

## NECESSARY AND SUFFICIENT CONDITIONS FOR MULTIPARAMETER BIFURCATION

JORGE IZE

**ABSTRACT.** Obstruction theory is used in order to give a complete characterization of linearized local and global bifurcation. In both cases there is a set of two topological invariants, depending only on the linear part, such that, if both are trivial, there is a nonlinear part with no local or global bifurcation. The nonvanishing of any of these invariants is sufficient for bifurcation for any nonlinearity.

**0. Introduction.** A bifurcation problem is the study of the zeros of the nonlinear map  $f(\lambda, x)$ , where  $\lambda$  belongs to the parameter space  $\Lambda$ ,  $x$  to a space  $E$  and  $f(\lambda, x)$  has values in another space  $F$  near a known family of solutions  $(\lambda, x(\lambda))$  called the trivial solutions. After linearization, one may assume that  $x(\lambda) \equiv 0$  and that  $f(\lambda, x)$  has the form

$$(1) \quad (A_0 - A(\lambda))x - g(\lambda, x)$$

where  $A(0) = 0$ ,  $g(\lambda, x) = o(\|x\|)$ . It is well known that a necessary condition for bifurcation is that  $A$  is non-invertible and if  $A$  is a Fredholm operator one may write (1), near  $(0, 0)$ , as

$$\begin{aligned} & (A_0 - QA(\lambda))(x_2 - (I - KQA(\lambda))^{-1}KQ(A(\lambda)x_1 + g(\lambda, x))) \\ & \oplus (I - Q)(A(\lambda)((I - KQA(\lambda))^{-1}x_1 \\ & + x_2 - (I - KQA(\lambda))^{-1}KQ(A(\lambda)x_1 + g(\lambda, x))) \\ & + (I - A(\lambda)KQ)^{-1}g(\lambda, x)), \end{aligned}$$

where  $x = x_1 \oplus x_2$ ,  $x_1$  in  $\ker A_0$ ,  $Q$  is a projection on  $\text{Range } A_0$  and  $K$  is the pseudo-inverse of  $A_0$  from  $\text{Range } A_0$  into  $X_2$ , the complement

---

1980 *Mathematics Subject Classification.* Primary 58B05, 58E07; Secondary 47H1S, 54F45.

*Key Words and Phrases.* Multiparameter bifurcation, obstruction theory.

This research was mostly done while the author was visiting the I.M.S.P., Université de Nice, and the University of Utah. Partly funded by CONACYT, PCCBBNA-022634.

of  $\ker A_0$ . Thus, the zeros of  $f(\lambda, x)$  are the zeros of the *bifurcation equation*

$$(2) \quad B(\lambda)x_1 + G(\lambda, x_1) = 0,$$

where

$$B(\lambda) = -(I - Q)A(\lambda)(I - KQA(\lambda))^{-1}$$

$$G(\lambda, x_1) = -(I - Q)(I - A(\lambda)KQ)^{-1}g(\lambda, x),$$

after one has solved the first piece for  $x_2$  as a function of  $\lambda$  and  $x_1$  if  $g$  is smooth enough. (See [5] or [7].) It is easily checked that  $G(\lambda, x) = o(\|x\|)$ ,  $B(0) = 0$ , and  $B(\lambda)$  is invertible if and only if  $A_0 - A(\lambda)$  is invertible.

The present paper will give necessary and sufficient conditions for bifurcation (in a sense to be made more precise below). In order to make this point clearer and to avoid technicalities, several simplifying hypotheses will be made:

- (a)  $B(\lambda)$  is  $C^1$  in  $\lambda$ ,
- (b)  $E = F = \mathbf{R}^N$ ,  $\Lambda = \mathbf{R}^k$  (the extension to infinite dimensional spaces, with some compactness, is standard. For the case of an infinite dimensional parameter space see [9]).
- (c) The index of  $A_0$  is 0. For a non-zero index one may use the ideas of [5], [7] or [9; Remark 4.8];  $\dim \ker A_0 = d$ .
- (d)  $B(\lambda)$  is invertible for  $\lambda$  small,  $\lambda \neq 0$ . (If this is not the case one may choose a hypersurface transversal to the set of  $\lambda$ 's where  $\det B(\lambda) = 0$  and use the normal space to get more dimension in the bifurcating set as in [9].) Note that if  $k > 1$ , this hypothesis may not be "generic" for certain classes of perturbations. However it is natural for problems like the Hopf bifurcation or if all the spaces are complex [5, 8]. See [3] for the generic approach.

From these hypotheses it follows that if  $\|\lambda\| = \rho$ , there is a positive number  $\varepsilon_0(\rho, G)$  such that if one has a solution of the bifurcation equation with  $\|\lambda\| = \rho$   $\|x_1\| \leq \varepsilon_0$ , then  $x_1 = 0$ .

**DEFINITION A.**  $(0, 0)$  is a point of linearized local bifurcation if and only if, for any  $G(\lambda, x_1) = o(\|x_1\|)$ , the bifurcation equation (2) has a solution with  $\|\lambda\| \leq \rho$   $\|x_1\| = \varepsilon$ , for any  $\varepsilon$ ,  $0 < \varepsilon < \varepsilon_0(\rho, G)$ .

Note that, to say that  $(0,0)$  is not a point of linearized local bifurcation means that there is a small nonlinearity such that the bifurcation equation has no other solution than the trivial one, for  $\|\lambda\| \leq \rho, \|x_1\| \leq \varepsilon_0(\rho, G)$ .

In a similar fashion, if  $C(g)$  is the continuum of non trivial solutions bifurcating from  $(0,0)$ , one has

**DEFINITION B.**  $(0,0)$  is a point of linearized global bifurcation if and only if, for any  $g(\lambda, x) = o(\|x\|)$ , the continuum  $C(g)$  is either unbounded or returns to a different bifurcation point.

Note again that  $(0,0)$  is not a point of linearized global bifurcation if one has a small nonlinearity  $g(\lambda, x)$ , with  $C(g)$  bounded and containing only  $(0,0)$  as a bifurcation point.

There are several ways, apparently equivalent from the point of view of analysis, to set the bifurcation problem:

1) Look for solutions of  $B(\lambda)x_0 + G(\lambda, x_0) = 0$ , with  $x_0$  fixed and  $\|x_0\| = \varepsilon$  in the set  $B_1 = \{\lambda / \|\lambda\| < \rho\}$ .

2) Look for solutions of  $(B(\lambda)x + G(\lambda, x) = 0, x - x_0 = 0)$ ,  $x_0$  as above in the set  $B_2 = \{(\lambda, x) / \|\lambda\| < \rho, \|x\| < 2\varepsilon\}$ .

3) Look for solutions of  $(B(\lambda)x + G(\lambda, x) = 0, \|x\| - \varepsilon = 0)$  in the set  $B_3 = \{(\lambda, x) / \|\lambda\| < \rho, \|x\| < 2\varepsilon\}$ .

4) Look for solutions of  $B(\lambda)x + G(\lambda, x) = 0$  in the set  $B_4 = \{(\lambda, x) / \|\lambda\| < \rho, \|x\| = \varepsilon\}$ .

The topological ideas behind the sufficient conditions (i.e., valid for all nonlinearities) for bifurcation are the following simple facts.

**FACT 1.** Let  $F(x) : \partial(B^{m+1}) \rightarrow \mathbf{R}^n \setminus \{0\}$ . Then any extension of  $F$  to  $B^{m+1}$  has a zero if and only if  $F$  is not deformable to a constant map.

**FACT 2.** Let  $A \subset X$  be two closed subsets of  $\mathbf{R}^{m+1}$ , and let  $F, G : A \rightarrow \mathbf{R}^n \setminus \{0\}$  be two homotopic maps. Then  $F$  extends to  $X$  without zeros if and only if  $G$  does.

The proofs of these facts can be found in any text on homotopy theory or in [7].

Here, in the four settings, the maps involved are non-zero on  $\partial B_i$  and one wishes to prove that they don't have a non-zero extension to  $B_i, i = 1, \dots, 4$ . From Fact 2, one may deform the bifurcation equation on  $\partial B_i$  to  $B(\lambda)x$  and to  $\|\lambda\| D(\lambda\rho/\|\lambda\|)$  if  $D(\lambda)$  and  $B(\lambda)$  are homotopic families of matrices from  $\{\lambda/\|\lambda\| = \rho\}$  into  $GL(\mathbf{R}^d)$ . In each case, one obtains a homomorphism from  $\prod_{k-1}(GL(\mathbf{R}^d))$  into the set of homotopy classes of non-zero maps defined on  $\partial B_i$ . In the first approach one has that, if  $P_*B(\lambda) \equiv [B(\lambda)x_0] \neq 0$  in  $\prod_{k-1}(S^{d-1})$ , then one has bifurcation in the direction  $x_0$  and in all directions (see [5; Chapter 2] and [7]); in the second case one obtains the  $d$ -fold suspension of  $P_*B(\lambda)$  (see Proposition 4.1 for an interpretation of this fact); the third approach gives the class of  $(B(\lambda)x, \|x\| - \varepsilon)$  in  $\prod_{k+d-1}(S^d)$ , which is called the Whitehead  $J$ -homomorphism [13]. For the case  $k < d$ , the Bott periodicity theorem and the characterization of  $\text{Im } J$  by Adams were used in [2] and [6] to give sufficient conditions for bifurcation: If  $JB(\lambda) \neq 0$ , then one has bifurcation. (If  $k = 1$ , then  $\det B(\lambda)$  changes sign) In [7, p. 182] it was shown that, for a sphere, there are no other ways of complementing the bifurcation equation. This fact is made clear in the present paper where the fourth approach is used in order to also get necessary conditions. The main results are the following two theorems.

**THEOREM A.** *One has linearized local bifurcation if and only if either  $[B(\lambda)x_0] \neq 0$  or  $JC(\lambda) \neq 0$  for all  $C(\lambda)$  in  $\prod_{k-1}(GL(\mathbf{R}^{d-1}))$  such that  $\begin{pmatrix} C(\lambda) & 0 \\ 0 & 1 \end{pmatrix}$  is deformable to  $B(\lambda)$ .*

**THEOREM B.** *One has linearized global bifurcation if and only if either  $\sum[B(\lambda)x_0] \neq 0$  or  $JB(\lambda) \neq 0$ .*

In §1, the local bifurcation problem is shown to be equivalent to an extension problem to  $B_4$ . In §2, this construction is made explicit for one real or complex parameter. In §3, Obstruction theory is used to compute the topological invariants. Note that in [1] there is a hint

that obstruction may be relevant for bifurcation problems. Another approach, cohomotopy theory, is sketched in the appendix. Global bifurcation is studied in §4. The basic references for the topology are [4], [12] and [15], however a short review of obstruction theory is included for the reader's convenience. The results of the paper were announced in [8] and the study for an equivariant problem will be published in a subsequent paper.

**1. Nonlinearities without bifurcation.** It has been seen that the nontriviality of  $JB(\lambda)$  is a sufficient condition for bifurcation and that it implied that  $B(\lambda)x$  had no non-zero extension from the set  $\{(\lambda, x) / \|\lambda\| = \rho, \|x\| = 1\}$  to the set  $\{(\lambda, x) / \|\lambda\| \leq \rho, \|x\| = 1\}$ . In this first section it will be shown that this condition is also "necessary" in the following sense.

**THEOREM 1.1.** *Let  $B(\lambda)$  be a  $C^1$ -family of  $d \times d$  matrices which are invertible for  $0 < \|\lambda\| \leq \rho$  and  $B(0) = 0$ . Then  $B(\lambda)x$  has a continuous non-zero extension  $B(\lambda, x)$  from the set  $\{(\lambda, x) / \|\lambda\| = \rho, \|x\| = 1\}$  to the set  $\{(\lambda, x) / \|\lambda\| \leq \rho, \|x\| = 1\}$  if and only if there is a continuous nonlinearity  $G(\lambda, x)$  such that the map  $F(\lambda, x) = B(\lambda)x + G(\lambda, x)$  has the properties:*

- (a)  $G(\lambda, x)$  is defined for  $\|\lambda\| \leq \rho$  and any  $x$ ;
- (b)  $G(\lambda, x) = o(\|x\|^2)$ , for fixed  $\|\lambda\|$ , and  $G(\lambda, x) = o(\|x\|)$  uniformly in  $\lambda$ , for  $\|\lambda\| \leq \rho$ ;
- (c) there is a  $\rho_1$  such that if  $F(\lambda, x) = 0$ , for  $\|\lambda\| \leq \rho_1$ , then  $x = 0$ . There is an  $\varepsilon > 0$  such that if  $F(\lambda, x) = 0$ , for  $\|\lambda\| \leq \rho, \|x\| \leq \varepsilon$ , then  $x = 0$ .

**PROOF.** (only if). From the hypotheses one derives the following facts:

- 1) If  $D(\lambda)$  is defined by  $B(\lambda) = \|\lambda\|^{1/2} D(\lambda)$ , with  $D(0) = 0$ , then there is a constant  $A$  such that  $\|D(\lambda)\| \leq A(\|\lambda\|)^{1/2}$  for  $\|\lambda\| \leq \rho$ .
- 2) There are constants  $B$  and  $C$  such that  $B \leq \|B(\lambda, \eta)\| \leq C$ , for  $\|\eta\| = 1$  and  $\|\lambda\| \leq \rho$ .
- 3) There is a continuous nondecreasing function  $g(\|\lambda\|)$ , with

$g(0) = 0$ , such that  $\|B(\lambda)^{-1}\| \leq g(\|\lambda\|)^{-1}$ . For  $0 \leq \|\lambda\| \leq \rho$  it is clear that  $\|B(\lambda)^{-1}\| \leq D \|\det B(\lambda)\|^{-1}$ . Define  $Dg(\|\lambda\|) = \min \|\det B(\mu)\|^{-1}$  for  $\|\lambda\| \leq \|\mu\| \leq \rho$ . (If  $\{\lambda_n\}$  goes to  $\lambda$ , take  $\{\mu_n\}$  where the minimum is achieved, take a subsequence converging to  $\mu$ ; then  $\|\mu\| \geq \|\lambda\|$  and  $Dg(\|\lambda\|) \leq \|\det B(\mu)\|^{-1}$ . If one has a strict inequality, then there is a  $\nu$  such that  $\|\lambda\| \leq \|\nu\|$  and  $\|\det B(\nu)\|^{-1}$  lies between the above two quantities. If  $\|\lambda\| < \|\nu\|$ , then  $\|\lambda_n\| < \|\nu\|$ , for large  $n$ ,  $Dg(\|\lambda_n\|) \leq Dg(\|\nu\|) < \|\det B(\mu)\|^{-1} = \lim Dg(\|\lambda_n\|)$ . If  $\|\lambda\| = \|\nu\|$ , then looking at  $\nu(1 + \varepsilon)$  will give the same contradiction for  $\varepsilon$  small enough). Note that if one wants to have a strictly increasing function, one may take  $\|\lambda\| \int_0^{\|\lambda\|} g(r) dr$ .

The next step in the proof is to construct the Urysohn's function  $\phi : \{0 \leq u \leq \rho, 0 \leq v\} \setminus \{0, 0\} \rightarrow [0, 1]$ :

$$\phi(u, v) = \begin{cases} 0 & \text{if } g(u)(u/\rho)^{1/2}C \\ & \leq v \leq (A/B)\rho^{1/2}u^{1/2} \\ 1 & \text{if } v \leq g(u)(u/\rho)^{1/2}/(2C) \\ & \text{or if } v \geq 2(A/B)\rho^{1/2}u^{1/2} \\ \frac{1}{2} - 2Cv(u/\rho)^{1/2}g(u)^{-1} & \text{if } g(u)(u/\rho)^{1/2}/(2C) \\ & \leq v \leq g(u)(u/\rho)^{1/2}/C \\ 1 - (B/A)\rho^{-1/2}u^{-1/2}v & \text{if } (A/B)(\rho u)^{1/2} \\ & \leq v \leq 2(A/B)(\rho u)^{1/2}. \end{cases}$$

The function  $\phi$  is continuous, except at  $(0, 0)$  where it remains between 0 and 1, and, if  $g$  is  $C^1$ , then  $\phi$  is locally Lipschitz continuous.

Define, for  $\|\lambda\| \leq \rho$  and any  $x$ ,

$$F(\lambda, x) = \|\lambda\|^{1/2} D(\lambda(\phi(\|\lambda\|, \|x\|) + (1 - \phi(\|\lambda\|, \|x\|))\rho/\|\lambda\|))x \\ + \|x\|^2 B(\lambda\rho/(\|\lambda\| + \|x\|^4), x/\|x\|).$$

The only point to check for the continuity of  $F(\lambda, x)$  is when  $\lambda$  goes to zero (then the argument in  $D(\cdot)$  has norm less than  $\rho$  and the first part of  $F$  goes to 0) and when  $x$  goes to 0 (then the argument in  $B(\cdot, \cdot)$  remains bounded and the second part of  $F$  goes to 0).

The proof of (b) is clear when  $\lambda = 0$ ; for  $\lambda \neq 0$  and  $\|x\| \leq g(\|\lambda\|)(\|\lambda\|/\rho)^{1/2}/(2C)$ , then  $\phi(\|\lambda\|, \|x\|) = 1$ , the first part of  $F$  is  $B(\lambda)x$  and the second part is bounded by  $C\|x\|^2$ . Furthermore,

$$\|F(\lambda, x) - B(\lambda)x\| \leq A\|\lambda\|^{1/2}\rho^{1/2}\|x\| + C\|x^2\| + A\|\lambda\|\|x\| \\ \leq (2A\rho^{1/2}h(\|x\|))^{1/2} + C\|x\|\|x\|$$

if  $\|\lambda\| \leq h(\|x\|)$ , where  $h(\|x\|)$  is the inverse function of  $g(\|\lambda\|)(\|\lambda\|/\rho)^{1/2}(2C)$ . Thus, for any  $(\lambda, x)$ ,  $\|F(\lambda, x) - B(\lambda)x\| \leq (2A\rho^{1/2}h(\|x\|)^{1/2} + C\|\lambda\|)\|x\|$  and, as such,  $G(\lambda, x) = o(\|x\|)$  uniformly in  $\lambda$ .

For the proof of (c) one observes that

( $\alpha$ )  $\|F(\lambda, x)\| \geq B\|x\|^2 - A\|\lambda\|^{1/2}\|\lambda\|\phi + (1-\phi)\rho\|x\|^{1/2}$   
 $\|x\| \geq B\|x\|^2 - A(\|\lambda\|\rho)^{1/2}\|x\|$  so that  $F(\lambda, x) \neq 0$  in the region  
 $\|x\| > (A/B)\rho\|\lambda\|^{1/2}$ , in particular for  $\lambda = 0$ .

( $\beta$ )

$$F(\lambda, x) = B(\lambda(\phi + (1-\phi)\rho/\|\lambda\|)(\|\lambda\|^{1/2}(\|\lambda\|\phi + (1-\phi)\rho)^{-1/2}x \\ + B^{-1}(\lambda(\phi + (1-\phi)\rho/\|\lambda\|)\|x\|^2 B(\cdot, \cdot)),$$

for  $\lambda \neq 0$ . Hence if  $F(\lambda, x) = 0$ , one has

$$\|\lambda\|^{1/2}\|x\|(\|\lambda\|\phi + (1-\phi)\rho)^{-1/2} = \|x\|^2\|B^{-1}(\cdot)B(\cdot, \cdot)\| \\ \leq \|x\|^2 Cg^{-1}(\|\lambda\|\phi + (1-\phi)\rho) \\ \leq \|x\| Cg^{-1}(\|\lambda\|),$$

since  $g$  is nondecreasing. Hence, if  $x \neq 0$ , then

$$\|\lambda\|^{1/2} \leq C(\|\lambda\|\phi + (1-\phi)\rho)^{1/2}g^{-1}(\|\lambda\|)\|x\| \\ \leq C\rho^{1/2}g^{-1}(\|\lambda\|)\|x\|,$$

which is not possible if  $\|x\| < (\|\lambda\|/\rho)^{1/2}g(\|\lambda\|)/C$ .

( $\gamma$ ) Thus the only possible zeros are in the region where  $\phi(\|\lambda\|, \|x\|) = 0$ . There,

$$F(\lambda, x) = (\|\lambda\|^{1/2} + \rho^{1/2}\|x\|)D(\lambda\rho/\|\lambda\|)x \\ + \|x\|^2(B(\lambda\rho/(\|\lambda\| + \|x\|^4), x/\|x\|) \\ - \rho^{1/2}D(\lambda\rho/\|\lambda\|)x/\|x\|).$$

Now, from the continuity of  $B(\nu, \eta)$ , there is a  $\delta$  such that  $\|B(\nu, \eta) - B(\mu, \eta)\| \leq g(\rho)/2$  if  $\|\nu - \mu\| \leq \delta < \rho/2$ ,  $\|\mu\| = \rho$ ,  $\|\nu\| \leq \rho$ , uniformly in  $\eta$ . Thus if  $F(\lambda, x) = 0$ , one gets  $(\|\lambda\|^{1/2} + \rho^{1/2}\|x\|)\|x\| \leq \|x\|^2\rho^{1/2}/2$  (which is impossible unless  $x = \lambda = 0$ )

provided  $\rho \|x\|^4 (\|\lambda\| + \|x\|^4)^{-1} \leq \delta$ , that is, for  $\|x\| \leq (\delta \|\lambda\| / (\rho - \delta))^{1/4}$ . But  $(A/B)(\rho \|\lambda\|)^{1/2} \leq (\delta \|\lambda\| / (\rho - \delta))^{1/4}$ , for  $\|\lambda\| \leq \delta / (\rho - \delta) (B/A)^4 \rho^{-2} \equiv \rho_1$ . Thus, for  $\|\lambda\| \leq \rho_1$ ,  $F(\lambda, x) \neq 0$  for  $x \neq 0$ . For  $\|\lambda\| \leq \rho$ , one gets the same result if  $\|x\| \leq (\delta / (\rho - \delta) \rho_1)^{1/4} \equiv \varepsilon$ .

(If). If  $F(\lambda, x)$  has the properties (a), (b), (c), then on the set  $\{(\lambda, x) / \|\lambda\| = \rho, \|x\| = 1\}$ ,  $F(\lambda, \varepsilon x)$  is deformable to  $B(\lambda)\varepsilon x$  (and to  $B(\lambda)x$ ) and has the non-zero extension  $F(\lambda, \varepsilon x)$ . Thus, from the Borsuk extension theorem,  $B(\lambda)x$  also has a non-zero extension  $B(\lambda, x)$  to the set  $\{(\lambda, x) / \|\lambda\| \leq \rho, \|x\| = 1\}$  (Fact 2).

REMARK 1.1.  $\|\lambda\|^{1/2} D(\lambda(t + (1-t)\rho / \|\lambda\|))$  gives a deformation from  $B(\lambda)$  to  $\|\lambda\|^{1/2} D(\lambda\rho / \|\lambda\|)$  via matrices which are invertible for  $0 < \|\lambda\| \leq \rho$ .

REMARK 1.2. If  $B(\lambda)$  has the form  $\|\lambda\|^{1/2} D(\lambda\rho / \|\lambda\|)$ , then one does not need the construction of  $\phi(\|\lambda\|, \|x\|)$  and thus  $G(\lambda, x) = \|x\|^2 B(\lambda / (\|\lambda\| + \|x\|^4), x / \|x\|)$  is uniformly  $O(\|x\|^2)$ .

REMARK 1.3. If, on the set  $\{\lambda, \|\lambda\| = \rho\}$ , the family of matrices  $B(\lambda)$  is deformable to  $I$  via  $B(\lambda, t)$  such that  $B(\lambda, 1) = B(\lambda)$ ,  $B(\lambda, 0) = I$ ,  $B(\lambda, t)$  invertible, then  $B(\lambda, \eta) = B(\lambda\rho / \|\lambda\|, \|\lambda\| / \rho)\eta$  is a good extension to  $\{(\lambda, \eta) / \|\lambda\| \leq \rho, \|\eta\| = 1\}$  (if  $\lambda$  goes to 0 then  $B(\lambda, 0) = I$  and one has continuity). In this case

$$F(\lambda, x) = (\|\lambda\|^{1/2} D(\lambda(\phi + (1-\phi)\rho / \|\lambda\|) + \|x\| B(\lambda\rho / \|\lambda\|, \|\lambda\| / (\|\lambda\| + \|x\|^4)))x$$

and with  $\phi$  replaced by 0 if  $B(\lambda) = \|\lambda\|^{1/2} D(\lambda\rho / \|\lambda\|)$ .

REMARK 1.4. If  $\Gamma$  is a compact Lie group, which acts linearly and via isometries on  $x$ , and  $B(\lambda)\gamma x = \gamma B(\lambda)x$ ,  $B(\lambda, \gamma x) = \gamma B(\lambda, x)$  for all  $\gamma$  in  $\Gamma$ , then  $F(\lambda, \gamma x) = \gamma F(\lambda, x)$  since the regions defined by  $\phi$  are invariant and the whole construction is equivariant. (See [8] and [10] for applications).

REMARK 1.5. For the general, maybe infinite dimensional, bifurcation problem,

$$A_0x - A(\lambda)x - g(\lambda, x) = 0$$

written as, for  $\|\lambda\|$  small enough,

$$\begin{aligned} & (A_0 - AQ(\lambda))(x_2 - (I - KQA(\lambda))^{-1}KQ(A(\lambda)x_1 + g(\lambda, x))) \\ & \ominus (I - Q)(A(\lambda)((I - KQA(\lambda))^{-1}x_1 + x_2 \\ & - (I - KQA(\lambda))^{-1}KQ(A(\lambda)x_1 + g(\lambda, x))) \\ & + (I - A(\lambda)KQ)^{-1}g(\lambda, x)) = 0, \end{aligned}$$

one may choose

$$\begin{aligned} Qg(\lambda, x) &= 0 \\ (I - Q)g(\lambda, x) &= -G(\lambda, x_1). \end{aligned}$$

Then the above equation reduces to

$$\begin{aligned} & (A_0 - QA(\lambda))(x_2 - (I - KQA(\lambda))^{-1}KQA(\lambda)x_1) \\ & \oplus -(I - Q)A(\lambda)(x_2 - (I - KQA(\lambda))^{-1}KQA(\lambda)x_1) \\ & + (B(\lambda)x_1 + G(\lambda, x_1)) = 0. \end{aligned}$$

This equation has the properties (a), (b), (c) of Theorem 1.1.

**2. Analyticity in one parameter.** If  $B(\lambda)$  is analytic in  $\lambda$  (as a real matrix if  $\lambda$  is real and as a complex matrix if  $\lambda$  is complex), then  $JB(\lambda)$  is zero if and only if  $\det B(\lambda) = a_0\lambda^m + \dots$  has  $m$  even, where  $m$  corresponds to the lowest non-zero term in the power series (in the complex case one is assuming that the complex dimension  $d$  is larger than 1). (See [5, p. 47].) In this section an explicit construction of the nonlinearity  $G(\lambda, x)$  will be given.

**THEOREM 2.1.** *If  $m$  is even, then one may construct a nonlinearity  $G(\lambda, x)$  with the properties of Theorem 1.1 such that*

(a)  $G(\lambda, x)$  is real analytic in both variables if  $\lambda$  is in  $\mathbf{R}$ ,  $x$  in  $\mathbf{R}^d$ ;

(b)  $G(\lambda, x)$  is real analytic in  $x$  and  $\lambda$  if  $\lambda \neq 0$  and Hölder continuous at  $\lambda = 0$  if  $\lambda$  is in  $\mathbf{C}$ ,  $x$  in  $\mathbf{C}^d$ . If  $m \neq 0, d = 1$  or  $G(\lambda, x)$  is complex analytic in  $x$  or  $G(\lambda, e^{i\phi}x) = e^{i\phi}G(\lambda, x)$ , then there is always bifurcation.

PROOF. For the last sentence see [5, pp. 47, 82], [8, p. 767].

The first step in the construction is to reduce the problem to the case where  $B(\lambda)$  is diagonal. Since  $\det B(\lambda)$  is analytic in  $\lambda$ , one has a  $\rho > 0$  such that  $B(\lambda)$  is invertible for  $0 < \|\lambda\| \leq \rho$ .

LEMMA 2.1. *There is a matrix deformation  $B(\lambda, t)$ , with  $B(\lambda, t)$  invertible for  $0 < \|\lambda\| \leq \rho$  such that  $B(\lambda, 0) = B(\lambda)$  and  $B(\lambda, 1) = \Lambda(\lambda)(A_1 + B_1(\lambda))$ , where  $\Lambda(\lambda) = \text{diag}(\lambda^{p_1}, \dots, \lambda^{p_d})$ ,  $\Sigma p_i$  is even,  $A_1$  is invertible,  $B_1(\lambda)$  is invertible for  $0 < \|\lambda\| \leq \rho$ ,  $B_1(0) = 0$ ,  $\det B_1(\lambda) = a_1 \lambda^{m_1} + \dots$ , and  $m_1$  is even.*

PROOF. By factoring out, from each row in  $B(\lambda)$ , the largest possible power of  $\lambda$  (except possibly the last row so that  $\Sigma p_i$  is even), one may write  $B(\lambda) = \Lambda(\lambda)(A_0 - A(\lambda))$ , where  $A_0$  has a non-zero element in each row (except maybe the last one) and  $A(0) = 0$ . Thus the rank of  $A_0$  is positive. Write  $x \equiv x_1 \oplus x_2$ , where  $x_1 \equiv Px$  is the projection of  $x$  onto  $\ker A_0$ ,  $x_2$  in a complement  $X_2$ . Let  $Q$  be the projection onto  $\text{Range}(A_0)$  and  $K$  be the inverse of  $A_0|_{X_2}$ . Then, as in [5, p. 43], one may write

$$B(\lambda)x = \Lambda(\lambda)((A_0 - QA(\lambda))(x_2 - (I - KQA(\lambda))^{-1}KQA(\lambda)x_1) \\ \ominus (I - Q)A(\lambda)((I - KQA(\lambda))^{-1}x_1 + x_2 - (I - KQA(\lambda))^{-1}KQA(\lambda)x_1))$$

(here  $(I - KQA(\lambda))^{-1}$  has to be interpreted as an infinite power series and  $\rho$  is then accordingly small). Let  $A_1$  be  $A_0(I - P)$ ,  $B_1(\lambda) = -(I - Q)A(\lambda)(I - KQA(\lambda))^{-1}P$ . From the fact that  $B(\lambda)$  is invertible, for  $\lambda \neq 0$ , it is clear that  $B_1(\lambda)$  is invertible, for  $0 < \|\lambda\| \leq \rho$ ,  $B_1(0) = 0$  and  $B_1(\lambda)$  is analytic in  $\lambda$ . Then if

$$B(\lambda, t) = \Lambda(\lambda)((A_0 - (1 - t)QA(\lambda)) \\ (I - P - (1 - t)(I - KQA(\lambda))^{-1}KQA(\lambda)P) \\ \oplus B_1(\lambda) - (1 - t)(I - Q)A(\lambda) \\ (I - P - (1 - t)(I - KQA(\lambda))^{-1}KQA(\lambda)P)) \\ \equiv \Lambda(\lambda)D(\lambda, t)$$

it is easy to verify that  $B(\lambda, t)$  is invertible for  $0 < \|\lambda\| \leq \rho$ . (One may also use the deformation  $A(\lambda(1 - t))$  in the above formula). Finally,

since the class of  $B(\lambda)$  must be the class of  $B(\lambda, 1)$  in  $\prod_0(\text{GL}(\mathbf{R}^d))$  or in  $\prod_1(\text{GL}(\mathbf{C}^d))$ , which is characterized by the degree of  $\det B(\lambda)$  as a mapping from  $S^0$  into  $\mathbf{R} \setminus \{0\}$  or from  $S^1$  into  $\mathbf{C} \setminus \{0\}$ ,  $m_1 + \sum p_i$  and  $m$  must have the same parity, in the real case, or must be equal, in the complex case.

Note that  $B_1(\lambda)$  has a similar deformation  $B_1(\lambda, t)$  and that one may replace  $B_1(\lambda)$  by  $B_1(\lambda, t)$  in the expression for  $B(\lambda, t)$  without losing the invertibility. The process stops in at most  $d$  steps, when  $B_1(\lambda) = b\lambda^{2p}(1 + \dots)$  which is linearly deformable to  $b\lambda^{2p}$ . In the complex case, if  $p \neq 0$ , one will consider the two-dimensional space generated by an eigenfunction of  $A_1$  (with eigenvalue  $a$ ) and by  $B_1(\lambda)$ . Deforming linearly to 0 the row corresponding to this eigenfunction, one will have a two dimensional diagonal matrix  $(a\lambda^{p_1}, b\lambda^{p_2+2p})$ , with  $p_1 + p_2$  even and  $p_1 \geq 1$  (the complex numbers  $a$  and  $b$  can be deformed to 1). The proof of Theorem 2.1 will be completed with the second step.

**LEMMA 2.2.**  $\Lambda(\lambda) = \text{diag}(\lambda^{p_1}, \lambda^{p_2}, \dots, \lambda^{p_d})$ , with  $\sum p_i$  even (considered as the equivalent  $2d \times 2d$  real matrix in the complex case), is deformable to  $\Lambda(\lambda, 1)$  via  $\Lambda(\lambda, t)$ , such that  $\Lambda(\lambda, 0) = \Lambda(\lambda)$ ,  $\Lambda(\lambda, t)$  is invertible for all  $\lambda, \|\lambda\| \leq \rho$  if  $t > 0$ .

*End of the proof of Theorem 2.1.* Define  $B(\lambda, \|x\|) = \Lambda(\lambda, \|x\|^2)D(\lambda, \|x\|^2)$  where, in  $D(\lambda, t)$ ,  $B_1(\lambda)$  has been replaced by  $\Lambda_1(\lambda, \|x\|^2)D_1(\lambda, \|x\|^2)$  and so on. Since  $\Lambda_1(\lambda, \|x\|^2)$  is invertible for  $\|x\| \neq 0$ ,  $B(\lambda, \|x\|)$  also has that property. Since  $D(\lambda, t)$  is polynomial in  $t$ , as well as  $\Lambda(\lambda, t)$ ,  $G(\lambda, x) = B(\lambda, \|x\|)x - B(\lambda)x = \|x\|^2 H(\lambda, \|x\|^2)x$  is real analytic in  $x$  and order of at least  $\|x\|^3$  near 0. The continuity in  $\lambda$  will follow from the proof of Lemma 2.2.

**PROOF OF LEMMA 2.2.** Order  $p_1, \dots, p_d$  such that  $p_1, \dots, p_{2\ell}$  are odd,  $p_{2\ell+1}, \dots, p_d$  are even. In the real case, couple the odd powers by matrices of the type

$$\begin{pmatrix} \lambda^{p_1} & t \\ -t & \lambda^{p_2} \end{pmatrix}$$

and the even powers by  $\lambda^{2n} + t$ , which have the property stated in the

lemma. In the complex case, if  $p_1 = p_2 + 2n, n \geq 0$ , then

$$\begin{pmatrix} \lambda^n & nt \\ -nt & \bar{\lambda}^{-n} |\lambda|^{-n+\varepsilon} \end{pmatrix} \begin{pmatrix} \lambda^{p_2+n} & z_1 + t\bar{z}_2 \\ \lambda^{p_2+n} |\lambda|^{-n-\varepsilon} & z_2 - t\bar{z}_1 \end{pmatrix},$$

with  $\varepsilon = 0$  if  $n = 0$  and  $\varepsilon = p_2/2$  if  $n > 0$ , gives the right kind of matrix when considered as a  $4 \times 4$  real matrix (see [5, p.23]). Thus if  $d$  is even, this construction will give  $\Lambda(\lambda, t)$ . Note that one has analyticity in  $\lambda$ , if  $n = 0$ , and  $C^\varepsilon$  if  $n > 0$ . However if  $d$  is odd ( $d \geq 3$  since  $d > 1$ ) this procedure will leave out one  $\lambda^p$ , with  $p$  even. Consider then  $d = 3$  and  $\Lambda(\lambda) = \text{diag}(\lambda^p, \lambda^q, \lambda^r)$ , with  $p \leq q \leq r$  and  $p + q + r = 2m$ . Write  $r = p + q + 2n$ , with  $2n = 2(m - p - q) \geq -p$ . Define  $\Lambda(\lambda, t)x$  by

$$\begin{pmatrix} 1 & 0 & n^2 t \\ n^2 t & \bar{\lambda}^{-n} |\lambda|^{-n+\varepsilon} & 0 \\ 0 & n^2 t & \lambda^n |\lambda|^{-n+\varepsilon} \end{pmatrix} \begin{pmatrix} \lambda^p x_1 + t\bar{x}_3 \\ \lambda^{q+n} |\lambda|^{-n-\varepsilon} x_2 + tx_1 \\ |\lambda|^{n-\varepsilon} \lambda^{p+q+n} x_3 + t\bar{x}_2 \end{pmatrix},$$

where  $\varepsilon = 0$  if  $n = 0$  and  $\varepsilon = q/2$  if  $n \neq 0$ . By taking the conjugate of its third component, the vector of the right hand side can be written as

$$\begin{pmatrix} \lambda^p & 0 & t \\ t & \lambda^{q+n} |\lambda|^{-n-\varepsilon} & 0 \\ 0 & t & |\lambda|^{n-\varepsilon} \bar{\lambda}^{p+q+n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

showing thus that  $\Lambda(\lambda, t)$  is invertible if  $t > 0$ . Note that one has real analyticity in  $\lambda$ , if  $n = 0$ , and  $C^\varepsilon$  if  $n \neq 0$ .

**3. The obstruction approach.** As has been seen in Theorem 1.1, there is a nonlinearity without bifurcation if  $B(\lambda)x$  has a non-zero extension from the set  $\{(\lambda, x) / \|\lambda\| = 1, \|x\| = 1\} = S^{k-1} \times S^{d-1}$  to the set  $\{(\lambda, x) / \|\lambda\| \leq 1, \|x\| = 1\} = B^k \times S^{d-1}$  (after scaling  $\lambda$  one may take  $\rho = 1$ ). This is a natural setting for obstruction theory. For the reader's convenience the main ideas and results needed in this paper will be recalled.

**3.1 Facts about obstruction theory.** Consider a finite cell complex  $K$  and a subcomplex  $L$  (here  $L = S^{k-1} \times S^{d-1}, K = B^k \times S^{d-1}$ ). Recall that a cell complex  $K$  is  $\cup_{q \leq p} \sigma_i^q$ ,  $q = 0, \dots, N = \dim K, i = 1, \dots, \alpha_q$ , such that

- (i)  $\sigma_i^q$  is a  $q$ -cell;
- (ii) if  $K^p = \cup_{q \leq p} \sigma_i^q$  is the  $p^{\text{th}}$  skeleton of  $K$ , then  $\sigma_i^q \cap K^{q-1} = \partial \sigma_i^q$  is the exact union of cells (called the faces of  $\sigma_i^q$ );
- (iii) if  $i \neq j$ , then  $\sigma_i^q \cap \sigma_j^q = \emptyset$ .

Cells are defined as the images of the unit ball under an "attaching map" which is a homeomorphism on the interior. Here the cell decomposition of  $S^{d-1}$  and  $S^{k-1}$  will be the equatorial decomposition with two cells in each dimension.

If  $f$  is a map from  $L$  into  $\mathbf{R}^n \setminus \{0\}$ , then one tries to extend  $f$  to  $K$  by extending it to  $\overline{K}^q \equiv K^q \cup L$  and increasing the dimension  $q$ . If  $f$  has been extended to  $\overline{K}^q$ , one has to extend it over all  $(q+1)$ -cells of  $K \setminus L$ . If  $\sigma^{q+1}$  is such a cell,  $f : \partial \sigma \rightarrow \mathbf{R}^n \setminus \{0\}$  is extendible to a map from  $\sigma$  into  $\mathbf{R}^n \setminus \{0\}$  if and only if  $f$  is deformable to a constant map in  $\prod_q (S^{n-1})$ . Since several  $(q+1)$ -cells may be attached to the boundary cells, one gets in fact a cochain  $C^{q+1}(f)$  in  $C^{q+1}(K, L; \prod_q (S^{n-1}))$ , which is a cocycle with the following properties:

1) If  $f_0$  and  $f_1$  are homotopic over  $\overline{K}^q$ , then  $C^{q+1}(f_0) = C^{q+1}(f_1)$  [4; Lemma 3.3., p. 177].

2) If  $f_0$  is as above, then  $f_0$  extends to  $\overline{K}^{q+1}$  if and only if  $C^{q+1}(f_0) = 0$  [4; Lemma 3.2, p. 177].

3) If  $f : \overline{K}^{q-1} \rightarrow \mathbf{R}^n \setminus \{0\}$  is extendible over  $\overline{K}^q$ , then the set of all  $(q+1)$ -dimensional *obstruction cocycles* form a *single* cohomology class  $\gamma(f_q)$  in  $H^{q+1}(K, L; \prod_q (S^{n-1}))$  and  $f$  extends over  $\overline{K}^{q+1}$  if and only if  $\gamma(f_q) = 0$  (i.e., one may have to modify the previous extension, but only on  $q$ -cells) [4, 5.1, p. 180; 12, 34.2, p. 174].

4) If  $\Theta^{q+1}(f) \subset H^{q+1}(K, L; \prod_q (S^{n-1}))$  is defined as empty if  $f$  is not extendible to  $\overline{K}^q$  and as the set of all  $\gamma^{q+1}(f_q)$  for all extensions  $f_q$  of  $f$  over  $\overline{K}^q$ , then this *obstruction set* has the following properties:

- (i) homotopic maps (over  $L$ ) have the same obstruction sets;
  - (ii)  $f$  is extendible to  $\overline{K}^q$  if and only if  $\Theta^{q+1}(f) \neq \emptyset$ ; and
  - (iii)  $f$  is extendible to  $\overline{K}^{q+1}$  if and only if 0 belongs to  $\Theta^{q+1}(f)$  [4, 6.1, p. 181, 6.4 and 6.5, p. 182; 12, 34.9, p. 176].
- 5) If  $q = n - 1$ , then the obstruction set  $\Theta_n(f)$  reduces to a single

element, the primary obstruction,  $\gamma^n(f) = \delta K^{n-1}(f)$ , where

$$\delta : H^{n-1}(L; \prod_{n-1}(S^{n-1})) \rightarrow H^n(K, L; \prod_{n-1}(S^{n-1}))$$

is the usual coboundary operator in the cohomology sequence for the pair  $(K, L)$  and  $K^{n-1}(f)$  is the characteristic element of  $f$  (given by  $f^*(K^{n-1}(S^{n-1}))$  with

$$f^* : H^{n-1}(S^{n-1}; \prod_{n-1}(S^{n-1})) \rightarrow H^n(L; \prod_{n-1}(S^{n-1}))$$

and  $K^{n-1}(S^{n-1})$  is the generator of the first group [4, 13.1, pp. 189, 193; 12, 35.4, p. 178].

6) Finally one has obstructions for lifting homotopies (with  $K$  replaced by  $K \times I$ ): If  $f_0, f_1 : \bar{K}^q \rightarrow \mathbf{R}^n \setminus \{0\}$  and if  $h_t$  is a homotopy of  $f_0 | \bar{K}^{q-1}$  to  $f_1 | \bar{K}^{q-1}$ , then one has a *difference cochain*  $d^q(f_0, h_t, f_1)$  in  $C^q(K, L, \prod_q(S^{n-1}))$  such that:

(a)  $\delta d^q(f_0, h_t, f_1) = C^{q+1}(f_0) - C^{q+1}(f_1)$ ,  $\delta$  the above coboundary operator [4, Lemmas 4.1, 4.2, p. 179].

(b)  $d^q(f_0, h_t, f_1) = 0$  if and only if  $h_t$  extends to a homotopy of  $f_0 | \bar{K}^q$  to  $f_1 | \bar{K}^q$  [12; 33.4, p. 171].

(c) If  $f$  and  $g$  are two extensions over  $\bar{K}^{q+1}$  such that  $f = g$  on  $L$ , and if  $h_t$  is a homotopy (relative to  $L$ ) of  $f | \bar{K}^{q-2}$  to  $g | \bar{K}^{q-2}$  which is extendible to a homotopy (relative to  $L$ )  $h'_t$  of  $f | \bar{K}^{q-1}$  to  $g | \bar{K}^{q-1}$ , then the set of obstruction cocycles  $d^q(f, h'_t, g)$  form a single cohomology class  $\delta^q(f, h_t, g)$  in  $H^q(K, L; \prod_q(S^{n-1}))$  and  $h_t$  is extendible to a homotopy of  $f | \bar{K}^q$  to  $g | \bar{K}^q$  if and only if  $\delta^q(f, h_t, g) = 0$  [4, 8.3, p. 184; 12, 34.6, p. 175] (i.e., one may have to modify  $h'_t$  on the  $(q-1)$  cells). As before, one may define obstruction sets and a primary obstruction for  $q = n-1$ :  $f$  and  $g$  are always homotopic on  $\bar{K}^{n-2}$ , and the obstruction set consists of a single element  $\delta^{n-1}(f, g)$  in  $H^{n-1}(K, L; \prod_{n-1}(S^{n-1}))$  with  $j^* \delta^{n-1}(f, g) = K^{n-1}(f) - K^{n-1}(g)$ , where  $j^*$  is the homomorphism  $H^{n-1}(K, L) \rightarrow H^{n-1}(K)$  induced by inclusion and  $K_{n-1}(f)$  is the characteristic element of  $f$  (as a mapping defined on  $K$ ). Note that this lifting of homotopies implies that the obstruction in the first non-vanishing group  $H^{q+1}(K, L; \prod_q(S^{n-1}))$  is

unique [12, 35.10 and 36.11].

**3.2. The cohomology groups and the obstructions.** Eventual obstructions will be elements of the groups

$$H^{q+1}(B^k \times S^{d-1}, S^{k-1} \times S^{d-1}; \prod_q (S^{d-1})),$$

where, in  $S^{d-1}$ , the cell  $B^p = \{(x_1, \dots, x_p) / \|x\|^2 = \sum x_i^2 \leq 1\}$  is sent into  $(x_1, \dots, x_p, \pm(1 - \|x\|^2)^{1/2}, 0, \dots, 0)$ . Relative cells, in  $K \setminus L$ , are of the form  $B^k \times B^p, B^p$  as above, attached to  $L$  by  $S^{k-1} \times B^p$ . Now  $H^{q+1}(B^k \times S^{d-1}, S^{k-1} \times S^{d-1}) \simeq H^{q+1}(S^{k+d-1}, S^{k-1} \times B^d)$ , by excision of the open set (in  $S^{k+d-1} = \partial(B^k \times B^d)$ )  $S^{k-1} \times (B^d \setminus S^{d-1})$ . Since  $B^d$  is contractible,  $S^{k-1} \times B^d$  has the cohomology of  $S^{k-1}$  and, from the exact sequence

$$\begin{aligned} &\rightarrow H^q(S^{k-1}) \rightarrow H^{q+1}(S^{k+d-1}, S^{k-1} \times B^d) \\ &\rightarrow H^{q+1}(S^{k+d-1}) \rightarrow H^{q+1}(S^{k-1}) \rightarrow, \end{aligned}$$

one derives easily that

$$H^{q+1}(B^k \times S^{d-1}, S^{k-1} \times S^{d-1}) = \begin{cases} 0 & \text{if } q+1 \neq k, k+d-1 \\ Z & \text{if } q+1 = k, \\ & \text{or } k+d-1, d > 1 \\ Z \oplus Z & \text{if } q+1 = k, d = 1. \end{cases}$$

Thus one will have at most two obstructions, one in  $\prod_{k-1}(S^{d-1})$  (in two copies of it, if  $d = 1$ ) and the other, in the top dimension, in  $\prod_{k+d-2}(S^{d-1})$ .

The first (unique) obstruction is given by the class of  $F(\lambda, x_0) : \{\lambda, \|\lambda\| = 1\} \rightarrow \mathbf{R}^d \setminus \{0\}$ , where  $x_0$  is a fixed point in  $S^{d-1}$  (two such points, if  $d = 1$ ). If  $d > 1$ ,  $F(\lambda, x_0)$  is deformable to  $F(\lambda, x_1)$ , for any other point  $x_1$  in  $S^{d-1}$ , and, for  $F(\lambda, x) = B(\lambda)x$  and  $d = 1$ , the class of  $B(\lambda)1$  and of  $B(\lambda)(-1)$  are the same (0 if  $k > 1$ ). This can be summarized as

**THEOREM 3.1.** *If  $B(\lambda)x_0 : S^{k-1} \rightarrow \mathbf{R}^d \setminus \{0\}$  is non-trivial (thus  $k \geq d$ ), one has bifurcation in all directions  $x_0$ .*

In order to compute the obstruction in  $\prod_{k+d-2}(S^{d-1})$ , one has first to extend  $F(\lambda, x)$  to  $B^k \times S^{d-2}$  ( $S^{d-2}$  the equator of  $S^{d-1}$ ).

Now, in the exact sequence

$$\rightarrow \prod_k (S^{d-1}) \xrightarrow{\delta} \prod_{k-1} (\text{GL}(\mathbf{R}^{d-1})) \xrightarrow{i_*} \prod_{k-1} (\text{GL}(\mathbf{R}^d)) \xrightarrow{P_*} \prod_{k-1} (S^{d-1}) \rightarrow$$

(since  $\text{GL}(\mathbf{R}^n)$  is homotopy equivalent to  $O(n)$  [12, p. 57]),  $P$  is the evaluation map  $P : B(\lambda) \rightarrow B(\lambda)x_0$ ,  $i$  the inclusion map  $i : C(\lambda) \rightarrow \begin{pmatrix} C(\lambda) & o \\ 0 & 1 \end{pmatrix}$  [12, p. 91]. Thus, if  $[B(\lambda)x_0] = 0$ ,  $B(\lambda)$  is deformable to  $\begin{pmatrix} C(\lambda) & 0 \\ 0 & 1 \end{pmatrix}$  and  $B(\lambda)x$  has the same obstruction sets as  $(C(\lambda)\tilde{x}, x_d)$ ,  $\tilde{x}$  in  $\mathbf{R}^{d-1}$ .

Consider the map

$$\tilde{F}(\lambda, x) = (\|\lambda\| C(\lambda/\|\lambda\|)\tilde{x}, x_d, -\varepsilon(1 - \|\lambda\|)), \quad 0 < \varepsilon < 1,$$

$$\tilde{F}(\lambda, x) : S^{k-1} \times S^{d-1} \cup B^k \times S^{d-2} \rightarrow \mathbf{R}^d \setminus \{0\}.$$

$\tilde{F}|_L$  is homotopic to  $B(\lambda)x$  and extends to the south hemisphere, thus the obstruction is the class of  $\tilde{F}$  on the north hemisphere,  $x_d = (1 - \|\tilde{x}\|^2)^{1/2}$ . Since, on the boundary of the cell, either  $\|\tilde{x}\| = 1$  or  $\|\lambda\| = 1$ , one may use the deformation

$$t((1 - \|\tilde{x}\|^2)^{1/2}\varepsilon(1 - \|\lambda\|) + (1 - t)(1 - 2\|\tilde{x}\|^2)$$

(for  $\|\lambda\| = 1$  and  $C(\lambda)\tilde{x} = 0$ , then  $\tilde{x} = 0$ ). Thus this obstruction vanishes if and only if  $JC(\lambda) = 0$ . Note that  $JB(\lambda) = J\Sigma C(\lambda) = -\Sigma JC(\lambda)$  [13] where the second  $\Sigma$  stands for the suspension map from  $\prod_{k+d-2}(S^{d-1})$  into  $\prod_{k+d-1}(S^d)$ . Hence, if  $JB(\lambda) \neq 0$  and  $[B(\lambda)x_0] = 0$ , one has bifurcation. Conversely  $JC(\lambda)$  is unique if it is the primary obstruction, i.e.,  $k < d$ , and  $\Sigma$  is an isomorphism, as well as  $i_*$ , if  $k < d - 1$  (both are onto if  $k = d - 1$ ); furthermore  $\Sigma$  is one to one if  $d = 2, 4$  or  $8$  [12, p. 112; 4, p. 327]. Thus if  $J(B(\lambda)) = 0$  and  $[B(\lambda)x_0] = 0$ ,  $k < d - 1$  or  $d = 2, 4$  or  $8$ ,  $JC(\lambda) = 0$  and one has an extension. In the other cases one may have more elements in the obstruction sets and  $C(\lambda)$  may not be unique. In fact, if  $\alpha$  is any element of  $\prod_k(S^{d-1})$ , then  $\delta(\alpha)C(\lambda)$  is also an element of  $\prod_{k-1}(\text{GL}(\mathbf{R}^{d-1}))$  with  $i_*(\delta(\alpha)C(\lambda)) = B(\lambda)$ . (The class of the product of matrices is the

sum of the classes : [12; Lemma 16.7, p. 88].) However one has the following result.

**THEOREM 3.2.** *If  $d > 2$ , the top obstruction set consists of all elements  $[JC(\lambda)]$ , where  $i_*[C(\lambda)] = [B(\lambda)]$ . For  $d = 2$ , if  $[JB(\lambda)] \neq 0$  then  $B(\lambda)x$  has no extension (thus  $k = 2$ ).*

**PROOF.** If  $d = 2$ , then if  $k > 2$ ,  $[B(\lambda)] = 0$  and if  $k = 2$ ,  $J$  and  $P_*$  are isomorphisms ( $J(\mu + i\nu)$  is the Hopf map). Suppose then that  $d > 2$ . In order to define the obstruction set in  $H^{k+d-1}(K, L; \prod_{k+d-2}(S^{d-1}))$  one has to look at all possible extensions to  $\overline{K}^{k+d-2}$ . In fact each extension is characterized by its behavior on  $\overline{K}^k = S^{k-1} \times S^{d-1} \cup B^k \times S^0$ .

**LEMMA 3.1.** *If  $F_0, F_1$  are two extensions of  $B(\lambda)x$  to  $\overline{K}^{k+d-2}$  such that  $F_0 \mid \overline{K}^k$  is homotopic, relative to  $L$ , to  $F_1 \mid \overline{K}^k$ , then  $F_0$  and  $F_1$  are homotopic, relative to  $L$ .*

**PROOF.** If  $x_0 = (1, 0, \dots, 0)$ , then  $\overline{K}^k$  has two relative cells  $B^k \times \{x_0\}$  and  $B^k \times \{-x_0\}$ . On  $\partial(B^k \times \{x_0\} \times I)$ , define the map  $F(\lambda, t)$  as  $F(\lambda, 0) = (F_0(\lambda), 0)$ ,  $F(\lambda, 1) = (F_1(\lambda), 1)$ ,  $F(\lambda, t) = (B(\lambda)x_0, t)$  for  $\|\lambda\| = 1$ . The class of  $F(\lambda, t)$  in  $\prod_k(S^d)$  is the obstruction for lifting the homotopy of  $F_0 \mid L$  and  $F_1 \mid L$  (here the identity) to  $L \cup B^k \times \{x_0\}$ . Together with the corresponding class for  $B^k \times \{-x_0\}$  they form the suspension of the generalized *primary* difference (since  $H^q(K, L) = 0$  for  $q < k$ ) [12, 36.11 and 36.8]. Now, if  $F_t(\lambda)$  is the homotopy from  $F_0$  to  $F_1$  on  $\overline{K}^k$ , one may extend  $F$  to  $B^k \times S^0 \times I$  via  $(F_t(\lambda), t)$ . Thus the primary difference vanishes and the next obstruction for lifting this homotopy will give cohomology classes in  $H^q(K, L; \prod_q(S^{d-1}))$  (since the obstructions for  $F_0$  and  $F_1$  vanish up to the level  $k + d - 2$ ). However, these groups are 0 for  $q \leq k + d - 2$ , thus one may lift the homotopy to  $\overline{K}^{k+d-2} = S^{k-1} \times S^{d-1} \cup B^k \times S^{d-2}$ , [12, 34.6]. Note that if  $d > 2$  and  $F_0(\lambda, x_0)$  is homotopic, relative to  $L$ , to  $F_1(\lambda, x_0)$ , then  $F_0$  and  $F_1$  are homotopic: since  $S^{d-2}$  is connected,  $F_i(\lambda, x_0)$  is homotopic to  $F_i(\lambda, -x_0)$ ,  $i = 0, 1$ , giving a homotopy, relative to  $L$ , on  $\overline{K}^k$ .

LEMMA 3.2. Assume  $B(\lambda)x_0 = x_0$  and let  $F(\lambda)$  be an extension of  $B(\lambda)x_0$  to  $\{\|\lambda\| \leq 1\}$ . Then there is  $\tilde{C}(\lambda)$  in  $\text{GL}(\mathbf{R}^{d-1})$ , for  $\|\lambda\| = 1$ , such that  $\Sigma[\tilde{C}(\lambda)] = [B(\lambda)]$  and an extension to  $\overline{K}^{k+d-2}$ ,  $B(\lambda, x)$ , of  $B(\lambda)x$  such that  $B(\lambda, x_0)$  is homotopic to  $F(\lambda)$ , relative to  $L$ , and the obstruction for extending  $B(\lambda, x)$  to  $K$  is  $J(\tilde{C}(\lambda))$ .

PROOF.  $F(\lambda) : (B^k, S^{k-1}) \rightarrow (\mathbf{R}^d \setminus \{0\}, x_0)$  defines an element of  $\prod_k (S^{d-1}, x_0)$  which is isomorphic to  $\prod_k (S^0(d), S^0(d-1))$ , via  $P_*$ , where  $PD(\lambda) = D(\lambda)x_0$ , [12, 17.2]. Thus there is  $D(\lambda) : (B^k, S^{k-1}) \rightarrow (S^0(d), (S^0(d-1)))$  with  $D(\lambda)x_0 = F(\lambda)$ ,  $D(\lambda) = \begin{pmatrix} 1 & O \\ 0 & C_1(\lambda) \end{pmatrix}$ , for  $\|\lambda\| = 1$ . Let

$$B(\lambda, x) = D(\lambda) \begin{pmatrix} 1 & \|\lambda\| B_1(\lambda/\|\lambda\|) \\ 0 & \|\lambda\| C_1^{-1}(\lambda/\|\lambda\|)C(\lambda/\|\lambda\|) \end{pmatrix} A(\lambda) \begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \\ x_d - \varepsilon(1 - \|\lambda\|) \end{pmatrix},$$

where  $B_1(\lambda)$  stands for the first line in  $B(\lambda)$  for the rows 2 through  $d$  and  $C(\lambda)$  corresponds to the rest of  $B$ .  $A(\lambda)$  is the rotation on the first and last coordinates:

$$A(\lambda) = \begin{pmatrix} \cos(1 - \|\lambda\|)\pi/2 & 0 & \cdots & 0 & \sin(1 - \|\lambda\|)\pi/2 \\ 0 & & I & & 0 \\ -\sin(1 - \|\lambda\|)\pi/2 & 0 & \cdots & 0 & \cos(1 - \|\lambda\|)\pi/2 \end{pmatrix},$$

$\tilde{C}(\lambda) = C_1^{-1}(\lambda)C(\lambda)$ . Since, for  $\|\lambda\| = 1$ ,  $B_1(\lambda)$  may be deformed to 0 in the matrix form of  $B(\lambda)$ ,  $\Sigma[C(\lambda)] = [B(\lambda)]$ . On the other hand

$$\begin{aligned} \Sigma[\tilde{C}(\lambda)] &= \left[ \begin{pmatrix} 1 & 0 \\ 0 & C_1^{-1}(\lambda) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C(\lambda) \end{pmatrix} \right] \\ &= [D^{-1}(\lambda)B(\lambda)] = [D^{-1}(t\lambda)B(\lambda)] = [D^{-1}(0)B(\lambda)] = [B(\lambda)], \end{aligned}$$

since  $D^{-1}(0)$  can be deformed to  $I$ . Clearly  $B(\lambda, x) = B(\lambda)x$ , for  $\|\lambda\| = 1$ , and if  $B(\lambda, x) = 0$ , then either  $\lambda \neq 0$ ,  $\tilde{x} = 0$  and  $x_d = \varepsilon(1 - \|\lambda\|)$ , which is not possible since  $\|x\| = 1$ , or  $\lambda = 0$  and  $x_d = \varepsilon$ . Thus  $B(\lambda, x)$  extends over the lower hemisphere. The

obstruction is the class of  $B(\lambda, x)$  for either  $\|\lambda\| = 1$  or  $\|\tilde{x}\| = 1$ , with  $x_d = (1 - \|\tilde{x}\|^2)^{1/2}$ . One may deform  $D(\lambda)$  to  $D(0)$  and then to  $I$  as above. Use, next, the deformation  $A(t\lambda)$  and send  $B_1(\lambda)$  to  $O$ . The obstruction is then the class of  $((1 - \|\tilde{x}\|)^{1/2} - \varepsilon(1 - \|\lambda\|), \|\lambda\| \tilde{C}(\lambda/\|\lambda\|)(x_2, \dots, x_{d-1}, -x_1)^T$ . A series of rotations will bring the vector  $(x_2, \dots, x_{d-1}, -x_1)$  to  $((-1)^{d+1}x_1, x_2, \dots, x_{d-1})$  and another series of  $d - 1$  rotations will take the function to  $J\tilde{C}(\lambda)$ . Finally, replacing  $\pi/2$  by  $(\pi/2)t$  in  $A(\lambda)$  and  $\varepsilon$  by  $\varepsilon t$ , one has that  $B(\lambda, x_0)$  is deformable, relative to  $L$ , to  $D(\lambda)x_0 = F(\lambda)$  (for  $\lambda = 0$ , the deformation is  $(\cos \pi/2t + \varepsilon t \sin \pi/2t, 0, \dots, 0)$ ).

*End of the proof of Theorem 3.2.* Since  $[B(\lambda)x_0] = 0, B(\lambda)$  is deformable to a matrix of the form  $\begin{pmatrix} 1 & 0 \\ 0 & C(\lambda) \end{pmatrix}$ , with the same obstruction sets. Take an element of the top obstruction set, realized by an extension  $F(\lambda, x)$ . By Lemmas 3.1 and 3.2 this extension is homotopic on  $\bar{K}^{k+d-2}$ , relative to  $L$ , to a  $B(\lambda, x)$  constructed as in Lemma 3.2, and the obstruction is  $J\tilde{C}(\lambda)$  where  $\Sigma[\tilde{C}(\lambda)] = [B(\lambda)]$ .

Computing the classes of  $J\tilde{C}(\lambda)$  for all possible  $C(\lambda)$  does not look very easy. Furthermore,  $JB(\lambda)$  seems to be a more “natural” invariant to use. The following result provides a partial answer to the extension problem.

**THEOREM 3.3.** *If  $JB(\lambda) = 0$  and  $B(\lambda)x_0$  is deformable to a constant, then  $B(\lambda)x$  extends to  $B^k \times S^{d-1}$  if one of the following conditions is satisfied:*

- 1)  $k \leq 2d - 2$  (“metastable range”).
- 2)  $d = 2, 4, 8$  (if  $d = 2, k > 2$ , then  $[B(\lambda)] = 0$ ).

**PROOF.** Note first that if  $d = 1$ , the only obstruction is the class of  $B(\lambda)1$  which is not trivial only if  $k = 1$  and  $B(\lambda)$  changes sign. Thus one may suppose that  $d > 1$ .

Consider the diagram

$$\begin{array}{ccccc}
 & & \prod_{k-1}(GL(R^{d-1})) \xrightarrow{i_*} \prod_{k-1}(GL(R^d)) & & \\
 & \nearrow \delta & \downarrow J & \searrow P_* & \\
 \prod_k(S^{d-1}) & & & & \prod_{k-1}(S^{d-1}) \\
 & \searrow [\cdot, i_{d-1}] & \downarrow J & & \downarrow P \\
 & & \prod_{k+d-2}(S^{d-1}) \xrightarrow{-\Sigma} \prod_{k|d-1}(S^d) & & \\
 \downarrow P & \nearrow & \downarrow H & \searrow & \\
 \prod_{k+d}(S^d, E^+, E^-) & & & & \prod_{k+d-1}(S^d, E^+, E^-) \\
 & \searrow h & \downarrow h & \nearrow & \\
 & & \prod_{k+d}(S^{2d-1}) & & \prod_{k+d-1}(S^{2d-1})
 \end{array}$$

The two rows are the exact sequences for orthonormal groups and the suspension triad for the sphere [11].  $E^\pm$  are the hemispheres of  $S^d$ ,  $j_*$  is induced by inclusion,  $\Delta$  is a double boundary operator,  $[\alpha, i_{d-1}]$  is the Whitehead product of  $\alpha$  in  $\prod_k(S^{d-1})$  with the generator of  $\prod_{d-1}(S^{d-1})$ .  $P$  and  $h$  are defined in [11] with the property that  $\Delta P(\alpha) = j\delta(\alpha) = [\alpha, i_{d-1}]$  (see also [14]),  $PP_*(B(\lambda)) = i_*JB(\lambda)$  [11; Theorem 1.5].  $H$  is the generalized Hopf invariant defined in [14]. And  $h$  has the property that  $hoP = \Sigma^d$ ; it is an isomorphism if  $d$  is even or if  $k \leq 2d-1$ ,  $k \neq 2d-3$  and  $d$  odd, in which case  $h$  is still onto [11; 7.2]. (Here  $h$  is considered from  $\prod_{k+d}(S^d, E^+, E^-)$  into  $\prod_{k+d}(S^{2d-1})$ . Note that  $\Sigma^d : \prod_k(S^{d-1}) \rightarrow \prod_{k+d}(S^{2d-1})$  is an isomorphism if  $k \leq 2d-4$  and onto if  $k = 2d-3$ ).  $Q$  is defined only when  $h$  is an isomorphism, giving rise to the  $Q\Sigma H$  ( $EHP$  in classical notation) exact suspension sequence.  $P$  is an isomorphism from  $\prod_k(S^{d-1})$  onto  $\prod_{k+d}(S^d, E^+, E^-)$  if  $k < 2d-3$ .  $P$  is onto if  $k = 2d-3$  and if  $k = 2d-2$  (if  $\prod_{2d-1}(S^d)$  has no element of Hopf invariant 1, i.e., according to Adams' results, if  $d = 2, 4, 8$  in which case  $\Sigma$  is one to one) (see [11; Theorems 7.3, 7.4, 7.5]). Thus if  $JB(\lambda) = -\sum JC(\lambda) = 0$ , then  $JC(\lambda)$  (if non-zero)

comes from an element in  $\prod_{k+d}(S^d, E^+, E^-)$ , by exactness. From the above result, for  $k \leq 2d - 2$  and  $\Sigma$  not one to one, there is an  $\alpha$  in  $\prod_k(S^{d-1})$  such that  $\Delta P\alpha = JC(\lambda) = [\alpha, i_{d-1}] = J\delta a$ . [11; Theorem 7.7] Hence  $J(\delta\alpha)^{-1}C(\lambda) = 0$  and  $i_*(\delta\alpha)^{-1}C(\lambda) = B(\lambda)$ , proving the Theorem (Part 2 comes from the fact that  $\Sigma$  is one to one and  $S0(2) \simeq S^1$  [12; 22.2]).

Note that, for  $k > 2d - 2$ ,  $P$  is not onto anymore, however, what is needed here is that the map  $[i_{d-1}]$  is onto  $\text{Im } J \cap \ker \Sigma$  (in the known examples  $\text{Im } J = 0$ ). This can be stated in the following form (in view of the “naturality” of the analysis problem).

CONJECTURE. *The map  $\alpha \rightarrow [\alpha i_{d-1}]$ , from  $\prod_k(S^{d-1})$  into  $\prod_{k+d-2}(S^{d-1})$  is onto  $\ker \Sigma \cap \text{Im } J$ .*

For  $k \leq 2d - 3$ , a summary of the results for local linearized bifurcation is given by

THEOREM 3.4. *If  $k \leq 2d - 3$ , the non-vanishing of  $J(B(\lambda))$  is a “necessary” and sufficient condition for local bifurcation. For  $d = 2$ , this is true for any  $k$ ; for  $d = 4$  or  $8$  one may allow  $k \leq 2d - 2$ .*

PROOF. The only point to check is for the case where  $P_*B(\lambda) = [B(\lambda)x_0] \neq 0$ . However,  $HJB(\lambda) = h \circ PP_*B(\lambda) = \Sigma^d[B(\lambda)x_0]$  and  $\Sigma^d$  is an isomorphism if  $k < 2d - 2$ , a one to one map if  $d = 4$  or  $8$  and  $k \leq 2d - 2$  (the first suspension is one to one, the others are isomorphisms). For  $d = 2$ ,  $\prod_{k-1}(S^1) = 0$  if  $k > 2$ , and for  $k = 2$  the suspension is an isomorphism.

REMARK 3.1. For  $k \geq 2d - 2$ , one may have  $[B(\lambda)x_0] \neq 0$  and  $JB(\lambda) = 0$ . Take  $d = 3$ ,  $k = 4$ . Then  $P_*$  is an isomorphism (from the exact sequence, this is true for  $k \geq 4$ ; for  $k = 3$ ,  $\prod_2(S0(3)) = 0$  and  $\prod_1(S0(3)) = Z_2$ ) [12, pp. 115-116]. Thus, if  $\rho$  generates  $\prod_3(S0(3)) = Z$ , then  $P_*\rho = \eta$  is the Hopf map.  $HJ(\rho) = \Sigma^3\eta$  generates  $\prod_6(S^5) = Z_2$ . Thus  $J(\rho)$  generates  $\prod_6(S^3) = Z_{12}$ . If  $B(\lambda)$

represents  $\rho^{12}$ , then  $P_*\rho^{12} = 12\eta \neq 0$  but  $J(\rho^{12}) = 0$ . Note that  $\Sigma P_*\rho^{12} = 0$ .  $P : \prod_3(S^2) \rightarrow \prod_6(S^3, E_+, E_-)$  is onto with kernel generated by  $3[i_2, i_2] = 6\eta$  [11; Theorem 7.4] since  $[i_2, i_2] = 2\eta$  generates  $\ker \Sigma$ . Also  $h$  is onto with kernel generated by  $2P\eta$  and of order 3. From this it is easy to see that  $\prod_6(S^3, E_+, E_-) = Z_6$ , generated by  $p\eta$ , and that  $j_*$  is onto with kernel generated by  $J(\rho^6)$ . Note that the suspension sequence is not exact since  $HJ(\rho^2) = 0$ , but  $J(\rho^2)$  is not a suspension.

**REMARK 3.2.** For the general Hopf construction, that is, for a map  $F(\lambda, x) : S^{k-1} \times S^{d-1} \rightarrow S^{d-1}$ , not necessarily linear in  $x$ , one has two possible obstructions for extension to  $B^k \times S^{d-1}$ . The primary obstruction is the class of  $F(\lambda, x_0)$  in  $\prod_{k-1}(S^{d-1})$ . If it does not vanish, then one has bifurcation in all directions for the problem  $G(\mu, y)$ , where  $G(\mu, 0) = 0$ ,  $G(\mu, y) \neq 0$  for  $0 < \|x\| \leq r$ ,  $\|\mu\| = p$ , taking  $F(\lambda, x) = G(\lambda\rho, \varepsilon x)$  with  $\|\lambda\| = \|x\| = 1$ ,  $0 < \varepsilon \leq r$ . If it vanishes, Lemma 3.1 is still valid and classifies all possible extensions to  $\overline{K}^{k+d-2}$  in terms of the class of  $F(\lambda, x_0)$  relative to  $L$ . In particular, if  $k < d$ , there is a unique generalized primary obstruction in  $\prod_{k+d-2}(S^{d-1})$ , the top obstruction. In fact the obstruction for lifting the homotopy of  $F_0(\lambda, x_0)$  to  $F_1(\lambda, x_0)$  lies in  $\prod_k(S^d) = 0$ , hence the generalized primary difference vanishes and the argument of Lemma 3.1 goes through. Of course the difficulty here is to find an extension to  $\overline{K}^{k+d-2}$  in order to compute the top obstruction.

#### 4. Global bifurcation and obstruction. For the equation

$$(A_0 - A(\lambda))x - g(\lambda, x) = 0,$$

let  $\mathbf{C}$  be the continuum of non-trivial solutions bifurcating from  $(0, 0)$  ( $\mathbf{C}$  maybe empty). Here  $g(\lambda, x) = o(\|x\|)$ ,  $A_0 - A(\lambda)$  is invertible for  $0 < \|\lambda\| \leq \rho$ ,  $A(0) = 0$  and  $\dim \ker A_0 = d$ , where  $x \in \mathbf{R}^N$ ,  $\lambda \in \mathbf{R}^k$ . (For the corresponding problem in infinite dimensions one has to add the usual hypotheses of compactness).

**DEFINITION 4.1.**  $(0, 0)$  is a point of linearized global bifurcation if and only if, for any  $g(\lambda, x)$ , the continuum  $\mathbf{C}$  is either unbounded or

returns to a different bifurcation point.

A sufficient condition for linearized global bifurcation is that  $J(A_0 - A(\lambda))$ , [2, 5, etc]. Note that  $J(A_0 - A(\lambda)) = \varepsilon \Sigma^{N-d} J(B(\lambda))$ , as in Remark 1.5, where  $\varepsilon = \pm 1$  depends on the orientations (see below). In this section it will be shown that, if  $N > d$ , the non-vanishing of  $J(A_0 - A(\lambda))$  is also necessary for linearized global bifurcation. As for the local bifurcation, an equivalence with extensions of maps will be first established.

**THEOREM 4.1.** *One has linearized global bifurcation if and only if the map*

$$\begin{aligned} ((A_0 - A(\lambda))\eta, r - \varepsilon) : \\ \{(\lambda, r, \eta) / \|\eta\| = 1, (\lambda, r) \in \partial(B^k \times I), B^k = \{\lambda / \|\lambda\| < \rho\}, I = [0, 2\varepsilon]\} \\ \rightarrow \mathbf{R}^{N+1} \setminus \{0\}, \end{aligned}$$

has no non-zero extension to  $B^k \times I \times S^{N-1}$ .

**PROOF.** (If). (Sketch). Suppose that one has no linearized global bifurcation. Then there is a nonlinearity  $g(\lambda, x)$  such that the corresponding continuum  $\mathbf{C}$  is bounded and does not meet any bifurcation point except  $(0, 0)$ . By standard arguments one constructs an open bounded set  $\Omega$ , containing  $\mathbf{C}$  and such that if  $(A_0 - A(\lambda))x - g(\lambda, x) = 0$  on  $\partial\Omega$ , then  $x = 0, \|\lambda\| \leq \rho$ . Taking a large box  $B = \{\|\lambda\| \leq R, \|x\| \leq R\}$  containing  $\Omega$ , then this equation, together with the condition  $\|x\| - \varepsilon$ , is non-zero on  $\partial\Omega \cup \{x = 0\}$  and, for  $\varepsilon$  large on  $(B \setminus \Omega) \cup \{x = 0\}$ . Thus, for any  $\varepsilon$ , the pair is inessential on  $\partial\Omega \cup \{x = 0\}$ , that is, it extends without zeros to  $(B \setminus \Omega) \cup \{x = 0\}$  (see [7] or [9; Remark 2.8], using the homotopy  $((A_0 - A(\lambda))x - g(\lambda, x), \|x\| - t\varepsilon - (1-t)2R)$ . Choose  $\varepsilon$  so small that, for  $\|\lambda\| = \rho$ , the linear part dominates the non-linear part if  $\|x\| \leq 2\varepsilon$  and such that the box  $\{\|\lambda\| \leq \rho, \|x\| \leq 2\varepsilon\}$  is contained in  $\Omega$  and  $\|x\| \geq 2\varepsilon$  outside the box but still in  $\Omega$ . Extend the pair without zeros to  $(B \setminus \Omega) \cup \{x = 0\}$ . By scaling, first in  $\lambda$ , then in  $r$ , the map  $((A_0 - A(\lambda))r\eta - g(\lambda, r\eta), r - \varepsilon)$  on  $\partial(B^k \times I) \times S^{N-1}$  is homotopic to the extension on  $\partial(\|\lambda\| \leq R, r \in [0, R]) \times S^{N-1}$ . Thus  $((A_0 - A(\lambda))\eta, r - \varepsilon)$ , which is clearly homotopic to the previous map

on  $\partial(B^k \times I) \times S^{N-1}$  also has an extension to  $B^k \times I \times S^{N-1}$ . For a more complete proof see [7] or [9; Proposition 4.1].

(Only if ). Suppose  $((A_0 - A(\lambda))\eta, r - \varepsilon)$  has the extension  $(A(\lambda, r, \eta), h(\lambda, r, \eta))$  to  $B^k \times I \times S^{N-1}$ . Define, for  $\|\lambda\| \leq \rho, \|x\| \leq 2\varepsilon$ ,

$$g(\lambda, x) = -\|x\| (A(\lambda, \|x\|, x/\|x\|) - (A_0 - A(\lambda)x/\|x\|)).$$

(take  $g(\lambda, x) \equiv 0$  outside this box). Since  $A(\lambda, r, \eta) - (A_0 - A(\lambda))\eta$  tends to 0 uniformly in  $(\lambda, \eta)$ , as  $r$  goes to 0, then  $g(\lambda, x) = o(\|x\|)$ . (In the infinite dimensional case this difference has to be compact, see [9]). Furthermore,  $g(\lambda, x) = 0$  for  $\|\lambda\| = \rho$  or  $\|x\| = 2\varepsilon$ . Let  $\Omega$  be the set  $\{(\lambda, x)/h(\lambda, \|x\|, x/\|x\|) < 0\}$ . Since  $h(\lambda, r, \eta)$  tends uniformly to  $-\varepsilon$ , when  $r$  goes to 0,  $h$  is continuous and  $\Omega$  is open. And  $\partial\Omega \subset \{(\lambda, x)/\|\lambda\| = \rho, \|x\| \leq \varepsilon\} \cup h^{-1}(0)$ . Since  $h = \varepsilon$ , for  $\|x\| = 2\varepsilon$ ,  $\partial\Omega$  separates  $(0, 0)$  from the level  $\|x\| = 2\varepsilon$ . Finally  $\mathbf{C}$  does not meet  $\partial\Omega$ , since if  $h = 0$ ,  $A(\lambda, \|x\|, x/\|x\|) \neq 0$  and if  $\mathbf{C}$  meets  $\|\lambda\| = \rho$  this could be only for  $x = 0$ , giving a bifurcation point for some  $\lambda$ , with  $\|\lambda\| = \rho$ , which contradicts the fact that  $A_0 - A(\lambda)$  is invertible for  $\|\lambda\| = \rho$ .

REMARK 4.2. It has been seen in [9; Proposition 4.5] that  $((A_0 - A(\lambda))x, \|x\| - \varepsilon)$  extends from  $\partial\{\|\lambda\| \leq \rho, \|x\| \leq 2\varepsilon\}$  to full ball if and only if  $J(A_0 - A(\lambda)) \neq 0$ . Here one wishes to maintain the map as  $(0, -\varepsilon)$  on  $x = 0$ , staying closer to the bifurcation idea. The next step is to use obstruction theory to know when one has an extension.

THEOREM 4.2. (a) *One has linearized global bifurcation if and only if either  $\Sigma(A_0 - A(\lambda))\eta_0 \neq 0$  or  $J(A_0 - A(\lambda)) \neq 0$ .*

(b) *If  $N > d$  or  $k \leq 2N - 2$ , one has linearized global bifurcation if and only if  $J(A_0 - A(\lambda)) = \varepsilon \Sigma^{N-d} JB(\lambda) \neq 0$ .*

PROOF. The possible obstructions for extending  $((A_0 - A(\lambda))\eta, r - \varepsilon)$  from  $L_1 \equiv \partial(B^k \times I) \times S^{N-1} \rightarrow \mathbf{R}^{N+1} \setminus \{0\}$  to  $K_1 \equiv B^k \times I \times S^{N-1}$  are in  $H^{q+1}(K_1, L_1; \prod_q(S^N))$ . These groups are zero except for  $q + 1 = k + 1$  or  $k + N$  if  $N > 1$ , in which case they are  $\prod_q(S^N)$ , and two copies

of  $\prod_k (S^1)$  if  $N = 1$ . The first obstruction, for the  $(k+1)$ -cells of  $K_1 \setminus L_1$ , is the class of  $((A_0 - A(\lambda))\eta_0, r - \varepsilon) : \partial(B^k \times I) \rightarrow \mathbf{R}^{N+1} \setminus \{0\}$ , that is  $\Sigma(A_0 - A(\lambda))\eta_0$ , where  $\eta_0 = (1, 0, \dots, 0)^T$ . For  $N = 1$  the classes for  $\eta_0$  and for  $-\eta_0$  are inverses of each other and non-zero only if  $k = 1$  and  $A(\lambda)$  changes sign as  $\lambda$  crosses 0 ( $A_0 = 0$  and one has degree 1 if  $\lambda A(\lambda) < 0$ ,  $-1$  if  $\lambda A(\lambda) > 0$  and 0 if  $A(\lambda)$  doesn't change sign).

In order to compute the second obstruction, in  $\prod_{k+N-1} (S^N)$ , one will rescale  $\lambda$  so that  $\rho = 1$  and replace  $r$  by  $\mu = (r - \varepsilon)/\varepsilon$  so that  $|\mu| \leq 1$ .

*Proof of (a) when  $N = d$ .* If  $N = d$ , then  $A_0 - A(\lambda) = B(\lambda)$  and  $(B(\lambda)\eta, \mu)$  is deformable, via  $((1-t)B(\lambda) + t\|\lambda\|B(\lambda/\|\lambda\|)\eta, \mu)$ , to  $(\|\lambda\|B(\lambda/\|\lambda\|)\eta, \mu)$  on  $\{\|\tilde{\lambda}\| \equiv \max(\|\lambda\|, |\mu|) = 1\} \times \{\|n\| = 1\}$  (any zero gives  $\mu = 0, \|\lambda\| = 1$ ). Since  $\Sigma B(\lambda)\zeta_0 = 0$ , the pair  $(\|\lambda\|B_d(\lambda/\|\lambda\|), \mu)$  has a non-zero extension  $(B_d(\lambda, \mu), h(\lambda, \mu))$  from the set  $\|\tilde{\lambda}\| \geq 1/2$  to the set  $\|\tilde{\lambda}\| \leq 1$ , where  $B_d(\lambda)$  is the last column of the matrix  $B(\lambda)$ . Now  $(\|\lambda\|B(\lambda/\|\lambda\|)\eta_0, \mu)$  is deformable to  $(\|\lambda\|B_d(\lambda/\|\lambda\|), \mu)$  since  $\eta_0$  is deformable to  $(0, \dots, 0, 1)^T$ . Let  $B(\lambda, \mu)$  be the matrix obtained from  $\|\lambda\|B(\lambda/\|\lambda\|)$  by replacing the last column by  $B_d(\lambda, \mu)$ . Let  $S = \{\tilde{\lambda}/\|\tilde{\lambda}\| \leq 1, h(\lambda, \mu) = 0, \det B(\lambda, \mu) = 0\}$ . For  $\|\tilde{\lambda}\| \geq 1/2$ ,  $h(\lambda, \mu) = \mu$ , so that if  $\mu = 0$ , then  $B(\lambda, \mu)$  is invertible. Let  $\phi : \{\tilde{\lambda}/\|\tilde{\lambda}\| \leq 1\} \rightarrow [0, 1]$  be such that  $\phi(S) = 0$ ,  $\phi(\lambda, \mu) = 1$  if  $\|\tilde{\lambda}\| \geq 1/2$ . Decompose  $\eta$  as  $(\tilde{\eta}, \eta_d)^T$  and define an extension of  $(\|\lambda\|B(\lambda/\|\lambda\|)\eta, \mu)$ , from  $L = S^k \times S^{d-1}$  to  $\bar{K}^{k+d-1} = L \cup B^k \times \{\eta_d = 0\}$ , by  $(B(\lambda, \mu)(\phi(\lambda, \mu)\tilde{\eta}, \eta_d - (1 - \|\tilde{\lambda}\|)/2)^T, h(\lambda, \mu))$ . Note that if this map is zero, then either one has  $\det B(\lambda, \mu) = 0, \phi = 0, \eta_d = (1 - \|\tilde{\lambda}\|)/2$  (since  $B_d(\lambda, \mu) \neq 0$  when  $h(\lambda, \mu) = 0$ ), or  $\det B(\lambda, \mu) \neq 0, \phi\tilde{\eta} = 0, \eta_d = (1 - \|\tilde{\lambda}\|)/2$ . Thus if  $\|\tilde{\lambda}\| = 1$  or  $\eta_d \leq 0$ , one has  $\eta_d = 0, \|\tilde{\lambda}\| = 1, \phi = 1, \|\tilde{\eta}\| = 1$ , leading to a contradiction.

The obstruction is then the class of

$$(B(\lambda, \mu)(\phi\tilde{\eta}, (1 - \|\tilde{\eta}\|^2)^{1/2} - (1 - \|\tilde{\lambda}\|)/2)^T, h(\lambda, \mu))$$

as a map from  $\partial(\|\tilde{\lambda}\| \leq 1, \|\tilde{\eta}\| \leq 1) = S^{k+d-1}$  into  $\mathbf{R}^{d+1} \setminus \{0\}$ . Perform the homotopy  $(1-t)((1 - \|\tilde{\eta}\|)^{1/2} - (1 - \|\tilde{\lambda}\|)/2) + t(2\|\lambda\|/2 - 1)$ ; if  $\|\tilde{\lambda}\| = 1$  and  $h(\lambda, \mu) = \mu = 0$ , then  $\|\lambda\| = 1, B(\lambda, \mu) = B(\lambda)$

is invertible, thus  $\phi = 1, \tilde{\eta} = 0$  and  $(1-t) + t = 1$  at a possible zero; if  $\|\tilde{\eta}\| = 1$  and  $h(\lambda, \mu) = 0$ , then  $\phi\tilde{\eta} = 0$  and  $-(1-t)(1 - \|\tilde{\lambda}\|)/2 + t(2\|\lambda\| - 1) = 0$  at a zero. Thus  $\phi = 0$  and, by construction,  $\|\lambda\| \leq \|\tilde{\lambda}\| < 1/2$ , a contradiction with the vanishing of the last expression.

Replace, next,  $2\|\lambda\| - 1$  by  $(1-t)(2\|\lambda\| - 1) + th(\lambda, \mu)$ ,  $h(\lambda, \mu)$  by  $(1-t)h(\lambda, \mu) - t(2\|\lambda\| - 1)$  and  $\phi(\lambda, \mu)$  by  $\phi(\lambda, \mu, t)$ , where  $\phi(\lambda, \mu, t) = 0$  if  $\det B(\lambda, \mu) = 0$  and  $(1-t)h - t(2\|\lambda\| - 1) = 0$ ,  $\phi(\lambda, \mu, 0) = \phi(\lambda, \mu)$ ,  $\phi(\lambda, \mu, 1) = 1$  and  $\phi(\lambda, \mu, t) = 1$  if  $\|\lambda\| \geq 1/2$ . (If  $t = 1$  and  $2\|\lambda\| - 1 = 0$ , or if  $\|\lambda\| \geq 1/2$ , then  $B(\lambda, \mu) = \|\lambda\| B(\lambda/\|\lambda\|)$  is invertible and  $\phi(\lambda, \mu, t)$  is well defined.) For a zero of the homotopy one would have  $2\|\lambda\| - 1 = 0$ ,  $h(\lambda, \mu) = \mu = 0$ ,  $\phi\tilde{\eta} = \tilde{\eta} = 0$ , which is not possible on the boundary of  $\{\|\tilde{\lambda}\| \leq 1, \|\tilde{\eta}\| \leq 1\}$ . The last homotopy is then  $(1-t)B_d(\lambda, \mu) + t\|\lambda\| B_d(\lambda/\|\lambda\|)$ ,  $(1-t)h(\lambda, \mu) + t\mu$ : for  $2\|\lambda\| - 1 = 0$ , this homotopy is just  $\|\lambda\| B_d(\lambda/\|\lambda\|, \mu)$ . The obstruction is thus the class of the map  $(\|\lambda\| B(\lambda/\|\lambda\|)(\tilde{\eta}, \mu)^T, 1 - 2\|\lambda\|)$ . Since one may use the linear homotopy  $(1-t)(1 - 2\|\lambda\|) + t(2\|\tilde{x}\| - 1)$ , with  $\tilde{x} = (\tilde{\eta}, \mu)$ , (at a zero one has either  $\lambda \neq 0$  and thus  $\tilde{x} = 0, \|\lambda\| = 1$  and the linear homotopy is negative, or  $\lambda = 0$ , thus  $\|\tilde{x}\| = 1$  and the linear homotopy is positive). One gets then the Whitehead homomorphism with the last component,  $2\|\tilde{x}\| - 1$ , with a different sign; its class is  $JB(\lambda)$  (Although a change of orientation in the range does not necessarily give an inverse for the class, in this case one might have chosen to replace  $2\|\lambda\| - 1$  by  $-h(\lambda, \mu)$  and  $h(\lambda, \mu)$  by  $2\|\lambda\| - 1$  in the second homotopy; the obstruction would have been the class of  $(\|\lambda\| B(\lambda/\|\lambda\|)(\tilde{\eta}, -\mu), 1 - 2\|\tilde{x}\|)$ : the change of orientation in the domain ( $\mu$  giving  $-\mu$ ) gives then the inverse. The orientations are given by  $(\lambda, \tilde{\eta}, \mu)$  in this order; this is how one proves  $\Sigma JB(\lambda) = -J\Sigma C(\lambda)$  with these orientations.)

*Proof of (b).* If  $N > d$ , then

$$\begin{aligned} (A_0 - A(\lambda))x &= B(\lambda)PX - (I - Q)A(\lambda)((I - P)x \\ &\quad - (I - KQA(\lambda))^{-1}KQA(\lambda)Px) \\ &\quad \oplus (A_0 - QA(\lambda))((I - P)x - (I - KQA(\lambda))^{-1}KQA(\lambda)Px). \end{aligned}$$

By replacing  $\lambda$  by  $(1-t)\lambda$  (except in  $B(\lambda)$ ) one may deform this map (for  $\|\lambda\| = \rho$ ), or this map with the side condition  $r - \varepsilon$  on  $\partial(B^k \times I)$ ,

to  $B(\lambda)Px \oplus A_0(I - P)x$ . Since  $A_0(I - P)$  can be deformed to  $(\text{sign det } A_0, I)$  (once bases have been chosen), one has that  $[A_0 - A(\lambda)\eta_0] = 0$ . If  $k < 2N - 2$  and  $\Sigma(A_0 - A(\lambda))\eta_0 = 0$ , then  $\Sigma$  is an isomorphism and again  $(A_0 - A(\lambda))\eta_0$  is deformable to a constant map. One could then proceed as before with  $(B(\lambda)P\eta, (\text{sign det } A_0, I)(I - P)\eta)$ , however it is simpler to note that, in both cases,  $B(\lambda)$  is deformable to  $\begin{pmatrix} C(\lambda) & 0 \\ 0 & 1 \end{pmatrix}$ .

One then gets an extension to  $\overline{K}^{N+k-1}$  given by  $(\|\lambda\| C(\lambda)/\|\lambda\| \|\tilde{\eta}\|, \eta_d - \varepsilon(1 - \|\tilde{\lambda}\|), \mu)$ , which will give, as in §3.2, the obstruction as the class of  $(\|\lambda\| C(\lambda)/\|\lambda\| \|\tilde{\eta}\|, 1 - 2\|\tilde{\eta}\|, \mu)$ . A rotation in the last two components will give  $(\|\lambda\| C(\lambda)/\|\lambda\| \|\tilde{\eta}\|, \mu, 2\|\tilde{\eta}\| - 1)$ , and a linear deformation will replace  $2\|\tilde{\eta}\| - 1$  by  $1 - 2\|\lambda\|$ .

Note that  $(\text{sign det } A_0, I)(I - P)x$  acts as a suspension, so that  $\varepsilon = \text{sign det } A_0$ . Thus one has a proof of the remaining cases for Part (a). If  $N = d, k \leq 2N - 2$ , then, since  $HJ(B(\lambda)) = \Sigma^N B(\lambda)\eta_0$  and  $\Sigma^{N-1}$  is an isomorphism, the vanishing of  $J(B(\lambda))$  then implies the vanishing of  $\Sigma B(\lambda)\eta_0$ , as in Theorem 3.3.

REMARK 4.2. The shorter argument for part (b) does not extend to the case where  $N = d$  and  $k \geq 2N - 2$ . One could have expected that the vanishing of  $\Sigma B(\lambda)\eta_0$  would have implied that  $\begin{pmatrix} B(\lambda) & 0 \\ 0 & 1 \end{pmatrix}$  could be deformed, on  $\|\lambda\| = 1$ , to

$$\begin{pmatrix} C(\lambda) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

using Stiefel Manifolds.

However, if  $k = 4, d = 3$ , and  $B(\lambda) = \rho^2$ , as in Remark 3.1, then  $\Sigma P^* \rho^2 = 0$  in  $\Pi_4(S^3)$ . And  $i_* \rho^2 = 2$  in  $\Pi_3(SO(4)) = \Pi_3(S^3) \oplus \Pi_3(SO(3))$  and  $i_* \rho^2$  cannot come from an element  $C(\lambda)$  in  $\Pi_3(SO(2)) = \Pi_3(S^1) = 0$ . Conversely, if  $\lambda = \mu + i\nu$ , then  $\begin{pmatrix} \lambda^2 & 0 \\ 0 & 1 \end{pmatrix}$  is deformable to  $I$  in  $\Pi_1(SO(3)) = Z_2$ , while  $\Sigma P_*(\lambda^2)$  has degree 2 in  $\Pi_2(S^2)$ .

REMARK 4.3. If  $N = 3$  or  $7$ , then  $\Sigma^{N-1}$  is one to one for  $k \leq 2N - 1$ , thus in this case,  $J(B(\lambda))$  is the only invariant. If  $N = 1, J(B(\lambda)) =$

$2\Sigma B(\lambda)\eta_0 (= 0 \text{ if } k > 1)$ ; if  $n = 2$ ,  $B(\lambda)$  is deformable to a constant when  $k > 2$ ,  $P_*$ ,  $J$ ,  $\Sigma$  are isomorphisms for  $k = 2$  and  $J$  is one to one for  $k = 1$ , within the range of Bott's and Adams' results.

REMARK 4.4. If  $\Sigma(A_0 - A(\lambda))\eta_0 \neq 0$ , then one has global bifurcation in the direction  $\eta_0$ , that is, the equation

$$(A_0 - A(\lambda))t\eta_0 - g(\lambda, t\eta_0) = 0$$

has a continuum of non-trivial ( $t \neq 0$ ) solutions branching from  $(0, 0)$  and going to another singular point of  $A_0 - A(\lambda)$  or to infinity; the suspension is the class of  $((A_0 - A(\lambda))t\eta_0, t - \varepsilon)$  on the set  $\partial(|\lambda| \leq \rho, |t| \leq 2\varepsilon)$ . Since the direction  $\eta_0$  can be deformed to any direction, one has global bifurcation in all directions of  $\mathbf{R}^N$ . One may give more precise information on this continuum, by

PROPOSITION 4.1. *If  $\Sigma^\ell(A_0 - A(\lambda))\eta_0 \neq 0$ , for some  $\ell, 1 \leq \ell \leq N$ , then the continuum has a connected subset  $\Sigma$ , bifurcating globally from  $(0, 0)$ , such that the local covering dimension at each point of  $\Sigma$  is at least  $\ell$ .*

PROOF. Consider the map

$$\begin{aligned} & (A_0 - A(\lambda))(r\eta_0, r\mu_2, \dots, r\mu_\ell, 0, \dots, 0)^T \\ & \quad - g(\lambda, r\eta_0, r\mu_2, \dots, r\mu_\ell, 0, \dots, 0) \\ & \equiv g_1(\lambda, r, \mu_2, \dots, \mu_\ell), \end{aligned}$$

where  $r, \mu_2, \dots, \mu_\ell$  are arbitrary parameters. Then  $g_1(\lambda, 0, \mu_2, \dots, \mu_\ell) = 0$  and  $g_1$  satisfies the hypotheses of Theorem 4.2 in [9]. Furthermore, on  $\partial(|\lambda| \leq \rho, |r| \leq 2\varepsilon, |\mu_i| \leq \rho)$ , the map  $(g_1(\lambda, r, \mu_2, \dots, \mu_\ell), \mu_2, \dots, \mu_\ell, r - \varepsilon)$  is deformable to  $(A_0 - A(\lambda))(\eta_0, 0, \dots, 0)^T, \mu_2, \dots, \mu_\ell, r - \varepsilon)$  which is just  $\Sigma^\ell(A_0 - A(\lambda))\eta_0$ . Thus the above map is 0-epi on the ball and one may apply Theorem 4.2 in [9]: there one has  $|r| - \varepsilon$  instead of  $r - \varepsilon$ , but it is easy to check that the argument goes through. Note that one could have chosen the type of solutions differently, and that one may have used the continuation result in [9] for the map  $(f(\lambda, x_1, \dots, x_\ell, 0, \dots, 0), x_1 - \varepsilon_1, \dots, x_\ell - \varepsilon_\ell)$ .

REMARK 4.5. It is clear that the same argument works for  $B(\lambda)$  instead of  $A_0 - A(\lambda)$ : If  $\Sigma^\ell B(\lambda)\eta_0 \neq 0$  one will get locally (i.e., as long as the Liapunov-Schmidt reduction is valid) an  $\ell$ -dimensional "surface". The following example illustrates this situation. Let  $A_0 - A(\lambda) = \begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix}$ , where  $\lambda$  is in  $\mathbb{C}$ ,  $N = 4$ ,  $d = 2$ . As in [5] it is easy to see that  $B(\lambda) = \lambda^2$ , so that  $B(\lambda)\eta_0 = \lambda^2\eta_0$  has degree 2, as well as  $\Sigma^2 B(\lambda)\eta_0$  (so that  $\ell = 2$ ).  $JB(\lambda)$  is twice the Hopf map, whose suspension (and thus  $J(A_0 - A(\lambda))$ ) is 0. Thus the invariants for the local bifurcation are non-zero, while both invariants for global bifurcation are trivial. Consider then the equations

$$\begin{aligned}\lambda z_1 - \phi(r^2)z_2 + r^2\bar{z}_2 &= 0 \\ \lambda z_2 - r^2\bar{z}_1 &= 0,\end{aligned}$$

where  $r^2 = |z_1|^2 + |z_2|^2$ ,  $\phi(t)$  is a non-increasing smooth function with  $\phi(t) = 1$  for  $t \leq 1/2$ ,  $\phi(t) = 0$  for  $t \geq 1$ ,  $\phi(t)$  is concave for  $t \leq t_0$ ,  $t_0 = \phi(t_0)$ . Multiplying the first equation by  $z_2$  and the second by  $z_1$ , one obtains  $r^4 = \phi(r^2)z_2^2$ , thus  $z_2$  is real;  $z_2(\phi(r^2) - r^2) = \bar{\lambda}z_1$ ,  $|\lambda|^2 = r^2(\phi(r^2) - r^2)$ , (if  $z_1 = 0$ , then  $z_2 = 0$ ). Thus any solution has  $r^2 \leq t_0$ ,  $\rho \leq 1/2$  (the function  $t(\phi(t) - t)$  has a unique maximum at  $t = 1/2$ ). Thus,  $|\lambda|$ ,  $|z_1|$ ,  $z_2$  are given in terms of  $r$  and if  $\lambda = |\lambda| e^{i\phi}$  is a solution, then  $z_1 = (\text{sign } z_2) |z_1| e^{-i\phi}$ . Thus the continuum bifurcating from  $(0, 0)$  is two dimensional and there is no global bifurcation. Note that, on the solutions,  $\partial(z_2(\phi(r^2) - r^2) - \bar{\lambda}z_1)/\partial z_2 = \phi - r^2 + 2r^4(\phi' - 1)/\phi$  is zero at  $r^2 = 1/2$ : there the Liapunov-Schmidt reduction fails.

REMARK 4.6. If  $\Sigma^\ell JB(\lambda) \neq 0$  for some  $\ell$ ,  $0 \leq \ell \leq N - d$ , then one has a local continuum branching from  $(0, 0)$ . Also, since  $J\Sigma^\ell B(\lambda) = (B(\lambda)x, \mu_1, \dots, \mu_\ell, \|x\| - \varepsilon)$  is non-trivial, one sees that any small perturbation of  $(B(\lambda)x, \mu_1, \dots, \mu_\ell)$ , in the variables  $\lambda, x, \mu_1, \dots, \mu_\ell$ , will have a global bifurcation. This could be used in cases where the reduction to  $B(\lambda)$  could be done in two steps with a larger range of applicability on the variables different from  $x, \mu_1, \dots, \mu_\ell$ . Note that the invariants  $B(\lambda)\eta_0$  and  $JB(\lambda)$  are linked by  $HJB(\lambda) = \Sigma^d B(\lambda)\eta_0$  and other properties for the suspensions: for example if  $d$  is odd and  $k \leq 3d - 1$ , then, if  $\Sigma JB(\lambda) = 0$ , then  $JB(\lambda)$  is a suspension [11; Theorem 10.6] and thus  $HJB(\lambda) = 0$ . However the usefulness of such

relations for the bifurcation problem is not clear.

The last result in this section will relate the local invariants on a bounded continuum. Let  $\mathbf{C}$  be the continuum of non-trivial zeros of  $f(\lambda, x) = (A_0 - A(\lambda))x - g(\lambda, x)$ , branching from  $(0, 0)$ , and  $\mathbf{C}(\eta_0)$  be the continuum for  $f(\lambda, t\eta_0), t > 0$ . Assume all bifurcation points,  $(\lambda_i, 0)$ , on  $\mathbf{C}$  or  $\mathbf{C}(\eta_0)$  have the property that  $A - A(\lambda)$  is invertible for  $\lambda$  close to  $\lambda_i, \lambda \neq \lambda_i$ . Thus, if  $\mathbf{C}$  is bounded, there is only a finite number of such bifurcation points.

**PROPOSITION 4.2.** *Assume  $\mathbf{C}$  or  $\mathbf{C}(\eta_0)$  is bounded, with the above hypothesis. Then*

$$\Sigma_I J(A_0 - A(\lambda))_i = 0, \text{ for } (\lambda_i, 0) \text{ on } \mathbf{C},$$

and

$$\Sigma_I \Sigma(A_0 - A(\lambda))_i = 0, \text{ for } (\lambda_i, 0) \text{ on } \mathbf{C}(\eta_0).$$

**PROOF.** Since the arguments have already been used in other contexts, the proof will be sketchy. Use first the fact that  $\mathbf{C}$ , (or  $\mathbf{C}(\eta_0)$ ) is a continuum to construct a bounded open set  $\Omega$  such that  $\mathbf{C}$  is contained in  $\Omega$  and  $f(\lambda, x)$  (or  $f(\lambda, t\eta_0)$ ) is non-zero on  $\partial\Omega$ , if  $x \neq 0$  ( $t \neq 0$ ). Complementing  $f(\lambda, x)$  by  $\|x\| - \varepsilon$  (or  $t - \varepsilon$ ), one gets a map which is inessential on  $\partial\Omega$  with respect to  $\overline{B} \setminus \Omega$ , where  $B = \{(\lambda, x) / \|\lambda\| < R_0, \|x\| < R\}$  contains  $\Omega$  (or  $B = \{(\lambda, t), \|\lambda\| < R_0, 0 < t < R\}$ ) and a non-zero extension to  $\overline{B} \setminus \cup_I B_i$  of this map (take the map to be  $(0, -\varepsilon)$  on  $\partial B$ ) where  $B_i = \{(\lambda, x) / \|\lambda - \lambda_i\| < \rho_i, \|x\| \leq 2\varepsilon \text{ (or } 0 < t < 2\varepsilon_i)\}$ . The result follows from the fact that one is dealing with primary obstructions for extensions from  $\partial(\overline{B} \setminus \cup_I B_i)$  to  $\overline{B} \setminus \cup_I B_i$  and some operations on cohomology groups which are given in [7, p. 159] and in [8, p. 789]: one has that the sum in  $\prod_{k+N-1} (S^N)$  (or in  $\prod_k (S^N)$ ) of the local obstructions has to be zero. Note that since this result belongs to the "sufficient" category one does not need to work with the set  $\{(\lambda, r, \eta) / \|\lambda\| < R_0, 0 < r < R, \|\eta\| = 1\}$  for which the obstructions (probably both of the above invariants) are more difficult to compute.

**REMARK 4.7.** One has to be careful with orientations when using Proposition 4.2 and relating  $J(A_0 - A(\lambda))$  to  $JB(\lambda)$ . Here is an easy

way to do it: observe first that, once the orientations have been chosen, the invariants for  $B(\lambda)$  are independent of the choices for  $P$  and  $Q$  (as in the proof of [5; Theorem 4.5, p. 33]). Next,  $(A_0 - A(\lambda))x$  is deformable to  $A_0(I - P)x \oplus B(\lambda)x$ , for any  $\lambda$  close to 0; thus the orientations have to be chosen such that  $\det A_0(I - P) \det B(\lambda)$  has the sign of  $\det(A_0 - A(\lambda))$  at each  $\lambda$  close to  $\lambda_0$  (in fact, if  $k > 1$ , this sign is invariant around 0; if  $k = 1$  one may have a change corresponding to the generalized algebraic multiplicity). In order to relate the signs of  $\det(A_0 - A(\lambda))$  near two distinct bifurcation points, one takes a path joining the two points and studies the set of points where  $A_0 - A(\lambda)$  is singular along the path. Note that the sign of  $\det(A_0 - A(\lambda))$  is the index of the zero solution at  $\lambda$ . Then  $J(A_0 - A(\lambda)) = \text{sign } \det A_0(I - P)JB(\lambda)$ , after choosing the orientation of the domain with  $((I - P)x, Px)$  and of the range with  $((I - Q)y, Qy)$  (there is then no change of sign when suspending).

#### APPENDIX: THE COHOMOTOPY APPROACH

One might have studied the extension problem in the more general (although related) framework of cohomotopy theory (see [4]). That is, since  $B(\lambda)x - g(\lambda, x)$  is non-zero on  $S^{k-1} \times S^{d-1} = \{\lambda/ \|\lambda\| = \rho\} \times \{x/ \|x\| = 2\varepsilon\}$  one has an element in  $\prod^{d-1}(S^{k-1} \times S^{d-1})$  (the set of all homotopy classes of maps from  $S^{k-1} \times S^{d-1}$  into  $\mathbf{R}^d \setminus \{0\}$ ). The extension problem to  $B^k \times S^{d-1}$  is then equivalent to the fact that  $[B(\lambda)x]$  is in the image of  $i^*$ ,

$$\prod^{d-1}(B^k \times S^{d-1}) \xrightarrow{i^*} \prod^{d-1}(S^{k-1} \times S^{d-1}),$$

where  $i^*$  is induced by restriction of the maps to  $S^{k-1} \times S^{d-1}$ . One has the sequence

$$j^* \prod^{d-1}(B^k \times S^{d-1}) \xrightarrow{i^*} \prod^{d-1}(S^{k-1} \times S^{d-1}) \xrightarrow{\delta} \prod^d(B^k \times S^{d-1}, S^{k-1} \times S^{d-1}) \xrightarrow{j^*}$$

with  $\text{Im } j^* = \ker i^*$ ,  $\text{Im } \delta^* \subset \ker j^*$ ,  $\text{Im } i^* \subset \ker \delta^*$ , [4]. The following facts are easily verified:

- (a)  $\text{Im } i^* = \ker \delta$ , if  $k < d$ , in which case these sets are groups.

(b)  $\prod^{d-1}(B^k \times S^{d-1}) \simeq Z$ , since  $B^k$  is contractible.

(c) If  $F(\lambda, x) : S^{k-1} \times S^{d-1} \rightarrow \mathbf{R}^d \setminus \{0\}$ , then  $\delta F(\lambda, x) = (\|\lambda\| F(\lambda/\|\lambda\|, x), 1 - 2\|\lambda\|^2)$  maps  $(B^k \times S^{d+1}, S^{k-1} \times S^{d-1})$  into  $(\mathbf{R}^{d+1} \setminus \{0\}, \mathbf{R}^{d+1} \setminus \mathbf{R}^+)$ , that is, the last component is negative for  $\|\lambda\| = \rho = 1$ .

(d) By excision of  $S^{k-1} \times B^d$ , one has an isomorphism  $\prod^d(B^k \times S^{d-1}, S^{k-1} \times S^{d-1}) \simeq \prod^d(S^{k+d-1}, S^{k-1}, \times \overline{B}^d)$

(e) the map  $\prod^d(S^{k+d-1}, S^{k-1} \times B^d) \xrightarrow{j^*} \prod^d(S^{k+d-1}) \simeq \prod_{k+d-1}(S^d)$  is an isomorphism for  $k < d$ .

(f)  $j^*\delta(B(\lambda)x) = JB(\lambda)$  is the Whitehead homomorphism; thus if  $JB(\lambda) \neq 0$ , then  $\delta(B(\lambda)x) \neq 0$  and  $B(\lambda)x \notin \text{Im } i^*$ .

(g) the following diagram commutes

$$\begin{array}{ccc} \prod^{d-1}(S^{k-1} \times S^{d-1}) & \xrightarrow{\delta} & \prod^d(B^k \times S^{d-1}, S^{k-1} \times S^{d-1}) \\ \downarrow P^* & & \downarrow P^* \\ \prod^{d-1}(S^{k-1}) & \xrightarrow{\Sigma} & \prod^d(B^k, S^{k-1}) \simeq \prod^d(S^k), \end{array}$$

where  $P^*F(\lambda, x) = F(\lambda, x_0)$ , i.e.,  $P^*\delta B(\lambda)x = \Sigma B(\lambda)x_0$  [4, p. 227]. One gets thus a relation between the two invariants, similar to the  $H$ -homomorphism.

(Note that in the example of Remark 3.1,  $J(\rho^{12}) = 0$ , thus  $j^*\delta(\rho^{12}x) = 0$  and  $\rho^{12}x$  does not belong to  $\text{Im } i^*$  since  $P_*\rho^{12} \neq 0$ . However it is not clear if  $\delta(\rho^{12}x)$  is 0 or not: this would give an example where the sequence would not be exact.)

However the non-exactness of the sequence, and the fact that  $j^*$  is not one to one, limit the application of this approach to the case where  $k < d$ , in which case  $P^*B(\lambda) = 0$ ,  $B(\lambda)$  is a suspension, one is in the stable range and the only invariant is  $JB(\lambda)$ . Thus, although the cohomotopy approach may seem to be more "natural", for the problem at hand the treatment with obstruction is simpler. Probably for more complicated sets or functions (for example the general Hopf construction) one could get some information from cohomotopy.

**Acknowledgment.** The research for this paper was primarily done while on sabbatical leaves at the Universities of Nice and Utah. Comments from members of both institutions were useful. I wish also to dedicate this paper to the memory of my colleague and friend D. Alcaraz, 1950 - September 19, 1985.

## REFERENCES

1. J.C. Alexander, *Bifurcation of zeros of parametrized functions*, J. Functional Analysis, **29** (1978), 37-53.
2. ——— and J.A. Yorke, *Calculating bifurcation invariants as elements in the homotopy of the general linear group*, J. Pure Appl. Algebra **13** (1978), 1-8.
3. E.N. Dancer, *On the existence of zeros of perturbed operators*, Nonlinear Anal. Theory, Methods and Appl. **7** (1983), 717-727.
4. S.T. Hu, *Homotopy Theory*, Academic Press, New York, 1959.
5. J. Ize, *Bifurcation Theory for Fredholm Operators*, Mem. Amer. Math. Soc. **7** No. 174, (1976).
6. ———, *Le Problème de bifurcation de Hopf*, Sem. Brezis-Lions, (1975).
7. ———, *Introduction to bifurcation theory*, in *Differential Equations*, G.G. de Figuereido and C.S. Honig, Ed. Lecture Notes in Math., Vol. **957**, Springer Verlag, Berlin and New York, (1982), 145-203.
8. ———, *Obstruction theory and multiparameter Hopf bifurcation*, Trans. Amer. Math. Soc. **289** (2) (1985), 757-792.
9. ———, I. Massabo, J. Pejsachowicz and A. Vignoli, *Structure and Dimension of global branches of solutions to multiparameter nonlinear equations*, Trans. Amer. Math. Soc. **291** (2) (1985), 383-435.
10. ———, ———, and A. Vignoli, *Global results on continuation and bifurcation for equivariant maps*, to appear in *Nonlinear Functional Analysis and its Applications*, S.P. Singh Ed., D. Reidel Pub.
11. I.M. James, *On the suspension sequence*, Annals of Math. **65** (1957), 74-107.
12. N. Steenrod, *Topology of fiber bundles*, Princeton Univ. Press, Princeton, N.J., 1965.
13. G.W. Whitehead, *On the homotopy groups of spheres and rotation groups*, Annals of Math. **43** (1942), 634-640.
14. ———, *A generalization of the Hopf invariant*, Annals of Math. **51** (1950), 192-237.
15. ———, *Elements of homotopy theory*, Graduate texts in Math. **61** Springer Verlag, Berlin and New York, 1978.

IIMAS-UNAM, AP 20-726, MEXICO 20 D.F., MEXICO,  
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UT  
84112

