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# STRONGLY EXTREME POINTS IN $L^p(\mu, X)$

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ABSTRACT. A natural characterization of strongly extreme points in the unit ball of $L^p(\mu, X)$, where $1 < p < \infty$, is given. This characterization is compared to similar known results concerning strongly exposed points and extreme points.

Sundaresan [8] and Johnson [6] considered the problem of characterizing extreme points in the unit ball of $L^p(\mu, X)$, where $X$ is a Banach space, $(S, \Sigma, \mu)$ is a measure space and $1 < p < \infty$. It is easily shown that sufficient conditions for $f$ to be such a point are that $\|f\|_p = 1$ and, for almost all $s$ in the support of $f$, the element $f(s)/\|f(s)\|$ is an extreme point of the unit ball of $X$. In [8] it is shown that these conditions are also necessary in the case that $X$ is a separable conjugate space, $S$ is a locally compact Hausdorff space and $\mu$ is a regular Borel measure; in [6] the same is shown in the case that $X$ is any separable Banach space, $S$ is a complete separable metric space and $\mu$ is a Borel measure. However, Greim [4] has produced an example of a nonseparable $X$ and a norm one $f$ in $L^p(\lambda, X)$, where $\lambda$ is Lebesgue measure on [0, 1], such that $f$ is an extreme point of the unit ball of $L^p(\lambda, X)$ but, for all $s$ in [0, 1], the element $f(s)/\|f(s)\|$ is not an extreme point of the unit ball of $X$.

Johnson [7] and Greim [5] considered the similar problem of characterizing strongly exposed points. In [7], a sufficient condition is given for $g$ in $L^q(\mu, X^*)$, where $p^{-1} + q^{-1} = 1$, to strongly expose $f$ of norm one in $L^p(\mu, X)$. In [5], the proof that this condition is sufficient is used to show, in the case that $X$ is a smooth Banach space, that $f$ in $L^p(\mu, X)$ is a strongly exposed point of the unit ball if $\|f\|_p = 1$ and, for almost all $s$ in the support of $f$, the element $f(s)/\|f(s)\|$ is a strongly exposed point of the unit ball of $X$. It is also shown in [5] that these conditions are necessary in the case that $X$ is a separable Banach space ($\mu$ arbitrary) and in the case that $\mu$ is a Radon measure on a locally compact Hausdorff space ($X$ arbitrary).

In this paper, the problem of characterizing strongly extreme points in $L^p(\mu, X)$ is considered. It will be shown (Theorem 1) that $f$ is a strongly extreme point of the unit ball of $L^p(\mu, X)$ if $\|f\|_p = 1$ and, for almost all

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$s$ in the support of $f$, the element $f(s)/\|f(s)\|$ is a strongly extreme point of the unit ball of $X$. Also, it will be shown that these conditions are necessary in the case $X$ is separable (Theorem 2) or in the setting that $\mu$ is a Radon measure on a locally compact Hausdorff space (Theorem 3). Thus, when characterizing a strongly extreme point, a notion lying strictly between that of strongly exposed point and extreme point, the natural sufficient conditions are obtained with no restriction on $X$ (for a strongly exposed point, $X$ was required to be smooth) and, in the simple setting of Lebesgue measure on [0, 1], the natural necessary conditions are obtained with no restriction on $X$ (for an extreme point, $X$ must be separable).

The notation and terminology is relatively standard: $(L^p(\mu, X), \|\cdot\|_p)$ denotes the Lebesgue-Bochner space of equivalence classes of $p$-integrable, $X$-valued, strongly measurable functions on the measure space $(S, \Sigma, \mu)$; without loss of generality, the measure $\mu$ is assumed to be complete; for $f$ in $L^p(\mu, X)$, the set $\{s \in S: f(s) \neq 0\}$ is denoted by $S_f$ and the function $s \to \|f(s)\|$ is denoted by $|f|$; and, whenever $X$ is the scalar field, $L^p(\mu, X)$ is denoted by $L^p(\mu)$.

Recall an element $x$ of the unit sphere of a Banach space $X$ is a strongly extreme point of the unit ball if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that the conditions $\|x \pm z\| < 1 + \delta$ imply $\|z\| < \varepsilon$; equivalently, every sequence $\{z_n\}$ such that $\|x \pm z_n\| \to 1$ converges to zero.

The following lemma may be known, but, since the author could not find a reference for it, a proof is included for completeness sake.

LEMMA. *Let $(S, \Sigma, \mu)$ be a measure space, $X$ a Banach space and $1 \leqq p < \infty$. For $\{f_n\}$ and $f$ in $L^p(\mu, X)$, if $\|f_n\|_p \to \|f\|_p$ and $f_n \to f$ almost everywhere, then $f_n \to f$ in $L^p(\mu, X)$.*

PROOF. Since $\|f_n\|_p \to \|f_p\|$ and $|f_n| \to |f|$ almost everywhere, it follows that (see [1, Exercise 4T])

$$(1) \qquad \lim_{n\to\infty} \int_E \|f_n\|^p d\mu = \int_E \|f\|^p d\mu, \qquad \text{for all } E \text{ in } \Sigma.$$

For each positive integer $n$, define $\lambda_n$ on $\Sigma$ by $\lambda_n(E) = \|f_n\chi_E\|_p^p$, for $E$ in $\Sigma$. In view of (1), the Vitali-Hahn-Saks theorem [3, p. 158] yields

$$(2) \qquad \lim_{\mu(E)\to 0} \lambda_n(E) = 0 \qquad \text{uniformly in } n.$$

Let $\varepsilon > 0$ given. Since $f \in L^p(\mu, X)$, there exists a set $F$ in $\Sigma$ of finite measure such that $\|f\chi_{S\setminus F}\|_p^p < \varepsilon/2$. By (1), there exists a positive $N$ such that $\|f_n\chi_{S\setminus F}\|_p^p < \varepsilon$, whenever $n \geqq N$. From this and the fact that $f_n \in L^p(\mu, X)$, for $n = 1, \ldots, N - 1$, there exists a set $A$ in $\Sigma$ of finite measure such that

$$(3) \qquad \int_{S\backslash A} \|f_n\|^p d\mu < \varepsilon, \qquad \text{for all } n.$$

In view of (2) and (3), since $f_n \to f$ almost everywhere, the Vitali convergence theorem [3, p. 150] yields that $f_n \to f$ in $L^p(\mu, X)$ and the proof is complete.

THEOREM 1. *Let $(S, \Sigma, \mu)$ be a measure space, $X$ a Banach space and $1 < p < \infty$. If $f$ in $L^p(\mu, X)$ has norm one and, for almost all $s$ in the support of $f$, the element $f(s)/\|f(s)\|$ is a strongly extreme point of the unit ball of $X$, then $f$ is a strongly extreme point of the unit ball of $L^p(\mu, X)$.*

PROOF. Suppose $\{g_n\}$ is a sequence in $L^p(\mu, X)$ such that $\|f \pm g_n\|_p \to 1$. Since $f$ has norm one, it follows that $\|2f \pm g_n\|_p \to 2$. By the triangle inequality in $X$ and $L^p(\mu)$,

$$(4) \qquad \|2f \pm g_n\|_p \leqq \|(|f| + |f \pm g_n|)\|_p \leqq \|f\|_p + \|f \pm g_n\|_p.$$

Since the left side and the right side of (4) tend toward two, the uniform rotundity of $L^p(\mu)$ yields that $|f \pm g_n| \to |f|$ in $L^p(\mu)$ and, hence, there exists a subsequence of $\{g_n\}$, called $\{g_n\}$ again, such that

$$(5) \qquad |f \pm g_n|(s) \to |f|(s), \qquad \text{for almost all } s \text{ in } S.$$

Thus, for almost all $s$ in $S_f$, it follows that $|f \pm g_n|(s)/\|f(s)\| \to 1$ and, hence, by the hypotheses, $g_n(s) \to 0$, for almost all $s$ in $S_f$. Also, by (5), for almost all $s$ in $S\backslash S_f$, it trivially follows that $g_n(s) \to 0$. So $\|f + g_n\|_p \to \|f\|_p$ and $f + g_n \to f$ almost everywhere. An application of the Lemma, with $f_n = f + g_n$, yields $g_n \to 0$ in $L^p(\mu, X)$, and the proof is complete.

The proofs of the next two results are modeled after the proofs given by Greim [5] for the corresponding results concerning strongly exposed points.

THEOREM 2. *Let $(S, \Sigma, \mu)$ be a measure space, $X$ a separable Banach space and $1 < p < \infty$. If $f$ is a strongly extreme point of the unit ball of $L^p(\mu, X)$, then $\|f\|_p = 1$ and, for almost all $s$ in the support of $f$, the element $f(s)/\|f(s)\|$ is a strongly extreme point of the unit ball of $X$.*

PROOF. Clearly, $\|f\|_p = 1$. For each positive integer $n$ and $s$ in $S_f$, let

$$E(s, n) = \{z \in X\colon \|(f(s)/\|f(s)\|) \pm z\| < 1 + 1/n\},$$
$$d(s, n) = \inf\{\varepsilon > 0\colon \text{every } z \in E(s, n) \text{ has } \|z\| < \varepsilon\},$$
$$e(s) = \inf\{d(s, n)\colon n \text{ is a positive integer}\}.$$

For $s$ in $S_f$, note that $f(s)/\|f(s)\|$ is a strongly extreme point of the unit ball of $X$ if only if $e(s) = 0$. Extend each $d(\cdot, n)$ and $e$ to all of $S$ by letting $d(s, n) = e(s) = 0$ for $s$ in $S\backslash S_f$. To show that $e$ is a measurable function, it suffices to show each $d(\cdot, n)$ is a measurable function, and, for this, it suffices to show $\{s \in S: d(s, n) > \delta\}$ is in $\Sigma$ for each $\delta > 0$. Fix such a $\delta$. For each positive integer $n$ and $z$ in $X$, let

$$A(n, z) = \{s \in S_f: z \in E(s, n)\}.$$

Then $A(n, z) \in \Sigma$ since $f$ is a measurable function. Let $D$ be a countable dense subset of $X$. Note that $d(s, n) > \delta$ if and only if $s \in A(n, z)$, for some $z$ in $D$ with $\|z\| > \delta$. Hence $\{s \in S: d(s, n) > \delta\}$ is the countable union of sets in $\Sigma$ and so is in $\Sigma$. To finish the proof, it remains to show that $\{s \in S: e(s) > 0\}$ has measure zero. If it does not, then there exists $\delta > 0$ such that $A \equiv \{s \in S: e(s) > \delta\}$ has positive measure. To obtain a contradiction, it suffices to produce a sequence $\{g_n\}$ in $L^p(\mu, X)$ such that

$$(6) \qquad \begin{aligned} \|g_n(s)\| &\geqq \delta\|f(s)\|, && \text{for all } s \text{ in } A, \text{ and} \\ \|f(s) \pm g_n(s)\| &\leqq (1 + 1/n)\|f(s)\|, && \text{for all } s \text{ in } S; \end{aligned}$$

for then, $\|g_n\|_p \geqq \delta\|f\chi_A\|_p > 0$, since $A \subset S_f$, and $\|f \pm g_n\|_p \leqq 1 + 1/n$, contradicting the hypothesis that $f$ is a strongly extreme point of the unit ball of $L^p(\mu, X)$. To this end, let $A(n, z)$ be defined as above and note

$$A \subset \{s \in S: d(s, n) > \delta\} = \bigcup \{A(n, z): z \in D \text{ with } \|z\| > \delta\}.$$

So $A$ may be written as the disjoint union of a sequence $\{B_m\}$ in $\Sigma$, where each $B_m$ is a subset of some $A(n, z_m)$, with $z_m$ in $D$ and $\|z_m\| > \delta$. Now, define $g_n$ by

$$g_n(s) = \|f(s)\| \sum_{m=1}^{\infty} z_m \chi_{B_m}(s),$$

for $s$ in $S$. Then $\{g_n\}$ satisfies the conditions in (6). This completes the proof.

THEOREM 3. *Let $\mu$ be a Radon measure on a locally compact Hausdorff space $S$, $X$ a Banach space and $1 < p < \infty$. If $f$ is a strongly extreme point of the unit ball of $L^p(\mu, X)$, then $\|f\|_p = 1$ and, for almost all $s$ in the support of $f$, the element $f(s)/\|f(s)\|$ is a strongly extreme point of the unit ball of $X$.*

PROOF. For the moment, suppose that $f$ is continuous and never zero on $S$. Proceed as in the proof of Theorem 2. The sets $A(n, z)$ are open and, hence, so is their union

$$\bigcup\{A(n, z): z \in X \text{ and } \|z\| > \delta\} = \{s \in S: d(s, n) > \delta\}.$$

This shows that $e$ is a measurable function. In order to show $\{s \in S: e(s) > 0\}$ has measure zero, replace the set $A$ in the proof of Theorem 2 by a compact subset with positive measure ($A \subset \{s \in S: e(s) > \delta\}$, $A$ compact and $\mu(A) > 0$) and continue as before. Then, by its compactness, $A$ is contained in a finite union of the sets $A(n, z)$, and defining the sequence

$\{g_n\}$ in a manner analogous to Theorem 2 completes the proof in this case. Now suppose that $f$ is arbitrary. Since $S_f$ is $\sigma$-finite, by repeated applications of Luzin's theorem [**2**, p. 335], it follows that $S_f$ may be written, up to a $\mu$-null set, as the disjoint union of a sequence $\{K_m\}$ of compact subsets of $S$ of positive measure such that $f$ restricted to each $K_m$ is continuous and never zero. Let $K$ be one of these sets and let $\mu_K$ denote $\mu$ restricted to $K$. Since $f$ is a strongly extreme point of the unit ball of $L^p(\mu, X)$, it follows that $f_K \equiv \|f\chi_K\|_p^{-1} f|_K$ is a strongly extreme point of the unit ball of $L^p(\mu_K, X)$; for, if $\{h_n\}$ is a sequence in $L^p(\mu_K, X)$ such that $\|f_K \pm h_n\|_p \to 1$, then, after extending each $h_n$ to be zero on $S\backslash K$, it follows, by direct computation, that $\|f \pm \|f\chi_K\|_p h_n\|_p \to 1$ and hence $\|f\chi_K\|_p h_n \to 0$ in $L^p(\mu, X)$ and so $h_n \to 0$ in $L^p(\mu_K, X)$. By applying the first case to $f_K$ in $L^p(\mu_K, X)$, it follows that, for almost all $s$ in $K$, the element $f_K(s)/\|f_K(s)\| = f(s)/\|f(s)\|$ is a strongly extreme point of the unit ball of $X$. This completes the proof.

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