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THE ULTRAFILTER THEOREM IN REAL ALGEBRAIC GEOMETRY

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Introduction. Let R be a real closed field and let $V \subset R^n$ be a real algebraic set, $A(V) = R[x_1, \dots, x_n]/I(V)$ the affine coordinate ring of V. The ultrafilter theorem says there is a natural bijective correspondence between ultrafilters of semi-algebraic subsets of V and points of the real spectrum of $A(V)$, [**1**, **2**].

The real spectrum, $\operatorname{Spec}_R(A)$, of a commutative ring A can be identified with, or defined as, the collection of prime cones in A: that is, subsets, α, of A which satisfy (i) $\alpha + \alpha \subset \alpha$, (ii) $\alpha \cdot \alpha \subset \alpha$, (iii) $\Sigma A^2 \subset \alpha$, where ΣA^2 denotes the sums of squares in A, (iv) $-1 \notin \alpha$, (v) $\alpha \cup -\alpha = A$, and (vi) $\alpha \cap -\alpha = p(\alpha)$ is a prime ideal of A. Given such an α, the residue ring $A/p(\alpha)$ is totally ordered, with non-negative elements being the image of α. Conversely, given a total ordering on A/p, $p \subset A$ a prime ideal, the inverse image, α, of its non-negative elements satisfies (i)-(vi). Of course, total orderings of rings must be compatible with the arithmetic operations in the usual way. Note $V \subset \operatorname{Spec}_R[A(V)]$, since a point of V can be identified with a maximal ideal of $A(V)$ with residue ring R.

Given an ultrafilter of semi-algebraic subsets of V, define $\alpha \subset A(V)$ by $f \in \alpha$ if $f(x) \geq 0$, all $x \in C$, for some semi-algebraic $C \subseteq V$ which belongs to the ultrafilter. Then one can show $\alpha \in \operatorname{Spec}_R[A(V)]$, and this is the correspondence of the ultrafilter theorem.

By a constructible subset of $\operatorname{Spec}_R(A)$, we mean any member of the smallest family of subsets closed under finite intersections, finite unions, and complements, and containing the sets $W(f) = \{\alpha \in \operatorname{Spec}_R(A) | f \in \alpha\}$. Note that $f \in \alpha$ just says that the image, $f(\alpha)$, of f in $A/p(\alpha)$ is non-negative. If $x \in V \subset \operatorname{Spec}_R[A(V)]$, then $x \in W(f)$ says $f(x) \geq 0 \in R$.

We offer the following proof of the ultrafilter theorem, which has certainly been noticed by others, for example, L. van den Dries, M.

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Coste, M. Marshall, and M.F. Roy (oral communications). First $\mathrm{Spec}_R(A)$ is a closed subspace of 2^A with the product, or Tychonoff, topology. This amounts to examining the defining conditions (i) - (vi) of $\mathrm{Spec}_R(A)$. Secondly, compactness of 2^A in the Tychonoff topology implies that each ultrafilter of constructible sets in the closed subspace $\mathrm{Spec}_R(A)$, for any ring A, is principal, that is, it consists precisely of the constructible sets containing some point $\alpha \in \mathrm{Spec}_R(A)$. Moreover, the correspondence from points to ultrafilters is bijective. Thirdly, there is an inclusion preserving, bijective correspondence between semi-algebraic subsets of V and constructible subsets of $\mathrm{Spec}_R[A(V)]$, induced by the inclusion $V \subset \mathrm{Spec}_R[A(V)]$. This is exactly the Artin-Lang homomorphism theorem in real algebra. Thus ultrafilters of semi-algebraic sets in V correspond to ultrafilters of constructible sets in $\mathrm{Spec}_R[A(V)]$, which correspond to points of $\mathrm{Spec}_R[A(V)]$.

The Artin-Lang theorem is surely indispensable in any proof of the ultrafilter theorem. The above proof first points out a rather easy abstract ultrafilter theorem in the real spectrum of any ring, then applies the Artin-Lang theorem. Compactness of $\mathrm{Spec}_R(A) \subset 2^A$ in the Tychonoff topology is useful for many purposes in real algebra. It is fair to say that principality of certain ultrafilters of sets is not just a consequence of the Tychonoff theorem, but actually *is* the Tychonoff theorem. Thus the abstract ultrafilter theorem for $\mathrm{Spec}_R(A)$ should be viewed as essentially identical to compactness. In fact, in §2, we review the standard proof of compactness of 2^A during our discussion of ultrafilters.

The main goal of this paper is a somewhat detailed discussion of filters, prime filters, and ultrafilters in rather general situations, which can be applied naturally in real algebra, but which is purely set theoretical in nature. Filters, prime filters and ultrafilters can be defined in any family, $\mathcal{B}$, of subsets of a set Y, which is closed under finite intersections. We can pass from $\mathcal{B}$ to the family of subsets $\mathcal{C} = \mathcal{C}(\mathcal{B})$ obtained as finite unions of sets in $\mathcal{B}$. An easy result is that prime filters (respectively ultrafilters) in the two families $\mathcal{B}$ and $\mathcal{C}$ correspond bijectively.

Here is the situation we find most relevant. We begin with an open set subbasis, $\mathcal{B}^\circ$, for a topology on Y which we call the *weak topology.* Let $\overline{\mathcal{B}}_0$ denote the complements of the sets in $\mathcal{B}^\circ{}_0\overline{\mathcal{B}}$, and let $\mathcal{B}^\circ$, and $\mathcal{B}$ denote the finite intersections of sets in $\mathcal{B}^\circ$, $\overline{\mathcal{B}}_0$, and $\mathcal{B}_0 = \mathcal{B}^\circ{}_0 \cup \overline{\mathcal{B}}_0$, respectively. Thus, $\mathcal{B}^\circ$ is a basis for the weak topology, and $\mathcal{B}$ is a basis for another topology which we call the *strong topology.* Let $\mathcal{C}^\circ$, $\overline{\mathcal{C}}$, and $\mathcal{C}$ denote the finite unions of sets in $\mathcal{B}^\circ, \overline{\mathcal{B}}$, and $\mathcal{B}$, respectively. Although it might seem like this is too many families of subsets to keep straight, we have tried to choose notation and terminology which is "self-explanatory" from a topological viewpoint.

$$\begin{aligned} \mathcal{B}^\circ &= \text{ Subbasic open sets} \\ \overline{\mathcal{B}}_0 &= \text{ Subbasic closed sets} \\ \mathcal{B}_0 &= \text{ Subbasic sets} \end{aligned}$$

$$\begin{aligned} \mathcal{B}^\circ &= \text{ Basic open sets} \\ \overline{\mathcal{B}} &= \text{ Basic closed sets} \\ \mathcal{B} &= \text{ Basic sets} \end{aligned}$$

$$\begin{aligned} \mathcal{C}^\circ &= \text{ Open constructible sets} \\ \overline{\mathcal{C}} &= \text{ Closed constructible sets} \\ \mathcal{C} &= \text{ Constructible sets} \end{aligned}$$

Our main abstract result is Proposition 1.8, which we certainly do not claim is new. It says that if the strong topology on Y above is compact, then prime filters in all six families $\mathcal{B}, \mathcal{B}^\circ, \overline{\mathcal{B}}, \mathcal{C}, \mathcal{C}^\circ, \overline{\mathcal{C}}$ are principal and correspond bijectively to points of Y. In even more general situations we identify filters in all six families with certain pro-sets in Y, that is arbitrary non-empty intersections of sets in the family.

We apply the results to $Y = \operatorname{Spec}_R(A) \subset 2^A$, for any ring A, with the sets of $\overline{\mathcal{B}}_0$ being the sets $W(f) = \{\alpha \in \operatorname{Spec}_R(A) \mid f \in \alpha\} = \{\alpha \in \operatorname{Spec}_R(A) \mid f(\alpha) \geq 0\}$, introduced earlier. The filters in the families $\mathcal{C}, \overline{\mathcal{C}}$, and $\mathcal{C}^\circ$ have nice topological interpretations as Tychonoff (or strong) closed subsets of $\operatorname{Spec}_R(A)$, Tychonoff closed subsets closed under specialization, (these are the weak closed subsets), and Tychonoff closed subsets closed under generalization, respectively. In fact, these descriptions are purely set-theoretical results which have nothing to do with rings. It does not seem so clear how to interpret $\mathcal{B}, \overline{\mathcal{B}}$, and $\mathcal{B}^\circ$ filters topologically. We do, however, interpret them as certain kinds of "geometric" precones and multiplicative sets in the ring A. These interpretations are essentially reformulations of the abstract Nichtnegativstellensatz and Positivstellensatz [**4**].

We conclude the paper by applying the main results to another collection of subsets of $\operatorname{Spec}_R[A(V)]$, for V an irreducible real algebraic variety. The hope is that many interesting geometric phenomena can be identified as properties of filters of sets.

We point out that the basic results apply routinely to the Zariski spectrum of a ring, $\operatorname{Spec}(A)$, and to various families of algebraic and constructible subsets of varieties over algebraically closed fields. Perhaps there are other algebraic structures on sets A which lead to natural closed subspaces of 2^A like $\operatorname{Spec}(A)$ or $\operatorname{Spec}_R(A)$. Also, we point out that the set-theoretic results fall naturally under the rubric "Stone duality."

1. Review of filters. Let Y be a set, $\mathcal{B}$ a collection of subsets of Y closed under finite intersections. Our convention will be that Y, the empty intersection, belongs to $\mathcal{B}$, but this is not too important.

A *$\mathcal{B}$-filter* is a non-empty subcollection $F \subset \mathcal{B}$, such that $\emptyset \notin \mathcal{F}, B_1, B_2 \in \mathcal{F}$, implies $B_1 \cap B_2 \in \mathcal{F}$, and $B_1 \subset B_2$, $B_1 \in \mathcal{F}$, $B_2 \in \mathcal{B}$, implies $B_2 \in \mathcal{F}$. The $\mathcal{B}$-filter $\mathcal{F}$ is *prime* if $B = \cup_{i=1}^k B_i$, $B \in \mathcal{F}$, $B_i \in \mathcal{B}$ implies some $B_j \in \mathcal{F}$. Equivalently, if $B \subset \cup_{i=1}^k B_i$, $B \in \mathcal{F}$, $B_i \in \mathcal{B}$, then some $B_j \in \mathcal{F}$. Namely, if $B \subseteq \cup B_i$, then $B = \cup(B \cap B_i)$. A $\mathcal{B}$-filter is a *$\mathcal{B}$-ultrafilter* if it is not properly contained in any other $\mathcal{B}$-filter.

If $\mathcal{F}$ is a $\mathcal{B}$-filter and $B \in \mathcal{B}$ then there exists a $\mathcal{B}$-filter containing $\mathcal{F}$ and B if and only if $B \cap C \neq \emptyset$ for all $C \in \mathcal{F}$. In particular, all $\mathcal{B}$-ultrafilters are prime since, whenever $B = \cup_{i=1}^k B_i$ with $B \in \mathcal{F}$, $B_i \in \mathcal{B}$, then some B_j meets every element of $\mathcal{F}$. By Zorn's Lemma, every $\mathcal{B}$-filter is contained in a $\mathcal{B}$-ultrafilter.

Given $\mathcal{B}$, let $\mathcal{C} = \mathcal{C}(\mathcal{B})$ denote the collection of finite unions of sets in $\mathcal{B}$. Our convention will be that $\emptyset$, the empty union, belongs to $\mathcal{C}$. Now, $\mathcal{C}$ is also closed under finite intersections, so $\mathcal{C}$-filters are defined. It is clear that if $\mathcal{F}'$ is a $\mathcal{C}$-filter, then $\mathcal{F} = \mathcal{F}' \cap \mathcal{B}$ is a $\mathcal{B}$-filter, and if $\mathcal{F}$ is a $\mathcal{B}$-filter, then $\mathcal{F}^{\mathcal{C}} = \{C \in \mathcal{C} \mid B \subseteq C$, some $B \in \mathcal{F}$ is a $\mathcal{C}$-filter with $\mathcal{F}^{\mathcal{C}} \cap \mathcal{B} = \mathcal{F}$. The following is an easy exercise.

LEMMA 1.1. *The correspondences $\mathcal{F}' \to \mathcal{F}' \cap \mathcal{B}$ and $\mathcal{F} \to \mathcal{F}^{\mathcal{C}}$ are inclusion preserving bijections, and mutual inverses, between the* *sets of prime $\mathcal{C}$-filters and prime $\mathcal{B}$-filters. In particular, $\mathcal{C}$-ultrafilters correspond to $\mathcal{B}$-ultrafilters.*

REMARK. Note that, since $\mathcal{C}$ is closed under finite unions, the definition of prime $\mathcal{C}$-filter can be simplified slightly to $C = C_1 \cup C_2$, $C \in \mathcal{F}'$, implies $C_1 \in \mathcal{F}'$ or $C_2 \in \mathcal{F}'$. This simpler definition is not appropriate for $\mathcal{B}$-filters.

If $\alpha \in Y$ then $\mathcal{F}_{\mathcal{B}}(\alpha) = \{B \in \mathcal{B} \mid \alpha \in B\}$ is clearly a prime $\mathcal{B}$-filter, provided it is not empty. We call the filters $\mathcal{F}_{\mathcal{B}}(\alpha)$, $\alpha \in Y$, the *principal* $\mathcal{B}$-filters.

Now impose a topology on Y by taking as open set basis the collection of all *complements* of members of $\mathcal{C}$. Thus, sets in $\mathcal{C}$ are closed and the family of all closed sets, $\mathcal{K}$, is the collection of arbitrary intersections of sets in $\mathcal{C}$.

REMARK 1.2. If $\alpha \in Y$, then $\cap_{B \in \mathcal{F}_{\mathcal{B}}(\alpha)} B$ is easily seen to be the closure of the point α in the above topology. In particular, if $\alpha, \beta \in Y$, then $\mathcal{F}_{\mathcal{B}}(\alpha) \subseteq \mathcal{F}_{\mathcal{B}}(\beta)$ if and only if β belongs to the closure of α. The same statements hold if $\mathcal{F}_{\mathcal{B}}(\alpha)$ is replaced by $\mathcal{F}_{\mathcal{C}}(\alpha)$ or $\mathcal{F}_{\mathcal{K}}(\alpha)$.

LEMMA 1.3. *The following are equivalent.*

(i) *Y is quasi-compact.*

(ii) *For any $\mathcal{K}$-filter, C-filter, or β-filter $\mathcal{F}, \cap_{D \in \mathcal{F}} D \neq \emptyset$.*

(iii) *All $\mathcal{K}, \mathcal{C}$ or $\mathcal{B}$-ultrafilters are principal.*

PROOF. Any family of closed sets, no finite subfamily of which has empty intersection, can be extended to a filter. Thus (ii) is just a reformulation of quasi-compactness. If $\mathcal{F}$ is, say, a $\mathcal{B}$-filter and $\alpha \in \cap_{B \in \mathcal{F}} \mathcal{B}$, then $\mathcal{F} \subset \mathcal{F}_{\mathcal{B}}(\alpha)$, so if $\mathcal{F}$ is a $\mathcal{B}$-ultrafilter, $\mathcal{F} = \mathcal{F}_{\mathcal{B}}(\alpha)$.

Let us return to our family of subsets $\mathcal{B}$ of Y, closed under finite intersection. Suppose $\mathcal{B}_0 \subset \mathcal{B}$ is a subcollection such that every element of $\mathcal{B}$ is a finite intersection of elements of $\mathcal{B}_0$. We refer to $\mathcal{B}_0$ as a *set of generators* of $\mathcal{B}$. Then, for every $\mathcal{B}$-filter $\mathcal{F}$, the collection $\mathcal{F}_0 = \mathcal{F} \cap \mathcal{B}_0$ is a set of generators of $\mathcal{F}$. □

In our main examples, we will have a set of generators $\mathcal{B}_0 = \mathcal{B}^\circ{}_0 \cup \overline{\mathcal{B}}_0$, where the sets in $\overline{\mathcal{B}}_0$ are exactly the complements in Y of the sets in $\mathcal{B}^\circ{}_0$. Let $\mathcal{B}^\circ$ (respectively $\overline{\mathcal{B}}$) denote the collection of finite intersections of sets in $\mathcal{B}^\circ{}_0$ (respectively $\overline{\mathcal{B}}_0$). If $\mathcal{F}$ is a $\mathcal{B}$-filter, let $\overline{\mathcal{F}} = \mathcal{F} \cap \overline{\mathcal{B}}$ and $\mathcal{F}^\circ = \mathcal{F} \cap \mathcal{B}^\circ{}_0$. Then $\mathcal{F}$ has generators $\mathcal{F}_0 = \mathcal{F} \cap \mathcal{B}_0 = \mathcal{F} \cap (\mathcal{B}^\circ{}_0) = \overline{B}_0\mathcal{F}^\circ \cup \overline{\mathcal{F}}$, where $\mathcal{F}^\circ = \mathcal{F}^\circ \cap \mathcal{B}^\circ$ and $\overline{\mathcal{F}}_0 = \overline{\mathcal{F}} \cap \overline{\mathcal{B}}_0$ are generators for $\mathcal{F}^\circ$ and $\overline{\mathcal{F}}$, respectively. In particular, $\mathcal{F}$ is determined by $\mathcal{F}^\circ{}_0$ and $\overline{\mathcal{F}}_0$.

If $\mathcal{F}$ is a prime $\mathcal{B}$-filter, then, for each complementary pair of generators $B' \in \mathcal{B}^\circ{}_0$, $B \in \overline{\mathcal{B}}_0$, exactly one of B and B' must belong to $\mathcal{F}$, since $Y = B \cup B'$. Therefore, $\mathcal{F}$ is necessarily a $\mathcal{B}$-ultrafilter, and $\mathcal{F}$ is determined by either $\mathcal{F}^\circ{}_0$ or $\overline{\mathcal{F}}_0$.

Conversely, suppose $\mathcal{F}^\circ$ is a prime $\mathcal{B}^\circ$-filter, with generators $\mathcal{F}^\circ{}_0 = \mathcal{F}^\circ \cap \mathcal{B}^\circ{}_0$. Let $\overline{\mathcal{F}}_0 \subset \overline{\mathcal{B}}_0$ be the complements of the sets in $\mathcal{B}^\circ{}_0 - \mathcal{F}^\circ{}_0$. Then we claim $\mathcal{F}_0 = \mathcal{F}^\circ{}_0 \cup \overline{\mathcal{F}}_0$ generates a $\mathcal{B}$-filter, which is necessarily a $\mathcal{B}$-ultrafilter by construction. It suffices to check $\cap_{i=1}^m B_i \cap \cap_{j=1}^n C'_j \neq \emptyset$ for $B_i \in \overline{\mathcal{F}}_0, C'_j \in \mathcal{F}^\circ{}_0$. If such an intersection were empty, we would get $\cap_{j=1}^n C'_j \subseteq \cup_{i=1}^m B'_i$. Since $\mathcal{F}^\circ$ is a prime $\mathcal{B}^\circ$-filter, some $B'_i \in \mathcal{F}^\circ{}_0$, contradicting $B_i \in \overline{\mathcal{F}}_0$, we have proved

LEMMA 1.4. *Assume $\mathcal{B}_0 = \mathcal{B}^\circ \cup \overline{\mathcal{B}}_0, \mathcal{B}, \mathcal{B}^\circ$, and $\overline{\mathcal{B}}$ are as above. Then the correspondences $\mathcal{F} \to \mathcal{F} \cap \overline{\mathcal{B}}$ and $\mathcal{F} \to \mathcal{F} \cap \mathcal{B}^\circ$ are bijections between the set of prime $\mathcal{B}$-filters and the sets of prime $\overline{\mathcal{B}}$ and $\mathcal{B}^\circ$-filters, respectively. All prime $\mathcal{B}$-filters are $\mathcal{B}$-ultrafilters.*

If we let $\mathcal{C}, \mathcal{C}^\circ$, and $\overline{\mathcal{C}}$ denote, respectively, the finite unions of sets in $\mathcal{B}, \mathcal{B}^\circ$, and $\overline{\mathcal{B}}$, then, combining Lemmas 1.1 and 1.4 we see that prime filters in all six families correspond bijectively. Since $\mathcal{C}$ is closed under complements, prime $\mathcal{C}$-filters are $\mathcal{C}$-ultrafilters, just as was the case for $\mathcal{B}$ in Lemma 1.4.

In the situation $\mathcal{B}_0 = \mathcal{B}^\circ{}_0 \cup \overline{\mathcal{B}}_0$ above, we impose *two* topologies on Y. The *weak topology* has open set subbasis $\mathcal{B}^\circ{}_0$, hence open set basis $\mathcal{B}^\circ$ or $\mathcal{C}^\circ = \mathcal{C}(\mathcal{B}^\circ)$. Sets in $\mathcal{C}^\circ$ are exactly the complements of sets in $\overline{\mathcal{C}} = \mathcal{C}(\overline{\mathcal{B}})$, hence the weak closed sets are the arbitrary intersections of sets in $\overline{\mathcal{C}}$. The *strong topology* has open set subbasis $\mathcal{B}_0$, hence basis $\mathcal{B}$ or $\mathcal{C} = \mathcal{C}(\mathcal{B})$. Since $\mathcal{C}$ is closed under complements, sets in $\mathcal{C}$ are both open and closed in the *strong topology.* The strong closed sets are the arbitrary intersections of sets in $\mathcal{C}$.

We name our nine families of sets as follows:

$$\mathcal{B}^\circ{}_0 = \text{ Subbasic open sets}$$
$$\overline{\mathcal{B}}_0 = \text{ Subbasic closed sets}$$
$$\mathcal{B}_0 = \text{ Subbasic sets}$$

$$\mathcal{B}^\circ{}_0 = \text{ Basic open sets}$$
$$\overline{\mathcal{B}}_0 = \text{ Basic closed sets}$$
$$\mathcal{B}_0 = \text{ Basic sets}$$
$$\mathcal{C}^\circ = \text{ Open constructible sets}$$
$$\overline{\mathcal{C}} = \text{ Closed constructible sets}$$
$$\mathcal{C} = \text{Constructible sets}$$

REMARK 1.5. If Y is quasi-compact in the strong topology, then the open constructible sets are exactly the constructible sets which are open in the weak topology and the closed constructible sets are the constructible sets which are closed in the weak topology. Moreover, in this case, all sets which are both open and closed in the strong topology are constructible.

REMARK 1.6. Just as in Remark 1.2, the intersection of all basic closed sets containing $\alpha \in Y$, that is, $\cap_{B \in \mathcal{F}_{\overline{\mathcal{B}}}(\alpha)} B$, is the weak closure of the point α. Analogously, the intersection of all basic open sets containing $\alpha, \cap B' \in \mathcal{F}_{\mathcal{B}^\circ}(\alpha)\mathcal{B}'$, is the set of $\beta \in Y$ such that α is in the weak closure of β. Write $\alpha \to \beta$, read β *specializes* α and α *generalizes* β, if β is in the weak closure of α.

REMARK 1.7. Suppose $\mathcal{F}$ is a prime $\mathcal{B}$-filter. Then $\mathcal{F}^\circ = \mathcal{F} \cap \mathcal{B}^\circ$ is a $\mathcal{B}^\circ$-ultrafilter if and only if each $B \in \mathcal{F}$ contains some $V \in \mathcal{F}^\circ$. In particular, B has non-empty weak interior. It suffices to check the assertion for $B \in \overline{\mathcal{F}^\circ} = \mathcal{F} \cap \overline{\mathcal{B}}_0$. But $B \in \overline{\mathcal{F}}_0$ means the complement $B' \notin \mathcal{F}^\circ{}_0 = \mathcal{F}^\circ \cap \mathcal{B}^\circ{}_0$. Now, $\mathcal{F}^\circ$ is a $\mathcal{B}$-ultrafilter if and only if, for all such B', $B' \cap V = \emptyset$, for some $V \in \mathcal{F}^\circ$. Thus $V \subset B$.

The following result summarizes the main conclusions of this section. We point out that since constructible sets are open and closed in the strong topology, if $\alpha, \beta \in Y$ and β belongs to the strong closure of α, then α belongs to the strong closure of β, and α and β belong to the same constructible sets. Write $\alpha \sim \beta$ in this case. The strong topology is Hausdorff precisely when no two distinct points are equivalent, and, in this case, Y is totally disconnected in the strong topology.

PROPOSITION 1.8. *Suppose in the situation* $\mathcal{B}_0 = \mathcal{B}^\circ{}_0 \cup \overline{\mathcal{B}}_0$ *above, that* Y *is quasi-compact in the strong topology. Then prime filters in all six families* $\mathcal{B}, \mathcal{B}^\circ, \overline{\mathcal{B}}, \mathcal{C}, \mathcal{C}^\circ, \overline{\mathcal{C}}$ *are principal, and correspond bijectively to equivalence classes of points of* Y. *In particular, if* Y *is Hausdorff in the strong topology, prime filters correspond bijectively to points of* Y. *If* $\alpha, \beta \in Y$, *then* $\mathcal{F}_{\overline{\mathcal{B}}}(\alpha) \subseteq \mathcal{F}_{\overline{\mathcal{B}}}(\beta)$ *and* $\mathcal{F}_{\mathcal{B}^\circ}(\beta) \subseteq \mathcal{F}_{\mathcal{B}^\circ}(\alpha)$ *exactly when* $\alpha \to \beta$. *In particular, if* Y *is Hausdorff in the strong topology, then* $\mathcal{F}_{\overline{\mathcal{B}}}(\alpha)$ *is a* $\overline{\mathcal{B}}$*-ultrafilter* (*respectively* $\mathcal{F}_{\mathcal{B}^\circ}(\alpha)$ *is a* $\mathcal{B}^\circ$*-ultrafilter*) *if and only if* α *has no non-trivial specializations* (*respectively generalizations*) *in* Y.

PROOF. By Lemma 1.4, prime $\mathcal{B}$-filters are $\mathcal{B}$-ultrafilters and, by Lemma 1.3, if Y is quasi-compact, $\mathcal{B}$-ultrafilters are principal. Clearly, $\mathcal{F}_{\mathcal{B}}(\alpha) = \mathcal{F}_{\mathcal{B}}(\beta)$ if and only if $\alpha \sim \mathcal{B}$.

Also by Lemma 1.4, the prime $\mathcal{B}^\circ$-filters and the prime $\overline{\mathcal{B}}$-filters are the principal filters $\mathcal{F}_{\mathcal{B}^\circ}(\alpha) = \mathcal{F}_{\mathcal{B}}(\alpha) \cap \mathcal{B}^\circ$ and $\mathcal{F}_{\overline{\mathcal{B}}}(\alpha) = \mathcal{F}_{\mathcal{B}}(\alpha) \cap \overline{\mathcal{B}}$, and these correspond bijectively to the prime $\mathcal{B}$-filters. The last two statements follow from Remark 1.6. The results for $\mathcal{C}, \mathcal{C}^\circ$, and $\overline{\mathcal{C}}$-filters all follow from Lemma 1.1. □

2. Compactness of power sets. Let A be a set, 2^A the power set of A. If $a \in A$, let $W(a) = \{\alpha \subseteq A \mid a \in \alpha\}$ and $W'(a) = \{\beta \subseteq A \mid a \notin \beta\}$. Thus $W(a)$ and $W'(a)$ are a complementary pair of subsets of 2^A.

We set $\mathcal{B}^\circ{}_0 = \{W'(a) \mid a \in A\}$ and $\overline{\mathcal{B}}_0 = \{W(a) \mid a \in A\}$. Let $\mathcal{B}_0 = \mathcal{B}^\circ{}_0 \cup \overline{\mathcal{B}}_0$ and $\mathcal{B}, \mathcal{B}^\circ, \overline{\mathcal{B}}, \mathcal{C}, \mathcal{C}^\circ, \overline{\mathcal{C}}$, all be as in the previous section. Then $\mathcal{B}$ is the standard basis for the Tychonoff, or product, topology on 2^A, which is the strong topology of §1. If $\alpha \in 2^A$, then the *weak* closure of α is $\cap_{a \in \alpha} W(a) = \{\beta \in 2^A \mid \alpha \subset \beta\}$. Similarly, if $\alpha' \in 2^A$, then $\cap_{b \in \alpha'} W'(b) = \{\beta \in 2^A \mid \beta \subset A - \alpha'\}$.

Now suppose $\mathcal{F}$ is a $\mathcal{B}$-filter. Let $\alpha(\mathcal{F}) = \{a \in A \mid W(a) \in \mathcal{F}\}$ and let $\alpha'(\mathcal{F}) = \{b \in A \mid W'(b) \in \mathcal{F}\}$. Then, since $\overline{\mathcal{F}}_0 = \{W(a) \mid a \in \alpha(\mathcal{F})\}$

and $\mathcal{F}^\circ{}_0 = \{W'(b) \mid b \in \alpha'(\mathcal{F})\}$ generate $\mathcal{F}$, we have $\alpha(\mathcal{F}) \cap \alpha'(\mathcal{F}) = \emptyset$ and $\cap_{B \in \mathcal{F}} B = \cap_{a \in \alpha(\mathcal{F})} W(a) \cap \cap_{b \in \alpha'(\mathcal{F})} W'(b) = \{\beta \in 2^A \mid \alpha(\mathcal{F}) \subseteq \beta \subseteq A - \alpha'(\mathcal{F})\}$. In particular, $\cap_{B \in \mathcal{F}} B \neq \emptyset$. We conclude from this observation and Lemma 1.3 that 2^A is *compact* in the strong topology. (The Hausdorff property is trivial.)

Let $Y \subseteq 2^A$ be any Tychonoff closed subspace. Then Y is also compact. We relativize all the above families of sets to Y. That is $\mathcal{B}(Y) = \{B \cap Y \mid B \in \mathcal{B}\}$, and similarly we have the other families of subsets of $Y : \mathcal{B}_0(Y) = \mathcal{B}^\circ{}_0(Y) \cup \overline{\mathcal{B}}_0(Y), \mathcal{B}^\circ(Y), \overline{\mathcal{B}}(Y), \mathcal{C}(Y), \mathcal{C}^\circ(Y)$, and $\overline{\mathcal{C}}(Y)$. All of the discussion of §1 applies to these families. In particular, Proposition 1.8 implies

PROPOSITION 2.1. *Let $Y \subseteq 2^A$ be any Tychonoff closed subspace. Then prime filters in each of the classes of subsets of $Y, \mathcal{B}(Y), \mathcal{B}^\circ(Y), \overline{\mathcal{B}}(Y), \mathcal{C}(Y), \mathcal{C}^\circ(Y), \overline{\mathcal{C}}(Y)$ are principal, and correspond bijectively to points of Y.*

REMARK 2.2. We can actually write down the point of Y corresponding to a prime filter. If $\mathcal{F}$ is a $\mathcal{B}(Y)$-filter, we have $\alpha(\mathcal{F}) = \{a \in A \mid W(a) \cap Y \in \mathcal{F}\}$ and $\alpha'(\mathcal{F}) = \{b \in A \mid W'(b) \cap Y \in \mathcal{F}\}$. Also, $\cap_{B \in \mathcal{F}} B = \{\beta \in Y \mid \alpha(\mathcal{F}) \subseteq \beta \subseteq A - \alpha'(\mathcal{F})\}$. If $\mathcal{F}$ is a prime $\mathcal{B}$-filter, then $\alpha(\mathcal{F}) \cup \alpha'(\mathcal{F}) = A$, so $\cap_{B \in \mathcal{F}} B = \{\alpha(\mathcal{F})\}$. (Note $\cap_{B \in \mathcal{F}} B \neq \emptyset$ guarantees $\alpha(\mathcal{F}) \in Y$.)

If $\overline{\mathcal{F}}$ is a $\overline{\mathcal{B}}$-filter, then $\alpha(\overline{\mathcal{F}})$ is defined and $\cap_{B \in \overline{\mathcal{F}}} B = \{\beta \in Y \mid \alpha(\overline{\mathcal{F}}) \subseteq \beta\}$. If $\mathcal{F}^\circ$ is a $\mathcal{B}^\circ$-filter, then $\alpha'(\mathcal{F}^\circ)$ is defined and $\cap_{B \in \mathcal{F}^\circ} B = \{\beta \in Y \mid \beta \subseteq A - \alpha'(\mathcal{F}^\circ)\}$. If $\overline{\mathcal{F}}$ is a prime $\overline{\mathcal{B}}$-filter, then $\overline{\mathcal{F}} = \mathcal{F} \cap \overline{\mathcal{B}}$ for a unique prime $\mathcal{B}$-filter $\mathcal{F}$, and $\alpha(\overline{\mathcal{F}}) = \alpha(\mathcal{F}) \in Y$. (This is just Lemma 1.4, or, more precisely, its proof.) If $\mathcal{F}^\circ$ is a prime $\mathcal{B}^\circ$-filter, then $\mathcal{F}^\circ = \mathcal{F} \cap \mathcal{B}^\circ$ for a unique prime $\mathcal{B}$-filter $\mathcal{F}$, and $A - \alpha'(\mathcal{F}^\circ) = \alpha(\mathcal{F}) \in Y$.

In the weak topology on Y, that is, the topology with basis $\mathcal{B}^\circ(Y)$, specialization $\alpha \rightarrow \beta$ (i.e., β in the weak closure of α) just means $\alpha \subseteq \beta$. This is, of course, consistent with our formulas $\cap_{B \in \mathcal{F}_{\overline{\mathcal{B}}}(\alpha)} B = \{\beta \in Y \mid \alpha \subseteq \beta\}$ and $\cap_{B \in \mathcal{F}_{\mathcal{B}^\circ}(\alpha)} B = \{\beta \in Y \mid \beta \subseteq \alpha\}$, for the principal filters $\mathcal{F}_{\overline{\mathcal{B}}}(\alpha)$ and $\mathcal{F}_{\mathcal{B}^\circ}(\alpha)$, $\alpha \in Y$. In particular, in accordance with Proposition 1.8, the $\overline{\mathcal{B}}$-ultrafilters correspond to points of Y which are maximal under inclusion and the $\mathcal{B}^\circ$-ultrafilters correspond to points of Y which are minimal under inclusion. It is a very simple exercise to show that, for any Tychonoff closed $Y \subseteq 2^A$, both the union and the intersection of a chain of points of Y under inclusion are themselves points of Y.

3. Pro-$\mathcal{B}$ sets. We return to our basic situation, $\mathcal{B}$ a family of subsets of Y closed under finite intersections, $\mathcal{C}$ the finite unions of sets in $\mathcal{B}$, including $\emptyset$. In this section, we make the following compactness assumption, which holds in all the examples $\mathcal{B}, \mathcal{B}^\circ, \overline{\mathcal{B}}$, of §1 if Y is quasi-compact in the strong topology.

ASSUMPTION 3.1. If a member of $\mathcal{C}$ contains the intersection of an arbitrary family of elements of $\mathcal{C}$, then it contains the intersection of a finite subfamily.

By a *pro-$\mathcal{B}$ set* we mean an arbitrary intersection of elements of $\mathcal{B}$. A *pro-$\mathcal{B}$ set* F is *irreducible* if $F = \cup_{i=1}^k F_i$, with F_i pro-$\mathcal{B}$ sets, implies $F = F_j$, some j. It is not difficult to show that this is equivalent to the condition $F \subseteq \cup_{i=1}^k B_i$, with $B_i \in \mathcal{B}$, implies $F \subseteq B_j$, some j.

PROPOSITION 3.1. *The assignments $\mathcal{F} \to F(\mathcal{F}) = \cap_{B\in\mathcal{F}} B$ and $F \to \mathcal{F}(F) = \{B \in \mathcal{B} \,|\, F \subseteq B\}$ are inclusion reversing bijections, and mutual inverses, between the set of all $\mathcal{B}$-filters and the set of all non-empty pro-$\mathcal{B}$ sets F. The $\mathcal{B}$-filter $\mathcal{F}$ is prime if and only if $F(\mathcal{F})$ is an irreducible pro-$\mathcal{B}$ set. The $\mathcal{B}$-ultrafilters correspond to the minimal non-empty pro-$\mathcal{B}$ sets.*

PROOF. First note that if $\mathcal{F}$ is a $\mathcal{B}$-filter, then $F = \cap_{B\in\mathcal{F}} B \neq \emptyset$, because every finite intersection is non-empty and Proposition 3.1 can be applied to $\emptyset \in \mathcal{C}$. Next, abusing notation slightly, we must show $F(\mathcal{F}(F)) = F$ and $\mathcal{F}(F(\mathcal{F})) = \mathcal{F}$. Three of the four set inclusions are tautologies which follow immediately from definitions. The only non-trivial inclusion is $\mathcal{F}(F(\mathcal{F})) \subseteq \mathcal{F}$. We need to know that if $\cap_{B\in\mathcal{F}} B \subseteq B'$ then $B' \in \mathcal{F}$. But this follows from Proposition 3.1.

Obviously, if $\mathcal{F}_1$ and $\mathcal{F}_2$ and $\mathcal{B}$-filters and $\mathcal{F}_1 \subset \mathcal{F}_2$, then $F_2 \subset F_1$, where $F_i = F(\mathcal{F}_i)$, and conversely. Thus our correspondences reverse inclusions, and ultrafilters correspond to minimal non-empty pro-$\mathcal{B}$ sets.

Finally, suppose $\mathcal{F}$ is a prime and suppose $F = \cap_{B\in\mathcal{F}} B \subseteq \cup_{i=1}^{k} B_i$. Then, by Proposition 3.1, $B \subseteq \cup_{i=1}^{k} B_i$, for some $B \in \mathcal{F}$. Hence $B_j \in \mathcal{F}$, some j, and therefore $F \subseteq B_j$, so F is irreducible. Conversely, if F is irreducible and $B \subseteq \cup_{i=1}^{k} B_i$, for some $B \in \mathcal{F}(F)$ (that is, $F \subseteq B$), then $F \subseteq \cup_{i=1}^{k} B_i$, so $F \subseteq B_j$, some j. Thus, $B_j \in \mathcal{F}(F)$ and $\mathcal{F}(F)$ is prime. □

In this situation $\mathcal{B}_0 = \mathcal{B}^\circ{}_0 \cup \overline{\mathcal{B}}_0$ etc., of §1, with Y quasi-compact in the strong topology, Proposition 3.2 also applies to the families $\mathcal{C}, \mathcal{C}^\circ$ and $\overline{\mathcal{C}}$, of constructible sets, open constructible set and closed constructible sets, respectively. We have

PROPOSITION 3.3. *The pro-$\mathcal{C}$ sets are exactly the strong closed subsets of Y. The pro-$\overline{\mathcal{C}}$ sets are exactly the weak closed subsets of Y. The weak closed subsets are the strong closed subsets, K, which are closed under specialization, that is, if $\alpha, \beta \in Y$, $\alpha \to \beta$, $\alpha \in K$, then $\beta \in K$. The pro-$\mathcal{C}^\circ$ sets are the strong closed subsets, L, which are closed under generalization, that is, if $\alpha, \beta \in Y, \alpha \to \beta$, and $\beta \in L$, then $\alpha \in L$.*

PROOF. The first two statements are trivial from the definitions and don't depend on quasi-compactness. If K is a strong closed subset of Y, certainly the weak closure of K contains all specializations of elements of K. Conversely, if $\beta \in Y$ is not a specialization of any $\alpha \in K$, then, for each $\alpha \in K$, choose a weak closed set $B_\alpha \in \overline{\mathcal{C}}$ with $\alpha \in B_\alpha$, $\beta \notin B_\alpha$. By strong quasi-compactness, finitely many of the B_α, say B_{α_i}, $1 \le i \le n$, cover K. Now if B'_{α_i}, is the complement of B_{α_i}, then $\cap_{i=1}^{n} B'_{\alpha_i}$ is a weak open neighborhood of β disjoint from K, so β is not in the weak closure of K. The proof of the characterization of pro-$\mathcal{C}^\circ$ sets, which are, of course, strong closed sets, is almost identical to this. □

REMARK 3.4. There does not seem to be a tidy topological characterization of pro-$\mathcal{B}$ sets, pro-$\overline{\mathcal{B}}$ sets, or pro-$\mathcal{B}^\circ$ sets. We discuss this further for the real spectrum of a ring in the next section. At least, we know the irreducible pro-sets, that is, the pro-sets corresponding to the prime, principal filters $\mathcal{F}_{\mathcal{B}}(\alpha)$, $\mathcal{F}_{\overline{\mathcal{B}}}(\alpha)$, and $\mathcal{F}_{\mathcal{B}^\circ}(\alpha)$. By Remarks 1.2 and 1.6, these are, respectively, the strong closure of α, the weak closure of α or set of specializations of α, and the set of generalizations of α. That is, $F(\mathcal{F}_{\mathcal{B}}(\alpha)) = \{\beta \in Y \mid \alpha \sim \beta\}$, $F(\mathcal{F}_{\overline{\mathcal{B}}}(\alpha)) = \{\beta \in Y \mid \alpha \to \beta\}$, and $F(\mathcal{F}_{\mathcal{B}^\circ}(\alpha)) = \{\beta \in Y \mid \beta \to \alpha\}$.

4. The real spectrum. If A is a commutative ring, its real spectrum, $\operatorname{Spec}_R(A)$, defined in the introduction, is a Tychonoff closed subset of 2^A. If $f \in A$ and $\alpha \in \operatorname{Spec}_R(A)$, then $f \in \alpha$, or $\alpha \in W(f)$, is the same as $f(\alpha) \geq 0$, where $f(\alpha)$ is the image of f in the totally ordered integral domain $A/p(\alpha)$, $p(\alpha) = \alpha \cap -\alpha$. Also, $f \notin \alpha$, or $\alpha \in W'(f)$, is the same as $f(\alpha) < 0$ in $A/p(\alpha)$. We will simplify notation and write $W(f)$ instead of $W(f) \cap \operatorname{Spec}_R(A)$ and write $U(f)$ instead of $W'(-f) \cap \operatorname{Spec}_R(A)$. Thus, $W(f) = \{\alpha \in \operatorname{Spec}_R(A) \mid f(\alpha) \geq 0\}$ and $U(f) = \{\alpha \in \operatorname{Spec}_R(A) \mid f(\alpha) > 0\}$. The weak topology is the Harrison topology on $\operatorname{Spec}_R(A)$, with open subbasis $\mathcal{B}^\circ{}_0 = \{U(f) \mid f \in A\}$ [**4**].

All the results of §1, §2, and §3 apply to $\operatorname{Spec}_R(A)$. In particular, prime filters in all six families of subsets $\mathcal{B}, \overline{\mathcal{B}}, \mathcal{B}^\circ, \mathcal{C}, \overline{\mathcal{C}}, \mathcal{C}^\circ$ are principal and correspond bijectively to points of $\operatorname{Spec}_R(A)$. In this section, we want to collect a few results about the real spectrum which are not so general.

The first result concerns $\mathcal{B}^\circ$-ultrafilters. Recall from Proposition 1.8 that $\overline{\mathcal{B}}$-ultrafilters correspond to closed points of $\operatorname{Spec}_R(A)$, that is, points with no proper specializations, and $\mathcal{B}^\circ$-ultrafilters correspond to points with no proper generalizations. We have the general result of Remark 1.7 that $\alpha \in \operatorname{Spec}_R(A)$ has no proper generalizations if and only if every (basic, closed) constructible set containing α, contains an open neighborhood of α.

PROPOSITION 4.1. *The point $\alpha \in \operatorname{Spec}_R(A)$ has no proper generalizations if every $g \in p(\alpha) = \alpha \cap -\alpha$ vanishes on an open set containing α. In the geometric case, $A = A(V)$, $V \subset R^n$ a real algebraic set, α has no proper generalizations if every* (*basic, closed*) *constructible set containing α has non-empty interior.*

PROOF. Suppose $\beta \subsetneq \alpha$, and $g \in \alpha$, $g \notin \beta$. Then also $-g \in \alpha$. (Otherwise, $\beta \in W'(g) \cap W'(-g) = U(-g) \cap U(g) = \emptyset$. This is just the well-known formula $\alpha = \beta \cup p(\alpha)$.) Now, if g vanishes on the open set V containing α, then we have $\beta \in V$, since $\beta \to \alpha$, and, thus, the contradiction $\beta \in W'(g) \cap V = U(-g) \cap V = \emptyset$.

In the geometric case, the interior, $\operatorname{int}(B)$, of any constructible B is an open constructible, by the Finiteness Theorem. Thus, if every B which contains α has interior, then, in fact, every such B contains an open neighborhood of α. Otherwise, $\alpha \in B - \operatorname{int}(B)$, but this last set has empty interior. □

REMARK 4.2. In the geometric case, there are only *finitely many* prime ideals $p_i \subset A(V)$ which support minimal points $\alpha \in \operatorname{Spec}_R(A)$. (The *support* of α is the prime ideal $p(\alpha) = \alpha \cap -\alpha$, or the irreducible subvariety corresponding to $p(\alpha)$.) These prime $\{p_i\}$ include the minimal primes of $A(V)$, that is, the irreducible components of V, then the irreducible components of $U = V - \overline{V}_{\text{reg}}$, then the irreducible components of $V - \overline{V}_{\text{reg}} - \overline{U}_{\text{reg}}$, and so on, where $\overline{V}_{\text{reg}}$ denotes the closure of the regular points of V. Each step lowers dimension, so the process terminates. These are exactly the primes such that the corresponding subvarieties of V contain open sets in V. By contrast, every subvariety of V supports closed points of $\operatorname{Spec}_R(A)$.

Next we discuss general $\overline{\mathcal{B}}$-filters. By Proposition 3.2, these correspond to non-empty pro-basic closed sets. However, it is not clear which weak closed sets are pro-basic closed sets. Any subset $X \subset \operatorname{Spec}_R(A)$ determines a minimum pro-basic closed set containing X, namely, $F(X) = \cap_{X \subseteq W(f)} W(f)$. Thus, if $f \in A$, then $f \geq 0$ on X if and only if $f \geq 0$ on $F(X)$. An easy compactness argument shows that any pro-basic closed set which is constructible is, in fact, a basic closed set. In the geometric case at least, the assignment $C \to F(C)$ is *injective* on the set of closed constructible subsets $C \subseteq \operatorname{Spec}_R(A(V))$. However, certain generalizations of points of C will belong to $F(C)$. For example, if $C = W(x) \cup W(y) \subset \operatorname{Spec}_R(R[x, y])$, then 2-dimensional points β which have 1-dimensional specializations on the negative x or y-axis must belong to $F(C)$.

Possibly, pro-basic closed sets can be characterized in terms of fans.

Pro-basic closed sets, F, are parametrized by the subsets of A,

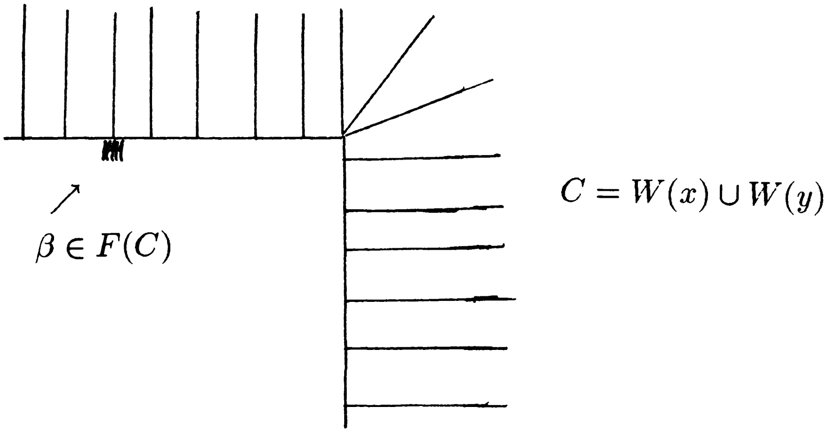

Figure 1

$\alpha(F) = \{f \in A \,|\, f \geq 0$ on $F\} = \{f \in A \,|\, F \subseteq W(f)\}$. Of course, this $\alpha(F)$ is the same as the set $\alpha(\mathcal{F})$ of §2, where $\mathcal{F}$ is the $\overline{\mathcal{B}}$-filter corresponding to F. These sets $\alpha(F) = \alpha(\mathcal{F})$ have a nice algebraic characterization. Recall that a *precone* is a subset $\rho \subset A$ such that (i) $\rho + \rho \subset \rho$, (ii) $\rho\rho \subset \rho$ (iii) $\Sigma A^2 \subset \rho$ (iv) $-1 \notin \rho$. We say ρ is a *geometric precone* if, in addition, (v) $(f^{2n} + p)f = f^{2n} + q$, p, $q \in \rho$ implies $f \in \rho$.

PROPOSITION 4.3. *The assignment $\mathcal{F} \to \alpha(\mathcal{F})$ is an inclusion reversing bijection between the set of $\overline{\mathcal{B}}$-filters and the set of geometric precones in A.*

PROOF. This result is simply a reformulation of the abstract Nichtnegativstellensatz [4]. Specifically, if $f \in A$ and $f \geq 0$ on $\cap_i W(g_i)$, then there is a formula $(f^{2n} + p)f \doteq f^{2n} + q$, some $n \geq 1$, p, $q \in \Sigma A^2[g_i]$. (Here, $\Sigma A^2[g_i]$ denotes the smallest subset of A closed under sums and products, and containing all squares and the g_i.) From this fact, the Proposition is easy. Another formulation of the Proposition is the statement that the intersection of all prime cones containing a given precone ρ, that is, $\cap_{\alpha \in \mathrm{Spec}_R(A), \rho \subseteq \alpha} \alpha$, is exactly the set

$\{f \in A \mid (f^{2n} + p)f = f^{2n} + q, \text{ some } n \geq 1,\ p,\ q \in \rho\}$. □

Various conditions on a $\overline{\mathcal{B}}$-filter can have interesting algebro-geometric interpretations. For example, we say a set $X \subseteq \operatorname{Spec}_R(A)$ is *Zariski dense* if $f \in A$ and $f = 0$ on X implies $f = 0$. The following is a trivial exercise.

PROPOSITION 4.4. *The following are equivalent conditions on a $\overline{\mathcal{B}}$-filter $\mathcal{F}$.*

(a) $\alpha(\mathcal{F}) \cap -\alpha(\mathcal{F}) = (0)$.

(b) $F = F(\mathcal{F}) = \cap_{B \in \mathcal{F}} B$ *is Zariski dense.*

(c) *Every* $B \in \mathcal{F}$ *is Zariski dense.*

(d) *For all* $g \neq 0, W(g) \notin \mathcal{F}$ *or* $W(-g) \notin \mathcal{F}$.

We now turn to $\mathcal{B}^\circ$-filters, which, by Proposition 3.2, correspond to pro-basic open sets. Any subset $X \subset \operatorname{Spec}_R(A)$ determines a pro-basic open set $V(X) = \cap_{X \subseteq W'(f)} W'(f) = \cap_{f<0 \text{ on } X} U(-f)$. Thus, $f < 0$ on X if and only if $f < 0$ on $V(X)$. If a pro-basic open set is constructible it is a basic open set, and, in the geometric case, the map $C \to V(C)$ is injective on the set of open constructibles C. The problem of topologically characterizing pro-basic open sets seems quite similar to the pro-basic closed case.

Analogous to Proposition 4.3, we can give a nice algebraic characterization of the sets $\alpha'(\mathcal{F}) = \{f \in A \mid f < 0 \text{ on } V\} = \{f \in A \mid W'(f) \in \mathcal{F}\}$, where $\mathcal{F}$ is a $\mathcal{B}^\circ$-filter and V the corresponding pro-basic open set. It is more convenient to work with $\sigma(\mathcal{F}) = -\alpha'(\mathcal{F}) = \{f \in A \mid f > 0 \text{ on } V\}$. Then $\sigma = \sigma(\mathcal{F})$ satisfies the following conditions: (i) $\sigma \cdot \sigma \subset \sigma$, (ii) $0 \notin \sigma$, (iii) $\sigma + \Sigma A^2[\sigma] \subset \sigma$, (iv) $sf = t,\ s,\ t \in \sigma$ implies $f \in \sigma$. We refer to such (non-empty) σ as *positive*, *saturated*, *multiplicative sets* in A. The conditions (ii) and (iii) guarantee that $\Sigma A^2[\sigma]$ is a precone. (Note $1 \in \sigma$ by (iv).) If we write $f < g$ to mean $0 \neq g - f \in \Sigma A^2[\sigma]$, then $f \in \sigma$ implies $0 < f$ and $f < g$, $f \in \sigma$ implies $g \in \sigma$, which explains the term "positive". Condition (iv) is the standard notion of saturated multiplicative set. It says that the elements of A which become units under the localization $A \to A_\sigma$ are exactly the elements of $A^* \cdot \sigma$, where A^* denotes the units of A.

PROPOSITION 4.5. *The assignment $\mathcal{F} \to \sigma(\mathcal{F}) = -\alpha'(\mathcal{F})$ is an inclusion reversing bijection between the set of $\mathcal{B}^\circ$-filters and the set of positive, saturated, multiplicative sets in A.*

PROOF. This result is a reformulation of the abstract Positivstellensatz [**4**] which implies that if $f \in A$ and $f > 0$ on $\cap_i U(g_i)$ then there is a formula $(s+p)f = s+q$ where s belongs to the multiplicative set generated by the g_i and $p, q \in \Sigma A^2[g_i]$. Note that $sf = t$ implies $(st+s^2)f = st+t^2$, so the conditions (iv) $sf = t,\ s,\ t \in \sigma$ implies $f \in \sigma$ and (iv)$'$ $(s+p)f = s+q,\ s \in \sigma,\ p,\ q \in \Sigma A^2[\sigma]$, implies $f \in \sigma$, are interchangeable. □

Next, we discuss briefly the $\mathcal{B}$-filters, which correspond to non-empty pro-basic sets. A pro-basic set is $F \cap V$ where F is pro-basic closed and V is pro-basic open. Then $F \cap V$ determines a $\mathcal{B}$-filter, $\mathcal{F}$, which determines and is determined by, according to §2, the $\overline{\mathcal{B}}$ and $\mathcal{B}^\circ$-filters $\overline{\mathcal{F}} = \mathcal{F} \cap \overline{\mathcal{B}}$ and $\mathcal{F}^\circ = \mathcal{F} \cap \mathcal{B}^\circ$. It seems difficult to characterize the pairs $(\overline{\mathcal{F}}, \mathcal{F}^\circ)$ corresponding to $\mathcal{B}$-filters $\mathcal{F}$. Note $F \cap V = F(\overline{\mathcal{F}}) \cap V(\mathcal{F}^\circ)$, but, in general, $F(\overline{\mathcal{F}}) \subsetneqq F$ and $V(\mathcal{F}^\circ) \subsetneqq V$. $F(\overline{\mathcal{F}})$ should be very close to weak closure of $F \cap V$, that is, the set of specializations of points of $F \cap V$, and $V(\mathcal{F}^\circ)$ should be very close to the set of generalizations of points of $F \cap V$.

We can write down compatibility conditions on a geometric precone ρ and a positive, saturated multiplicative set σ such that $(\rho, \sigma) = (\alpha(\overline{\mathcal{F}}),\ -\alpha'(\mathcal{F}^\circ))$ for some $\mathcal{B}$-filter $\mathcal{F}$. Consider the conditions

$$(*) \qquad (sg^{2n}+p)g = sg^{2n}+q,\ s \in \sigma,\ p,\ q \in \Sigma A^2[\rho, \sigma] \text{ implies } g \in \rho;$$

$$(**) \qquad (s+p)h = s+q,\ s \in \sigma,\ p,\ q \in \Sigma A^2[\rho, \sigma] \text{ implies } h \in \sigma.$$

Then the following is a consequence of the abstract Nichtnegativstellensatz and Positivstellensatz.

PROPOSITION 4.6. *We have $(\rho, \sigma) = (\alpha(\overline{\mathcal{F}}),\ -\alpha'(\mathcal{F}^\circ))$ for some $\mathcal{B}$-filter $\mathcal{F}$, with $\overline{\mathcal{F}} = \mathcal{F} \cap \overline{\mathcal{B}}$ and $\mathcal{F}^\circ = \mathcal{F} \cap \mathcal{B}^\circ$, if and only if (*) and (**) hold for the geometric precone ρ and positive, saturated multiplicative set σ.*

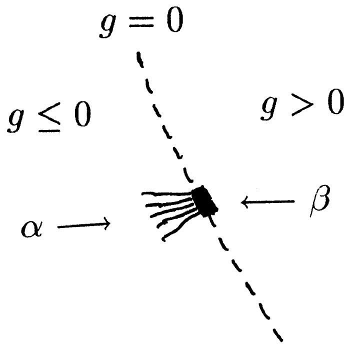

Figure 2

It seems likely that there are other families of subsets of real spectra which have interesting filters, prime filters and ultrafilters, besides the families considered in this paper. For example, in [**3**] we considered the family, $\mathcal{V}^\circ$, of finite intersections of sets in $\mathcal{V}^\circ{}_0 = \{V(f) = \overline{U(f)}^{\circ} \cap \overline{X_{\mathrm{reg}}}\ f \in A(X)\}$, where X is an *irreducible* real algebraic variety, $\overline{X_{\mathrm{reg}}}$ is the closure of simple points, and $\overline{U}(f)^{\circ}$ is the interior of the closure of $U(f)$. Thus, $V(f)$ is the set of points of $\overline{X_{\mathrm{reg}}}$ where f is "locally nowhere negative". We showed that $\mathcal{V}^\circ$-filters correspond bijectively to partial orders in the function field $k(X)$, (which are just the precones in $k(X)$), and that $\mathcal{V}^\circ$-ultrafilters correspond to total orders of $k(X)$. What about the prime $\mathcal{V}^\circ$-filters? The complements of the $V(f)$ are the sets in $\overline{\mathcal{V}}_0 = \{V'(f) = \overline{U(-f)} \mid f \in A(X)\}$. All these sets are semi-algebraic, hence constructible in $\operatorname{Spec}_R(A(X))$. If $Y \subseteq \operatorname{Spec}_R(A(X))$ is the constructible corresponding to $\overline{X_{\mathrm{reg}}} \subseteq X$, then we have families of subsets of Y, as in §1, $\mathcal{V}^\circ, \overline{\mathcal{V}}, \mathcal{V}, \mathcal{C}(\mathcal{V}^\circ), \mathcal{C}(\overline{\mathcal{V}}), \mathcal{C}(\mathcal{V})$, and a weak and a strong topology, with bases $\mathcal{V}^\circ$ and $\mathcal{V}$, respectively. Proposition 1.8 applies, and we deduce that prime $\mathcal{V}^\circ$-filters are principal, $\mathcal{F}_{\mathcal{V}^\circ}(\alpha)$, $\alpha \in Y$. What is not so immediate is that the strong $\mathcal{V}$-topology is Hausdorff.

Since the $V(f)$ are open, $\mathcal{V}^\circ \subset \mathcal{C}(\mathcal{B}^\circ)$ by the Finiteness Theorem. We claim, conversely, that $\mathcal{B}^\circ \subset \mathcal{C}(\mathcal{V}^\circ)$, so, in fact, $\mathcal{C}(\mathcal{V}^\circ) = \mathcal{C}(\mathcal{B}^\circ)$ and the weak and strong $\mathcal{V}$-topologies are the same as the weak and strong topologies already studied. Here is the delicate separation lemma which is needed to establish this fact, which we state without proof.

LEMMA 4.7. *Suppose* $\alpha \neq \beta \in Y$ *and* $\alpha \to \beta$. *Then there exists* $g \in A(X)$ *such that* $\beta \in \overline{U(g)}$ *and* $\alpha \notin \overline{U(g)}$.

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