ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 3, Summer 1989

AUTOMORPHISMS AND ISOMORPHISMS OF HENSELIAN FIELDS

RON BROWN

We announce here some results on automorphisms and isomorphisms of formally real Henselian valued fields. Proofs and further results (including generalizations to fields which are not formally real) will appear elsewhere [**3, 4**].

1. Automorphisms. We begin by recalling two standard facts about real closed fields.

(A) The field $\mathbf{R}$ of real numbers (and indeed any Archimedean real closed field) has no nontrivial automorphisms.

(B) A real closure of a field F admits no nontrivial automorphisms fixing F.

Contrary to the impression these results might give, in general a real closed field can admit many automorphisms. One has a bijective Galois correspondence between the set of fixed fields of a real closed field K with respect to sets of automorphisms of K and the set of groups of automorphisms of K of the form $\mathrm{Aut}\,(K/F)$ for some set $F \subset K$. Can the fixed fields be characterized intrinsically? It is easy to see such a field must be algebraically and topologically closed in K. (Facts (A) and (B) above are corollaries of this observation.) We announce a partial converse.

Recall that a real closed field K admits a canonical valuation with Archimedean real closed residue class field [**1**, §4]. The order topology on K is just the valuation topology for this valuation, unless the valuation is trivial.

Received by the editors on October 28, 1986.
Partially supported by an NSF grant.

THEOREM C. *Let K be a real closed field which is a maximal valued field with respect to its canonical valuation. Then a subfield F of K is the fixed field of a set of automorphisms of K (so that $F = K^{\mathrm{Aut}(K/F)}$) if and only if F is algebraically and topologically closed in K.*

For example the field "R" of real algebraic numbers is a fixed field of any nonarchimedean real closed maximal valued field.

Theorem C can be considerably generalized.

THEOREM D. *Let K be a maximal valued field with Archimedean real closed residue class field. A subfield F of K is the fixed field of a set of automorphisms of K if and only if F is topologically closed in K and the algebraic closure of F in K is a multiquadratic extension of F.*

The topology on K referred to above is the order topology for any ordering on K (all such topologies are the same). The algebraic closure of F in K is the set of elements of K which are algebraic over F. The theorem implies, for example, that R and $R((x^2))$ are fixed fields of the field $\mathbf{R}((x))$ of Laurent series over $\mathbf{R}$.

We cannot replace the hypothesis in Theorem D that K is a maximal valued field by the weaker hypothesis that K is Henselian. A counterexample can be obtained by letting F be the Henselization of $\mathbf{R}(x_1^2, x_2, x_3)((x_4))$ in the iterated Laurent series field

$$U := \mathbf{R}((x_1))((x_2))((x_3))((x_4))$$

and letting K be the Henselization of $F(x_1, y)$ in U, where

$$y = \sum_{n>0} (x_2 x_3^2)^{n!} + x_1 \sum_{n>0} (x_2 x_3)^{n!}.$$

The proof of Theorem D has two main parts. One involves the construction of homomorphisms from "small" extensions of F into K; heavy use is made here of Kaplansky's analysis of maximal valued fields [**5**]. The other involves extension theorems for isomorphisms, which brings us to our second topic.

2. Isomorphisms. We assume throughout this section that K and L are Henselian valued extensions with real closed residue class fields of a valued field F. We want to determine when K and L are F-isomorphic. The next example shows that it does not suffice to require that K and L give isomorphic value group and residue class field extensions of the value group and residue class field of F.

EXAMPLE E. Let $F = \mathbf{R}((x))$, $K = F[\sqrt{x}]$ and $L = F[\sqrt{-x}]$. Then K and L are not F-isomorphic since $x \in K^2 \cap F$ but $x \notin L^2$. However, K and L have the same residue class field (namely $\mathbf{R}$) and the same value group extension (namely, $(1/2)\mathbf{Z}$) of the value group of F (namely, $\mathbf{Z}$).

Let vF, vK and vL denote the value groups of F, K and L. If K and L are algebraic extensions of F, then we can and do canonically identify vK and vL with subgroups of a fixed divisible hull of vF.

THEOREM F. *Say K and L are algebraic extensions of F. The following statements are equivalent.*

(1) *K and L are F-isomorphic (as valued fields).*

(2) *K and L are F-isomorphic (as fields).*

(3) *$K^n \cap F = L^n \cap F$ for all integers $n > 0$.*

(4) *$K^2 \cap F = L^2 \cap F$ and $vK = vL$.*

(5) *Some ordering of F extends to an ordering of K and to an ordering of L, and $vK = vL$.*

The implication "(5)⇒(2)" above generalizes the fact that real closures K and L of an ordered field F are isomorphic (note that in this situation vK and vL are both equal to the divisible hull of vF). One can easily deduce Becker's condition [**1**, Theorem 4.4 (ii)] for the conjugacy of generalized real closures from the implication "(3)⇒(1)".

Most of the nontrivial content of Theorem F is in the following result.

THEOREM G. *Suppose that K and L are algebraic extensions of F and that P_K and P_L are orderings of K and L which induce the same ordering on F. Then there exists an F-isomorphism $\Delta : K \to L$ with $\Delta(P_K) = P_L$ if and only if $vK = vL$. Such an isomorphism, if it exists, is unique.*

The uniqueness part of Theorem G yields the following computation of $\mathrm{Aut}\,(K/F)$, the group of F-automorphisms of K.

THEOREM H. *Suppose K and L are algebraic extensions of F. There is a group isomorphism $\Phi : \mathrm{Aut}\,(K/F) \to \mathrm{Hom}\,(vK/vF, \mathbf{Z}^{\cdot})$ such that, for all $\tau \in \mathrm{Aut}\,(K/F)$, $\Phi(\tau)$ maps the image in vK/vF of any $a \in K^{\cdot}$ to the "sign" ($+1$ or -1) of the residue class of $a/\tau^{-1}(a)$.*

One can formulate a generalization [**3**, §5] of Theorem G which allows P_K and P_L to be orderings of higher level [**1**, §2]. Some restriction needs to be put on the exact levels of P_K and P_L, as is shown by the fields in Example E, for which Becker has observed that every ordering of higher level of F extends to orderings of higher level of K and of L. We give here only a corollary of this generalization.

THEOREM I. *Let (K, P_K) and (L, P_L) be generalized real closures of (F, P), where P is an ordering of F of higher level. There exists an F-isomorphism $\Delta : K \to L$ with $\Delta(P_K) = P_L$ if and only if $vK = vL$.*

(The hypotheses of Theorem I imply all the covering hypotheses of this section, where the valuations are those canonically associated with P_K, P_L and P [**1**, §2].)

Theorem G has an analogue [**3**, §4] for field extensions which are not algebraic, but which are maximal valued fields. We will state here only a corollary of this analogue, so as to illustrate the kind of isomorphism extension theorem used in the study of automorphisms discussed in §1.

THEOREM J. *Suppose S and T are orderings of a maximal valued field K with Archimedean real closed residue class field. Suppose θ is a homomorphism from a subfield E of K into K. Suppose $\theta(E \cap S) \subset T$ and $v\theta = v|E$ where v is the valuation on K. Then θ extends to an automorphism θ' of K with $\theta'(S) = T$ and $v\theta' = v$.*

The hypothesis $\theta(E \cap S) \subset T$ above guarantees that $v\theta$ and $v|E$ are equivalent, but does not guarantee they are equal!

3. Fields which are not formally real. We describe briefly here a generalization of some of the theory in the previous section to fields which are not necessarily formally real. Details and further results will appear in [**4**]. Our motivation is to provide a unified setting for studying formally real and formally p-adic fields. For both these types of fields a central role is played by places into a field with a distinguished absolute value (namely, the field of real numbers or an algebraic extension of the field of p-adic numbers). Extended absolute values [**2**] provide a convenient formalism for studying exactly this kind of situation. Using this concept, one can define "orderings" and "order closures" for fields of characteristic zero. Just as in the Artin-Schreier theory, isomorphism classes of order closures correspond bijectively to orderings. Examples of order closed fields include real closed fields, algebraically closed fields, and p-adically closed fields [**6**]. Properly reformulated, Theorem F is valid (all five parts) in this generalized context. The theorem of Prestel and Roquette [**6**, Corollary 3.11] on the isomorphism of Henselian p-valued algebraic extensions of a field is a special case of part of the generalized theorem.

REFERENCES

1. E. Becker, *Extended Artin-Schreier theory of fields,* Rocky Mountain J. Math. **14** (1984), 881-897.

2. R. Brown, *Extended prime spots and quadratic forms,* Pacific J. Math. **51** (1974), 379-395.

3. ———, *Automorphisms and isomorphisms of real Henselian fields,* Trans. AMer. Math. Soc. **307** (1988), 675-703.

4. ———, *Orderings and order closures of not necessarily formally real fields,* in preparation.

5. I. Kaplansky, *Maximal fields with valuations,* Duke Math. J. **9** (1942), 303-321.

6. A Prestel and P. Roquette, *Formally p-adic Fields,* Lecture notes in Mathematics, no. 1050. Springer-Verlag, Berlin, 1984.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII, HONOLULU, HI 96822