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COMPLETELY MONOTONIC FUNCTIONS OF THE FORM $s^{-b}(s^2+1)^{-a}$

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ABSTRACT. The function $s^{-b}(s^2+1)^{-a}$ is shown to be completely monotonic for $b \geq 2a \geq 0$, for $b \geq a \geq 1$. or for $0 \leq a \leq 1$, $b \geq 1$. Moreover this function is proven not to be completely monotonic for $0 \leq b < a$, nor for $a = b$, $0 < a < 1$. This proves some conjectures of Askey [**1**], and extends some of the results of [**2**], [**3**], and [**4**].

1. Introduction. In recent years Askey, Gasper, Ismail, and others have looked into the problem of determining the nonnegativity of the Bessel function integrals $\int_0^t (t-s)^c s^d J_\nu(s)ds$, as well as some ${}_1F_2's$. See [**2,3**]. This is related to the complete monotonicity of $s^{-a}(s^2+1)^{-b}$ as we shall see in this article.

The definition of complete monotonicity used in this paper is:

DEFINITION. A function $f(s)$ is completely monotonic (C.M.) if

$$(-)^n f^{(n)}(s) \geq 0, s > 0, n = 0, 1, 2, \cdots.$$

The main result we will need is the Hausdorff–Bernstein–Widder theorem [**8**].

THEOREM A. *$f(s)$ is completely monotonic if and only if it is the Laplace Transform of a positive measure on* $(0, \infty)$.

Accordingly, we will make the following definitions.

DEFINITION. Let $\mathcal{L}$ denote the Laplace transform operator and let $\mathcal{L}^{-1}$ denote its inverse. We define:

$$S_{a,b}(t) = \mathcal{L}^{-1}(s^{-a}(s^2+1)^{-b}). \tag{1.1}$$

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$$f * g(t) = \int_0^t f(s)g(t-s)ds \tag{1.2}$$

$$ {}_1F_2(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n (c)_n n!}, \tag{1.3}$$

where $(a)_0 = 1$, and $(a)_n = a(a+1)(a+2)\cdots(a+n-1), n \geq 1$.
The other Series ${}_2F_1, {}_3F_2$, etc., are defined similarly.

Using the elementary theory of Laplace transforms, these results follow immediately:

$$S_{a,b}(t) = \frac{t^{2a+b-1}}{\Gamma(2a+b)} {}_1F_2(a, a+b/2, a+b/2+1/2; -t^2/4). \tag{1.4}$$

$$S_{a,b}(t) = \frac{\pi^{\frac{1}{2}} t^{a+b-1/2}}{\Gamma(a)\Gamma(b)2^{a-\frac{1}{2}}} \int_0^1 (1-u)^{b-1} u^{a-\frac{1}{2}} J_{a-\frac{1}{2}}(tu)du. \tag{1.5}$$

$$S_{0,b}(t) = \frac{t^{b-1}}{\Gamma(b)}. \tag{1.6}$$

$$S_{a,0}(t) = \frac{\pi^{\frac{1}{2}} t^{a-\frac{1}{2}}}{\Gamma(a)2^{a-\frac{1}{2}}} J_{a-\frac{1}{2}}(t) \tag{1.7}$$

$$S_{a,b}(t) * S_{c,d}(t) = S_{a+c,b+d}(t). \tag{1.8}$$

The problem is to determine the set of all (a,b) such that $S_{a,b}(t)$ is nonnegative on $(0,\infty)$. One result that follows immediately from the above equations is:

LEMMA 1. *If $s^{-b}(s^2+1)^{-a}$ is C.M. then $s^{-c}(s^2+1)^{-a}$ is C.M. for all $c > b$.*

2. The positive results.

There is a useful sum due to George Gasper [**5**, (3.1)],

THEOREM B.

$$\begin{aligned} &{}_1F_2(a, a+b/2, a+b/2+1/2; -x^2y) = \\ &\Gamma^2(\nu+1)\left(\frac{2}{x}\right)^{2\nu} \sum_{n=0}^{\infty} \left(\frac{(2\nu+1)_n (2n+2\nu)}{n!(n+2\nu)} J^2_{n+\nu}(x) \right. \end{aligned} \tag{2.1}$$

$$ {}_4F_3\left(\begin{matrix}-n,\ n+2\nu,\ \nu+1,\ a\\ \nu+1/2,\ a+b/2,a+b/2+1/2\end{matrix};y\right)\Bigg), \nu \geq 0. $$

One result that can be obtained from Theorem B is:

THEOREM 1.

$$ s^{-b}(s^2+1)^{-a} \text{ is C.M. for } b=1 \text{ and } 0 \leq a \leq 1. $$

PROOF. Using (1.4) and (2.1) with $b=1, \nu = a/2$, and $y=1$, it suffices to show that:

(2.2)

$$ {}_4F_3\left(\begin{matrix}-n, n+a, a/2+1, a\\ a/2+1/2, a+1/2, a+1\end{matrix};1\right) \geq 0, 0 \leq a \leq 1,\ n=0,1,2,\cdots . $$

Now we use a result of Bailey [7, (4.3.5.1)]: ${}_4F_3\left(\begin{smallmatrix}x,y,z,-n\\ u,v,w\end{smallmatrix};1\right) = \frac{(v-z)_n(w-z)_n}{(v)_n(w)_n}{}_4F_3\left(\begin{smallmatrix}u-x,u-y,z,-n\\ 1-v+z-n,1-w+z-n,w\end{smallmatrix};1\right)$, provided $u+v+w=1+x+y+z-n$. Set $x=n+a, y=a/2+1, z=a, v=a/2+1/2, w=a+1/2$, and $u=a+1$, and the ${}_4F_3$ becomes

$$ \frac{(1/2-a/2)_n(1/2)_n}{(a/2+1/2)_n(a+1/2)_n}{}_4F_3\left(\begin{matrix}1-n,\ a/2,\ a,\ -n\\ 1/2+a/2-n, 1/2-n, a+1/2\end{matrix};1\right). $$

For $0 \leq a \leq 1$, the terms of the ${}_4F_3$ series are positive which implies that (2.2) holds. The theorem is proved.

One result proved by Fields and M. Ismail [3], is:

THEOREM C.

$$ s^{-b}(s^2+1)^{-a} \text{ is C.M. for } b \geq a \geq 1. $$

Also, Askey and Pollard [2] proved that $s^{-b}(s^2+1)^{-a}$ is C.M. for $b \geq 2a$ using a theorem of Schoenberg:

THEOREM D. *Let $f(s)$ be a continuous function defined on $[0,\infty)$ such that $f(0)=1$. Then $(f(s))^\lambda$ is C.M. for all $\lambda > 0$ if and only if there is a completely monotonic function $g(t)$ such that:*

$$ f(s) = \exp\left(-\int_0^s g(t)dt\right). \tag{2.3} $$

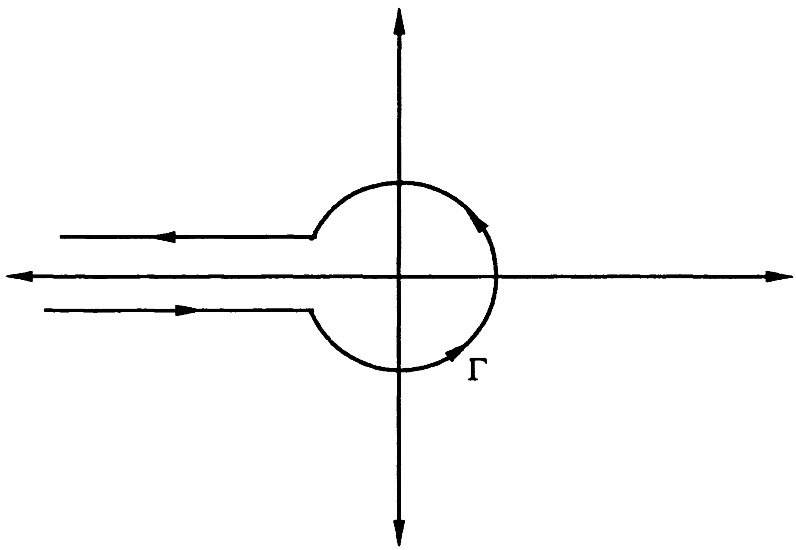

Figure 1.

Now all the positive results have been established.

3. The negative results. The main tool we will use is the following:

THEOREM E. (Watson's lemma for loop contours) *Let f be analytic in an open neighborhood, U, of $(-\infty, 0]$ except for a branch cut at $(-\infty, 0]$. Suppose that*

$$f(s) \sim \sum_{n=0}^{\infty} a_n s^{n-a}, \text{ as } s \to 0, \tag{3.1}$$

in a neighborhood of 0, *and let* Γ *be the loop that starts at* $-\infty$, *goes around the origin then goes back to* $-\infty$ *as depicted in* Fig. 1. *Then,*

$$\frac{1}{2\pi i}\int_\Gamma e^{st} f(s)ds \sim \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(a-n)} t^{a-n-1}, \tag{3.2}$$

$$\text{as } |t| \to \infty, |\arg(t)| \le \pi/2 - \varepsilon, \varepsilon > 0.$$

A proof of this theorem can be found in Olver [**6**]. One important consequence of this theorem is

THEOREM 2.

(3.3)

$$S_{a,b}(t) \sim \frac{2^{1-a}}{\Gamma(a)}\Big(\cos(t-\pi a/2-\pi b/2)\sum_{n=0}^{\infty}\frac{(-)^n(a)_{2n}(1-a)_{2n}}{2^{2n}(2n)!}$$
$$\cdot {}_2F_1(-2n,b,1-a-2n;2)t^{a-2n-1}+\sin(t-\pi a/2-\pi b/2)$$
$$\cdot\sum_{n=0}^{\infty}\frac{(-)^n(a)_{2n+1}(1-a)_{2n+1}}{2^{2n+1}(2n+1)!}\,{}_3F_1(-2n-1,b,-a-2n;2)t^{a-2n-2}\Big)+$$
$$\frac{1}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n(1-b)_{2n}(-)^n}{n!}t^{b-2n-1},\ \text{as } |t|\to\infty, |\arg(t)|\le\pi/2-\varepsilon, \varepsilon>0.$$

PROOF. We use the inversion formula for the Laplace transform:

$$S_{a,b}(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}s^{-b}(s+i)^{-a}(s-i)^{-a}ds, c>0. \tag{3.4}$$

We will assume throughout that the principal values of the powers and logs will be taken. The contour can be deformed into $\Gamma\cup(\Gamma+i)\cup(\Gamma-i)$. Translating the integrals over $\Gamma+i$ and $\Gamma-i$ to integrals over Γ, we obtain

$$\begin{aligned} S_{a,b}(t) &= \frac{1}{2\pi i}\int_\Gamma e^{st}s^{-b}(s^2+1)^{-a}ds+ \\ &2\mathrm{Re}\Big(\frac{1}{2\pi i}\int_\Gamma e^{st+it}(s+i)^{-b}s^{-a}(s+2i)^{-a}ds\Big). \end{aligned} \tag{3.5}$$

We now use:

$$(s^2+1)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n(-)^n}{n!}s^{2n}, |s|<1, \text{ and} \tag{3.6}$$

$$\begin{aligned}(s+i)^{-b}(s+2i)^{-a} &= e^{-\pi(a+b)i/2}2^{-a} \\ &\cdot\sum_{n=0}^{\infty}\frac{(b)_n}{n!}(is)^n\sum_{n=0}^{\infty}\frac{(a)_n}{n!}(\frac{is}{2})^n, |s|<1.\end{aligned} \tag{3.7}$$

$$\begin{aligned}(s+i)^{-b}(s+2i)^{-a} &= e^{-\pi(a+b)i/2}2^{-a} \\ &\cdot\sum_{n=0}^{\infty}\frac{(a)_n}{n!2^n}{}_2F_1(-n,b,1-a-n;2)i^ns^n, |s|<1.\end{aligned} \tag{3.8}$$

Now use Theorem E on each integral in (3.5) with the series expansions obtained in (3.6) and (3.8) and the result follows.

Theorem 2 has some interesting consequences, among them being:

COROLLARY 1. *$s^{-b}(s^2+1)^{-a}$ is not C.M. for $0 < b < a$.*

PROOF. It is evident from (3.3) that

$$S_{a,b}(t) \sim 2^{1-a} t^{a-1} \cos(t - \pi b/2 - \pi a/2)/\Gamma(a), \text{ as } t \to \infty, \tag{3.9}$$

i.e., the ratio of the two sides goes to one as $t \to \infty$. There are arbitrarily large values of t where the right side of (3.9) is negative, so the same is true for $S_{a,b}(t)$. Hence $S^{-b}(s^2+1)^{-a}$ is not C.M. for $0 < b < a$.

Another consequence of Theorem 2 is,

COROLLARY 2. *$s^{-b}(s^2+1)^{-a}$ is not C.M. for $a = b, 0 < a < 1$.*

PROOF. The two dominant terms of (3.2) yield:

$$S_{a,a}(t) \sim [2^{1-a} \cos(t - \pi a) + 1] t^{a-1}/\Gamma(a). \tag{3.10}$$

For $0 < a < 1$, $2^{1-a} > 1$ which implies that there are arbitrarily large values of t for which the right side of (3.10) is negative. Hence $S_{a,a}(t)$ must be negative somewhere. So $s^{-a}(s^2+1)^{-a}$ is not C.M. for $0 < a < 1$.

4. Conclusion. At this point we know where $s^{-b}(s^2+1)^{-a}$ is or is not C.M. in the first quadrant of the (a, b) plane, except in the interior of the triangle with vertices $(0,0), (1,1)$, and $(1/2, 1)$. In that triangle there is a boundary curve of complete monotonicity, where $s^{-b}(s^2+1)^{-a}$ is C.M. on or above it, but not C.M. below it. There the numerical evidence suggests that this curve increases monotonically from $(0,0)$ to $(1,1)$ in a concave down fashion with a slope of 2 at $(0,0)$. It remains an open challenge to determine this curve more explicitly.

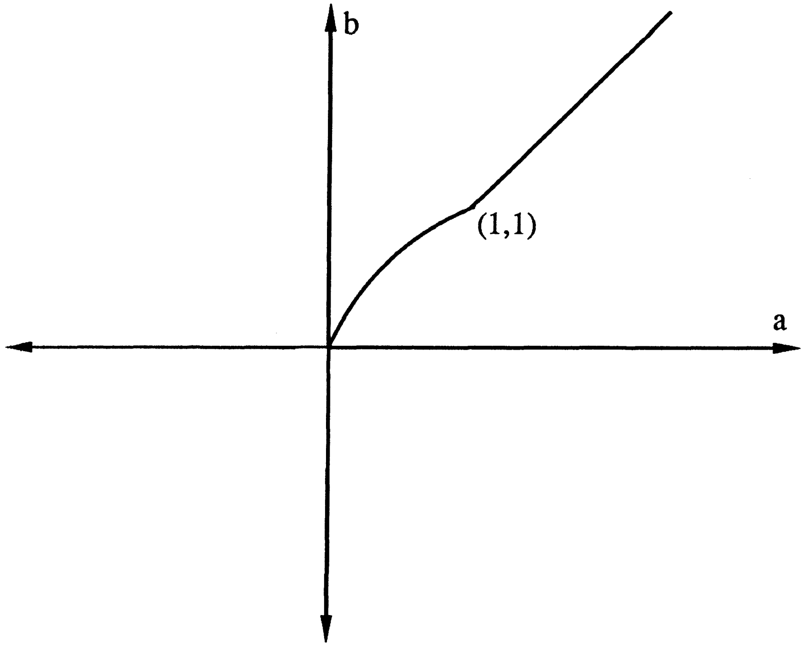

Figure 2.

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