, Fund. Math. **150** (1996), 97–112.
8. M. Kojman and S. Shelah, *Almost isometric embedding between metric spaces*, Israel J. Math. **155** (2006), 309–334.
9. J. van Mill, *Infinite-dimensional topology, prerequisites and introduction*, North-Holland Publishing Co., Amsterdam, 1989.
10. ———, *The infinite-dimensional topology of function spaces*, North-Holland Publishing Co., Amsterdam, 2001.
11. J.R. Munkres, *Topology*, 2nd ed., Prentice-Hall, Upper Saddle River, N.J., 2000.
12. J. Väisälä, *Lipschitz homeomorphisms of the Hilbert cube*, Topology Appl. **11** (1980), 103–110.
13. B. Zamora Avilés, *Espacios numerablemente densos homogéneos*, Master's thesis, Universidad Michoacana de San Nicolás de Hidalgo, 2003 (in Spanish).

FACULTEIT DER EXACTE WETENSCHAPPEN / AFDELING WISKUNDE, VRIJE UNIVERSITEIT, DE BOELELAAN 1081A, 1081 HV AMSTERDAM, THE NETHERLANDS
Email address: dijkstra@cs.vu.nl