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# STABILITY PROPERTIES OF PERIODIC SOLUTIONS OF PERIODICALLY FORCED NON-DEGENERATE SYSTEMS

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**1. Introduction.** In this paper we study the stability properties of T-periodic solutions of the ordinary differential equation

$$x' = f(x) + \varepsilon F(t,x) \tag{1.1}$$

where $'$ denotes $\frac{d}{dt}$ and $\varepsilon \in \mathbf{R}$ is a small parameter. We make the following hypotheses about (1.1):

1. Let $U \subseteq \mathbf{R}^2$ be open, $0 \in U$. $f : U \to \mathbf{R}^2$ is of class $C^2$ and $f(x) = 0$ if and only if $x = 0$.

2. $F : \mathbf{R} \times U \to \mathbf{R}^2$ is of class $C'$ on its domain and $F(t,x) = F(t+T,x)$ for all $(t,x) \in \mathbf{R} \times U$.

3. 0 is a center of

$$x' = f(x), \tag{1.2}$$

that is, there exists a continuum $C$ of periodic orbits of (1.2) contained in $U$ and enclosing the origin. Moreover, $C$ contains a nontrivial periodic solution of least period $T$ which will be denoted by $u$.

4. $u$ is non-degenerate where we define non-degenerate periodic solutions as follows.

Let $v$ be a nontrivial $q$-periodic solution of (1.2). Associated with (1.2) and $v$ we have the linear variational equation

$$y' = f_x(v(t))y, \tag{1.3}$$

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where $f_x(v(t))$ denotes the Jacobian matrix of $f$ evaluated at $v(t)$.

DEFINITION 1.1. $v$ is degenerate if and only if every solution of (1.3) is $q$-periodic.

We shall use non-degenerate to mean "not degenerate."

DEFINITION 1.2. We say that (1.2) is degenerate if and only if each member of $C$ is degenerate.

The next proposition, which we will state without proof, relates the concept of degeneracy to the periods of the elements of $C$.

PROPOSITION 1.1. (1.2) *is degenerate if and only if every element of* $C$ *has the same minimum period.*

At this point we introduce some notation. The symbol $\cdot$ will denote the scalar product on $\mathbf{R}^2$ and $|\ |$ will denote the absolute value of a real number, the Euclidean norm on $\mathbf{R}^2$, or the induced norm of a matrix or linear operator $\mathbf{R}^2 \to \mathbf{R}^2$; which one will be clear from context. A vector $x = (x_1, x_2) \in \mathbf{R}^2$ will be identified with its column representation, $\operatorname{col}(x_1, x_2)$. $x^t$ will denote the row vector $[x_1, x_2]$. A linear operator $L : \mathbf{R}^2 \to \mathbf{R}^2$ will be identified with its matrix representation with respect to the canonical basis of $\mathbf{R}^2$. In particular, the letter $I$ will denote both the identity operator on $\mathbf{R}^2$ and the $2 \times 2$ identity matrix. $A$ will denote the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$x(t, x_0, \varepsilon)$ will denote the solution of (1.1) such that $x(0, x_0, \varepsilon) = x_0$. We will use $x_{x_0}$ to denote the derivative of $x$ with respect to the intial condition coordinate and $x_\varepsilon$ to denote the derivative of $x$ with respect to the parameter coordinate.

$f_{xx}$ will denote the second derivative of $f$ with respect to its variable, a symmetric bilinear mapping from $\mathbf{R}^2 \times \mathbf{R}^2$ into $\mathbf{R}^2$.

We gather some results from elementary Floquet theory into a proposition which will be stated without proof.

PROPOSITION 1.2.

1. *If $\mu_1(0)$ and $\mu_2(0)$ are the characteristic multipliers of (1.3) with $v = u$, then $\mu_1(0) = \mu_2(0) = 1$.*

2.

$$\int_0^T \operatorname{tr}[f_x(u(t))]dt = 0. \tag{1.4}$$

3. *If $t = 0$ is chosen, without loss of generality, such that $u'(0)$ is parallel to the horizontal $x_1$ axis, in the positive direction, then the principal matrix solution (1.3) with $v = u$ is given by*

$$X(t) = \left[\frac{u'(t)}{|u'(0)|}, p(t) + Kt\frac{u'(t)}{|u'(0)|}\right] \tag{1.5}$$

*where $p : \mathbf{R} \to \mathbf{R}^2$ is $C'$, $T$-periodic, and has $p(0) = col(0,1)$ and $K$ is a constant.*

REMARK. $u$ is degenerate if and only if $K = 0$.

Finally, we establish a local coordinate system about $u$:

$$\hat{t}(t) = \frac{u'(t)}{|u'(t)|}\ ;\ \hat{n}(t) = A\hat{t}(t). \tag{1.6}$$

Then [**3**], the following theorems are proved concerning the existence of $T$-periodic solutions of (1.1).

THEOREM 1.2. *There exist $\varepsilon_0 > 0$ and $R : (-\varepsilon_0, \varepsilon_0) \times \mathbf{R} \to \mathbf{R}$ of class $C'$ and $T$-periodic in the second variable such that*

$$\hat{t}(s) \cdot [x(T, u(s) + R(\varepsilon, s)\hat{n}(s), \varepsilon) - u(s) - R(\varepsilon, s)\hat{n}(s)] = 0 \tag{1.7}$$

*for all $s \in \mathbf{R}$. Furthermore, $R(0, s) = 0$ for all $s \in \mathbf{R}$.*

REMARK. The geometric meaning of Theorem 1.2 is that solutions of (1.1) with initial point $u(s) + R(\varepsilon, s)\hat{n}(s)$ return to the line through $u(s)$ normal to $\{u(t) | 0 \le t \le T\}$ after time $T$.

We define the $C'$ function $\gamma : (-\varepsilon_0, \varepsilon_0) \times \mathbf{R} \to \mathbf{R}^2$ by $\gamma(\varepsilon, s) = u(s) + R(\varepsilon, s)\hat{n}(s)$.

THEOREM 1.3. *Let $g(s)$ be defined by*

$$g(s) = \hat{n}(s) \cdot x_\varepsilon(T, u(s), 0). \tag{1.8}$$

*Assume that there exists an $s_0 \in [0, T]$ such that*

$$g(s_0) = 0 \quad \text{and} \tag{1.9}$$

$$g'(s_0) \neq 0. \tag{1.10}$$

*Then there exists an $\varepsilon_1 > 0$ and a $C'$ function $\tilde{s} : (-\varepsilon_1, \varepsilon_1) \to \mathbf{R}$, $\tilde{s}(0) = s_0$, such that for any $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$,*

$$\gamma(\varepsilon, \tilde{s}(\varepsilon)) = u(\tilde{s}(\varepsilon)) + R(\varepsilon, \tilde{s}(\varepsilon))\hat{n}(\tilde{s}(\varepsilon)) \tag{1.11}$$

*is the initial condition of a $T$-periodic solution of (1.1). Furthermore, if*

$$z(t, \varepsilon, s_0) = x(t, u(\tilde{s}(\varepsilon)) + R(\varepsilon, \tilde{s}(\varepsilon))\hat{n}(\tilde{s}(\varepsilon)), \varepsilon) \tag{1.12}$$

*denotes this family of $T$-periodic solutions, then*

$$z(t, \varepsilon, s_0) = u(t + s_0) + \varepsilon\beta(t, s_0) + \rho_0(t, \varepsilon, s_0) \tag{1.13}$$

*where $\lim_{\varepsilon \to 0} |\rho_0(t, \varepsilon, s_0)|/\varepsilon = 0$ uniformly in $t$. Moreover, $\beta$ is a $T$-periodic solution of*

$$\begin{aligned} y' &= f_x(u(t + s_0))y + F(t, u(t + s_0)) \\ y(0) &= \frac{\partial R}{\partial \varepsilon}(0, s_0)\hat{n}(s_0) + \tilde{s}'(0)|u'(s_0)|\hat{t}(s_0). \end{aligned} \tag{1.14}$$

*Finally, $\beta$ is given explicitly by*

$$\beta(t, s_0) = x_\varepsilon(t, u(s_0), 0) + x_{x_0}(t, u(s_0), 0)\beta(0, s_0). \tag{1.15}$$

REMARK. Let

$$e(t) = \exp\Big[\int_0^t \mathrm{tr}(f_x(u(s)))ds\Big]. \tag{1.16}$$

Then it is a straightforward calculation to show that

$$g(s_0) = 0, \ g'(s_0) \neq 0$$

if and only if

$$h(s_0) = 0, \ h'(s_0) \neq 0$$

where

$$h(s) = \int_0^T \frac{[Au'(s+\sigma)]^t}{e(s+\sigma)} F(\sigma, u(s+\sigma))d\sigma. \tag{1.17}$$

Moreover, if $g(s_0) = 0$,

$$\operatorname{sgn}[g'(s_0)] = \operatorname{sgn}[h'(s_0)]. \tag{1.18}$$

In §2 we will state and prove our main result which gives sufficient conditions for the asymptotic stability, or instability, of $z(t, \varepsilon, s_0)$. Finally, in §3, we analyze an example which illustrates the theory presented here.

**2. Stability**. In this section, we study the stability properties of the $T$-periodic solution

$$z(t, \varepsilon, s_0) = u(t+s_0) + \varepsilon\beta(t, s_0) + \rho_0(t, \varepsilon, s_0) \tag{1.13}$$

of

$$x' = f(x) + \varepsilon F(t, x) \tag{1.1}$$

where $s_0$ is determined by (1.9) and (1.10). We begin with the relevant definitions. (see, for example, [**2**, p.26].)

For the purposes of the next definition, let $x(t, t_0, x_0, \varepsilon)$ denote the solution of (1.1) with $x(t_0, t_0, x_0, \varepsilon) = x_0$.

DEFINITION 2.1. The solution $x(t, t_0, x_0, \varepsilon)$ of (1.1) is said to be Lyapunov stable, if and only if for any $\gamma > 0$ and for any $t_0 \geq 0$, there exists $\delta = \delta(\gamma, t_0)$ such that $|x_0 - y_0| < \delta$ implies $|x(t, t_0, x_0, \varepsilon) - x(t, t_0, y_0, \varepsilon)| < \gamma$ for $t \in [t_0, \infty)$. The solution $x(t, t_0, x_0, \varepsilon)$ is asymptotically stable if and only if it is stable and there exists $b = b(t_0)$ such that $|x_0 - y_0| < b$ implies $|x(t, t_0, x_0, \varepsilon) - x(t, t_0, y_0, \varepsilon)| \to 0$ as $t \to \infty$. The solution $x(t, t_0, x_0, \varepsilon)$ is unstable if and only if it is not stable.

REMARK. Since (1.1) is periodic in $t$, if a solution $x(t, t_0, x_0, \varepsilon)$ is stable (asymptotically stable) then $\delta(\gamma, t_0)(b(t_0))$ can be chosen independently of $t_0$. (see [**2**, Lemma 4.1, p.27]).

The stability properties of $z(t, \varepsilon, s_0)$ are determined by the characteristic multipliers $\mu_1(\varepsilon), \mu_2(\varepsilon)$ of the linear variational equation

$$y' = f_x(z(t, \varepsilon, s_0))y + \varepsilon F_x(t, z(t, \varepsilon, s_0))y. \tag{2.1}$$

PROPOSITION 2.1.

a. *if both* $|\mu_1(\varepsilon)|, |\mu_2(\varepsilon)| < 1$, *then* $z(t, \varepsilon, s_0)$ *is asymptotically stable; and*

b. *if one of* $|\mu_1(\varepsilon)|, |\mu_2(\varepsilon)| > 1$, *then* $z(t, \varepsilon, s_0)$ *is unstable.*

PROOF. See, for example, Th. 2.1., p.322, and Th. 1.2, p.317 of [**1**].

REMARK. If both $|\mu_1(\varepsilon)|, |\mu_2(\varepsilon)| = 1$ or if $|\mu_1(\varepsilon)| = 1$ and $|\mu_2(\varepsilon)| < 1$, then stability cannot be determined by the linear approximation.

Let $Z(t, \varepsilon, s_0)$ be the fundamental matrix solution of (2.1) with $Z(0, \varepsilon, s_0) = X(s_0)$, where $X$ is defined by (1.5). Then $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$ are the eigenvalues of

$$Z^{-1}(0, \varepsilon, s_0)Z(T, \varepsilon, s_0) = X^{-1}(s_0)Z(T, \varepsilon, s_0). \tag{2.2}$$

In order to study $Z(t, \varepsilon, s_0)$ further, we begin by observing that

$$\begin{aligned} &f_x[u(t+s_0) + \varepsilon\beta(t, s_0) + \rho_0(t, \varepsilon, s_0)] \\ &\quad + \varepsilon F_x[t, u(t+s_0) + \varepsilon\beta(t, s_0) + \rho_0(t, \varepsilon, s_0)] \\ &\quad = f_x(u(t+s_0)) + \varepsilon f_{xx}(u(t+s_0))\beta(t, s_0) \\ &\quad + \varepsilon F_x(t, u(t+s_0)) + \rho_1(t, \varepsilon, s_0) \end{aligned} \tag{2.3}$$

where $\lim_{\varepsilon\to 0} |\rho_1(t, \varepsilon, s_0)|/\varepsilon = 0$, uniformly in $t$. Using (2.3), (2.1) can be rewritten as

$$\begin{aligned} y' &= f_x(u(t+s_0))y + \varepsilon f_{xx}(u(t+s_0))\beta(t, s_0)y \\ &\quad + \varepsilon F_x(t, u(t+s_0))y + \rho_1(t, \varepsilon, s_0)y. \end{aligned} \tag{2.4}$$

Applying the variation of constants formula to (2.4), we obtain

$$
\begin{aligned}
Z(t,\varepsilon,s_0) &= X(t+s_0) + \varepsilon X(t+s_0)\int_0^t X^{-1}(s+s_0) \\
&[f_{xx}(u(s+s_0))\beta(s,s_0) + F_x(s,u(s+s_0))]Z(s,\varepsilon,s_0)ds \\
&\quad + \rho_2(t,\varepsilon,s_0)
\end{aligned}
\tag{2.5}
$$

where $\lim_{\varepsilon\to 0}|\rho_2(t,\varepsilon,s_0)|/\varepsilon = 0$ uniformly for $t\in[0,T]$. From (2.4) and Ch. 2, Theorem 4.1 of [**1**], we see that

$$
Z(t,\varepsilon,s_0) = X(t+s_0) + \rho_3(t,\varepsilon,s_0)
\tag{2.6}
$$

where $\lim_{\varepsilon\to 0}|\rho_3(t,\varepsilon,s_0)| = 0$, uniformly for $t\in[0,T]$. Substituting (2.6) into (2.5) yields

$$
\begin{aligned}
Z(t,\varepsilon,s_0) &= X(t+s_0) + \varepsilon X(t+s_0)\int_0^t X^{-1}(s+s_0) \\
&\cdot [f_{xx}(u(s+s_0))\beta(s,s_0) + F_x(s,u(s+s_0))]X(s+s_0)ds \\
&\quad + \rho_4(t,\varepsilon,s_0)
\end{aligned}
\tag{2.7}
$$

where $\lim_{\varepsilon\to 0}|\rho_4(t,\varepsilon,s_0)|/\varepsilon = 0$ uniformly for $t\in[0,T]$. Using (2.7) at $t=T$ we obtain

$$
\begin{aligned}
&Z^{-1}(0,\varepsilon,s_0)Z(T,\varepsilon,s_0) \\
&\quad = M + \varepsilon M\int_0^T X^{-1}(s+s_0)[f_{xx}(u(s+s_0))\beta(s,s_0) \\
&\qquad + F_x(s,u(s+s_0))]X(s+s_0)ds + \rho_5(\varepsilon,s_0)
\end{aligned}
\tag{2.8}
$$

where $M = \begin{pmatrix} 1 & KT \\ 0 & 1 \end{pmatrix}$ and $\lim_{\varepsilon\to 0}|\rho_5(\varepsilon,s_0)|/\varepsilon = 0$.

Hence $\mu_1(\varepsilon),\mu_2(\varepsilon)$ are eigenvalues of a matrix of the form

$$
\begin{pmatrix} 1+\varepsilon d_{11}(s_0)+\rho_{11}(\varepsilon,s_0) & KT+\varepsilon d_{12}(s_0)+\rho_{12}(\varepsilon,s_0) \\ \varepsilon d_{21}(s_0)+\rho_{21}(\varepsilon,s_0) & 1+\varepsilon d_{22}(s_0)+\rho_{22}(\varepsilon,s_0) \end{pmatrix}
\tag{2.9}
$$

where $\lim_{\varepsilon\to 0}|\rho_{ij}(\varepsilon,s_0)|/\varepsilon = 0$, $i,j=1,2$.

The characteristic equation of (2.9) is

$$
(\lambda-1)^2-\varepsilon(d_{11}(s_0)+d_{22}(s_0))(\lambda-1)-\varepsilon KTd_{21}(s_0)+\rho_6(\varepsilon,s_0)=0
\tag{2.10}
$$

where $\lim_{\varepsilon\to 0}|\rho_6(\varepsilon, s_0)|/\varepsilon = 0$. The roots of (2.10) are

$$\begin{aligned}\lambda =& 1+\frac{\varepsilon}{2}[d_{11}(s_0)+d_{22}(s_0)]\pm\\ &\frac{1}{2}\sqrt{\varepsilon^2(d_{11}(s_0)+d_{22}(s_0))^2+4\varepsilon KTd_{21}(s_0)+4\rho_6(\varepsilon,s_0)}.\end{aligned}\tag{2.11}$$

Examination of (2.11) yields

a. If $\varepsilon Kd_{21}(s_0) > 0$, then the roots are both real, one less than 1 and the other greater than 1, and $z(t, \varepsilon, s_0)$ is unstable for $\varepsilon$ sufficiently small; and

b. If $\varepsilon Kd_{21}(s_0) < 0$, the roots are complex conjugates for $\varepsilon$ small and the square of their common modulus is

$$\det[Z^{-1}(0, \varepsilon, s_0)Z(T, \varepsilon, s_0)].$$

In this case, $z(t, \varepsilon, s_0)$ is unstable or asymptotically stable as $\det[Z^{-1}(0, \varepsilon, s_0)Z(T, \varepsilon, s_0)]$ is greater than or less than 1.

Fortunately, reasonably simple expressions can be derived for $d_{21}(s_0)$ and $\det[Z^{-1}(0, \varepsilon, s_0)Z(T, \varepsilon, s_0)]$.

In order to calculate $d_{21}(s_0)$, let $z_1(t, \varepsilon, s_0)$ denote the first column of $Z(t, \varepsilon, s_0)$. That is,

$$z_1(t, \varepsilon, s_0) = Z(t, \varepsilon, s_0)\mathrm{col}(1, 0).\tag{2.12}$$

Using (2.12) and (2.8), we calculate

(2.13)
$$\begin{aligned}&Z^{-1}(0, \varepsilon, s_0)z_1(T, \varepsilon, s_0)\\ &= \mathrm{col}(1,0) + (\varepsilon/|u'(0)|)M\int_0^T X^{-1}(s+s_0)[f_{xx}(u(s+s_0))\beta(s, s_0)\\ &\qquad + F_x(s, u(s+s_0))]u'(s+s_0)ds + \rho_7(\varepsilon, s_0)\end{aligned}$$

where $\lim_{\varepsilon\to 0}|\rho_7(\varepsilon, s_0)|/\varepsilon = 0$.

At this point, let us recall the differential equation for $\beta$:

$$\beta' = f_x(u(t+s_0))\beta + F(t, u(t+s_0)).\tag{2.14}$$

Differentiate (2.14) with respect to $t$ to obtain

$$\begin{aligned}\beta'' =& f_x(u(t+s_0))\beta' + [f_{xx}(u(t+s_0))\beta \\ &+ F_x(t,u(t+s_0))]u'(t+s_0) + F_t(t,u(t+s_0))).\end{aligned} \tag{2.15}$$

(2.15) can be solved using the variation of constants formula to obtain

$$\begin{aligned}&\beta'(t,s_0) = X(t+s_0)X^{-1}(s_0)\beta'(0,s_0) \\ &+ X(t+s_0)\int_0^t X^{-1}(s+s_0)\{[f_{xx}(u(s+s_0))\beta(s,s_0) \\ &\quad + F_x(s,u(s+s_0))]u'(s+s_0) + F_t(s,u(s+s_0))\}ds.\end{aligned} \tag{2.16}$$

Since $\beta$ is T-periodic, $\beta'$ is T-periodic also so that

$$\begin{aligned}&[I - X(T+s_0)X^{-1}(s_0)]\beta'(0,s_0) \\ &\quad - X(T+s_0)\int_0^T X^{-1}(s+s_0)F_t(s,u(s+s_0))ds \\ &= X(T+s_0)\int_0^T X^{-1}(s+s_0)[f_{xx}(u(s+s_0))\beta(s,s_0) \\ &\quad + F_x(s,u(s+s_0))]u'(s+s_0)ds.\end{aligned} \tag{2.17}$$

Using (2.17) in (2.13), we obtain

$$\begin{aligned}&Z^{-1}(0,\varepsilon,s_0)z_1(T,\varepsilon) \\ &= \mathrm{col}(1,0) + \varepsilon M\{[X^{-1}(T+s_0) - X^{-1}(s_0)]\beta'(0,s_0)/|u'(0)| \\ &- (1/|u'(0)|)\int_0^T X^{-1}(s+s_0)F_t(s,u(s+s_0))ds\} + \rho_7(\varepsilon,s_0).\end{aligned} \tag{2.18}$$

Since

$$[0,1]Z^{-1}(0,\varepsilon,s_0)z_1(T,\varepsilon,s_0) = \varepsilon d_{21}(s_0) + \rho_{21}(\varepsilon,s_0),$$

(2.18) implies

$$d_{21}(s_0) = -([0,1]/|u'(0)|)\int_0^T X^{-1}(s+s_0)F_t(s,u(s+s_0))ds. \tag{2.19}$$

We remark that it is a short calculation using the definition of $X$ to show that

$$d_{21}(s_0) = -(1/|u'(0)|^2)\int_0^T \frac{[Au'(s+s_0)]^t}{e(s+s_0)}F_t(s,u(s+s_0))ds. \tag{2.20}$$

By the standard theory of linear systems

$$\tag{2.21} \begin{aligned} &\det[Z^{-1}(0,\varepsilon,s_0)Z(T,\varepsilon,s_0)] \\ &\quad = \exp\left\{ \int_0^T \operatorname{tr}[f_x(z(s,\varepsilon,s_0)) + \varepsilon F_x(s, z(s,\varepsilon,s_0))]ds \right\}. \end{aligned}$$

Thus

$$\det[Z^{-1}(0,\varepsilon,s_0)Z(T,\varepsilon,s_0)] > 1 \text{ or } < 1$$

as

$$\tag{2.22} \begin{aligned} Q(\varepsilon,s_0) = &\int_0^T \operatorname{tr}[f_x(z(s,\varepsilon,s_0)) + \varepsilon F_x(s, z(s,\varepsilon,s_0))]ds \\ &> 0 \text{ or } < 0. \end{aligned}$$

The preceding discussion has proved

THEOREM 2.1. *Let $d_{21}(s_0)$ be given by* (2.19) *or* (2.20) *and let $Q(\varepsilon, s_0)$ be given by* (2.22). *Then*

a. *If $\varepsilon K d_{21}(s_0) > 0$, then $z(t,\varepsilon,s_0)$ is unstable for $\varepsilon$ sufficiently small;*

b. *If $\varepsilon K d_{21}(s_0) < 0$ and $Q(\varepsilon,s_0) > 0$, then $z(t,\varepsilon,s_0)$ is unstable for $\varepsilon$ sufficiently small; and*

c. *If $\varepsilon K d_{21}(s_0) < 0$ and $Q(\varepsilon,s_0) < 0$, then $z(t,\varepsilon,s_0)$ is asymptotically stable for $\varepsilon$ sufficiently small.*

In order to study $Q(\varepsilon,s_0)$ further, we use (2.3) and (1.4) to obtain

$$Q(\varepsilon,s_0) = \varepsilon Q_0(s_0) + \rho_8(\varepsilon,s_0).$$

where

$$\tag{2.23} Q_0(s_0) = \int_0^T \operatorname{tr}[f_{xx}(u(s+s_0))\beta(s,s_0) + F_x(s, u(s+s_0))]ds$$

and $\lim_{\varepsilon\to 0} |\rho_8(\varepsilon,s_0)|/\varepsilon = 0$.

We then have, as a corollary to Theorem 2.1, COROLLARY 2.2. *Let $d_{21}(s_0)$ be given by* (2.19) *or* (2.20) *and let $Q_0(s_0)$ be given by* (2.23). *Then*

a. *If $\varepsilon K d_{21}(s_0) > 0$, then $z(t,\varepsilon,s_0)$ is unstable for $\varepsilon$ sufficiently small;*

b. *If $\varepsilon K d_{21}(s_0) < 0$ and $\varepsilon Q_0(s_0) > 0$, then $z(t,\varepsilon,s_0)$ is unstable for $\varepsilon$ sufficiently small; and*

c. *If $\varepsilon K d_{21}(s_0) < 0$ and $\varepsilon Q_0(s_0) < 0$, then $z(t,\varepsilon,s_0)$ is asymptotically stable for $\varepsilon$ sufficiently small.*

Finally, we examine the relationship between $d_{21}(s_0)$, as given by (2.19) and $g'(s_0)$, where $g$ is the branching function defined in Theorem 1.3.

THEOREM 2.3. *Let $d_{21}(s_0)$ be given by* (2.19) *and let $g$ be given by* (1.8). *Then*

$$\operatorname{sgn}[g'(s_0)] = \operatorname{sgn}[d_{21}(s_0)]. \tag{2.24}$$

PROOF. By direct calculation

$$g(s) = \hat{n}(s)\cdot p(s)[0,1]\int_0^T X^{-1}(t+s)F(t,u(t+s))dt. \tag{2.25}$$

Since $\hat{n}(0)\cdot p(0) = 1$ and since $X(s)$ is invertible for all $s$, we have that $\hat{n}(s)\cdot p(s) > 0$ for all $s$. Thus, $g(s_0) = 0$ if and only if $[0,1]\int_0^T X^{-1}(t+s)F(t,u(t+s))dt = 0$ and

$$\operatorname{sgn}[g'(s_0)] = \operatorname{sgn}\left[\frac{d}{ds}[0,1]\int_0^T X^{-1}(t+s)F(t,u(t+s))dt\right]\Bigg|_{s=s_0}. \tag{2.26}$$

Next we examine

$$\begin{aligned} I(s) &= \frac{d}{ds}\left\{[0,1]\int_0^T X^{-1}(t+s)F(t,u(t+s))dt\right\} \\ &= [0,1]\int_0^T \frac{d}{ds}[X^{-1}(t+s)]F(t,u(t+s))dt \\ &\quad + [0,1]\int_0^T X^{-1}(t+s)F_x(t,u(t+s))u'(t+s)dt. \end{aligned} \tag{2.27}$$

We integrate the first term by parts after observing that

$$\frac{d}{ds}[X^{-1}(t+s)] = \frac{d}{dt}[X^{-1}(t+s)]$$

to obtain

$$\begin{aligned} I(s) = {} & [0,1][X^{-1}(t+s)F(t,u(t+s))]_{t=0}^{T} \\ & - [0,1]\int_0^T X^{-1}(t+s)[F_t(t,u(t+s)) \\ & \qquad\qquad + F_x(t,u(t+s))u'(t+s)]dt \\ & + [0,1]\int_0^T X^{-1}(t+s)F_x(t,u(t+s))u'(t+s)dt. \end{aligned} \tag{2.28}$$

Since the second row of $X^{-1}(t+s)$ is $T$-periodic in its argument, the first member on the right hand side of (2.28) evaluates to 0. Thus, $I(s)$ becomes simply

$$I(s) = -[0,1]\int_0^T X^{-1}(t+s)F_t(t,u(t+s))dt. \tag{2.29}$$

Using (2.19), (2.26), (2.27), and (2.29), we see that

$$\operatorname{sgn}[g'(s_0)] = \operatorname{sgn}[I(s_0)] = \operatorname{sgn}[d_{21}(s_0)], \tag{2.30}$$

thus completing the proof of the theorem.

In [**5**], W.S. Loud studied the equation

$$x'' + g(x,x') = \varepsilon f(t,x,x',\varepsilon) \tag{2.31}$$

where $x$ is a scalar. If $f$ in (2.31) does not depend on $\varepsilon$, (2.31) can be recast as a particular instance of system (1.1). In this case, Theorem 3.9 of [**5**] which discusses stability when $s_0 = 0$ can be deduced from Corollary 2.2 of this paper. The case $s_0 \neq 0$, treated in Theorem 3.12 of [**5**], can also be obtained from Corollary 2.2 in part. However, Theorem 3.12 there contains additional information about stability when $\varepsilon K d_{21}(s_0) > 0$ and $Q_0(s_0) = 0$, obtained by use of the $\varepsilon^2$ term in the power series for $z(t,\varepsilon,s_0)$, which Corollary 2.2 here does not provide.

**3. An illustrative example.** In this section, we apply the theory developed above to the system

$$
\begin{aligned}
x_1' &= x_2(x_1^2 + x_2^2) + \varepsilon(\lambda_1 x_1 + a_1 \mathrm{cos} t + b_1 \mathrm{sin} t + F_1(t)) \\
x_2' &= -x_1(x_1^2 + x_2^2) + \varepsilon(\lambda_2 x_2 + a_2 \mathrm{cos} t + b_2 \mathrm{sin} t + F_2(t))
\end{aligned} \tag{3.1}
$$

where $\lambda_1; \lambda_2 > 0$, $F_j : \mathbf{R} \to \mathbf{R}$, $j = 1, 2$, are $C'$, $2\pi$-periodic, and

$$
\int_0^{2\pi} F_j(t) \mathrm{sin} t \, dt = \int_0^{2\pi} F_j(t) \mathrm{cos} t \, dt = 0, \quad j = 1, 2. \tag{3.2}
$$

The unperturbed system

$$
\begin{aligned}
x_1' &= x_2(x_1^2 + x_2^2) \\
x_2' &= -x_1(x_1^2 + x_2^2)
\end{aligned} \tag{3.3}
$$

was studied in [**4**] and was shown to be nondegenerate there.

To place (3.1) and (3.3) into the context of the previous theory, let

$$
\begin{aligned}
x = \mathrm{col}(x_1, x_2), f(x) &= \mathrm{col}(x_2(x_1^2 + x_2^2), -x_1(x_1^2 + x_2^2)), \\
F(t, x) = \mathrm{col}(\lambda_1 x_1 &+ a_1 \mathrm{cos} t + b_1 \mathrm{sin} t \\
&+ F_1(t), \lambda_2 x_2 + a_2 \mathrm{cos} t + b_2 \mathrm{sin} t + F_2(t)),
\end{aligned}
$$

and let $x(t, x_0, \varepsilon)$ denote the solution of (3.1) passing through $x_0$ at $t = 0$. Then $f, F$ satisfy the hypotheses detailed in the Introduction.

It is clear that (3.3) possesses $u(t) = \mathrm{col}(\sin t, \cos t)$ as a $2\pi$-periodic solution. The linear variational equation for (3.3) associated with this solution is given by

$$
y' = A(t)y \tag{3.4}
$$

where $y = \mathrm{col}(y_1, y_2)$ and

$$
A(t) = \begin{pmatrix} 2 \mathrm{sin}\, t \mathrm{cos}\, t & 1 + 2\mathrm{cos}^2 t \\ -1 - 2\mathrm{sin}^2 t & -2\mathrm{sin}\, t \mathrm{cos}\, t \end{pmatrix} \tag{3.5}
$$

Clearly, $u'(t) = \mathrm{col}(\cos t, -\sin t)$ is a solution of (3.4) and it can be shown that $\mathrm{col}(\sin t + 2t\cos t, \cos t - 2t\sin t)$ is a second linearly independent solution.

Thus

$$X(t) = \begin{pmatrix} \cos t & \sin t + 2t\cos t \\ -\sin t & \cos t - 2t\sin t \end{pmatrix} \tag{3.6}$$

is the principal matrix solution of (3.4) and, in the notation of (1.5), $p(t) = \mathrm{col}(\sin t \cos t)$ and $K = 2$. Finally, the local coordinate system about $u(t)$ is given by

$$\hat{t}(t) = \mathrm{col}(\cos t, -\sin t), \quad \hat{n}(t) = \mathrm{col}(\sin, t, \cos t). \tag{3.7}$$

According to Theorem 1.3, $2\pi$-periodic solutions of (3.1) branch from translates of $u, u(t + s_0)$, where $s_0$ is given by

$$g(s_0) = 0, \quad g'(s_0) \neq 0 \tag{3.8}$$

and $g$ is defined by

$$g(s) = \hat{n}(s) \cdot x_\varepsilon(2\pi, u(s_0), 0). \tag{3.9}$$

Using (1.17), (3.8) is equivalent to

$$h(s_0) = 0, \quad h'(s_0) \neq 0 \tag{3.10}$$

where

$$h(s) = \int_0^{2\pi} [\sin(s + \sigma), \ \cos(s + \sigma)] F(\sigma, u(s + \sigma)) d\sigma. \tag{3.11}$$

It is a brief calculation to see that

$$h(s) = (\lambda_1 + \lambda_2)\pi + (\sin s)(a_1 - b_2)\pi + (\cos\ s)(b_1 + a_2)\pi \text{ and}$$

$$h'(s) = (\cos\ s)(a_1 - b_2)\pi - (\sin\ s)(b_1 + a_2)\pi. \tag{3.12}$$

In order to solve (3.10), it is convenient to write

$$h(s)/\pi = \lambda_1 + \lambda_2 + R\cos\ (s - \phi) \tag{3.13}$$

where

$$R^2 = (a_1 - b_2)^2 + (b_1 + a_2)^2 \text{ and}$$

$$\tag{3.14} \tan\phi = \begin{cases} -\frac{b_1+a_2}{a_1-b_2} & a_1 \neq b_2 \\ \pi/2 & , a_1 = b_2. \end{cases}$$

We can conclude from (3.13) and (3.14)

a. if $(\lambda_1 + \lambda)^2 > (a_1 - b_2)^2 + (b_1 + a_2)^2$, then (3.10) has no solutions; and

b. if $(\lambda_1 + \lambda_2)^2 < (a_1 - b_2)^2 + (b_1 + a_2)^2$, then (3.10) has two solutions of the form

$$s_0 = \phi + \cos^{-1}(\frac{\lambda_1 + \lambda_2}{R}) \text{ and}$$

$$\tag{3.15} s_1 = \phi - \cos^{-1}(\frac{\lambda_1 + \lambda_2}{R}).$$

In the case that solutions exist,

$$\tag{3.16} h'(s)/\pi = -R \sin(s - \phi)$$

and it is clear that

$$\tag{3.17} h'(s_0)/\pi = -h'(s_1)/\pi$$

with

$$\tag{3.18} |h'(s_0)| = |h'(s_1)| = \pi R.$$

Thus, if $(\lambda_1 + \lambda_2)^2 < (a_1 - b_2)^2 + (b_1 + a_2)^2$, (3.1) possesses two $2\pi$-periodic solutions branching from $u(t + s_0)$ and $u(t + s_1)$.

In order to study the stability properties of these solutions, we need more information about $s_0$ and $s_1$. From (3.15) and (3.16), one can see that $h'(s_0) < 0$ and $h'(s_1) > 0$.

We recall that in light of Theorem 2.3, (1.18), and the fact that $K = 2, \text{sgn}[\varepsilon K d_{21}(s)] = \text{sgn}[\varepsilon h'(s)]$ where $s = s_0$ or $s_1$.

Finally, we must calculate $Q$, defined by (2.22). It is clear that

$$Q(\varepsilon, s_0) = Q(\varepsilon, s_1) = 2\pi\varepsilon(\lambda_1 + \lambda_2). \tag{3.19}$$

We apply Theorem 2.1 and summarize the preceding discussion in

THEOREM 3.1. *Let* $(\lambda_1 + \lambda_2)^2 < (a_1 - b_2)^2 + (b_1 + a_2)^2$ *and let* $z(t, \varepsilon, s_0)$ *and* $z(t, \varepsilon, s_1)$ *represent the solutions branching, respectively, from* $u(t + s_0)$ *and* $u(t + s_1)$, *with* $s_0, s_1$ *defined by (3.15). Then*

a. *If* $\lambda_1 + \lambda_2 > 0$, $z(t, \varepsilon, s_0)$ *is unstable for all* $\varepsilon \neq 0$ *sufficiently small and* $z(t, \varepsilon, s_1)$ *is unstable for* $\varepsilon > 0$ *sufficiently small and asymptotically stable for* $\varepsilon < 0$ *sufficiently small; and*

b. *If* $\lambda_1 + \lambda_2 < 0$, $z(t, \varepsilon, s_1)$ *is unstable for all* $\varepsilon \neq 0$ *sufficiently small and* $z(t, \varepsilon, s_0)$ *is unstable for* $\varepsilon < 0$ *sufficiently small and asymptotically stable for* $\varepsilon > 0$ *sufficiently small.*

REMARK. If $\varepsilon = 0$, $z(t, 0, s_i) = u(t + s_i), i = 0, 1$, which is always unstable.

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