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REGULARIZED RITZ APPROXIMATIONS FOR FREDHOLM EQUATIONS OF THE FIRST KIND

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ABSTRACT. Some results of Groetsch, King and Murio on a general regularized finite element method for Fredholm equations of the first kind are improved in this note. A sufficient condition for weak convergence of the approximations is also given. Taken together, the main results of this paper are exact finite element analogues of classical results on Tikhonov regularization in infinite dimensional spaces.

1. Introduction. Suppose $K: H_1 \to H_2$ is a compact linear operator from a Hilbert space H_1 into a Hilbert space H_2. It is well known that the equation of the first kind

$$Kx = g \tag{1}$$

is ill-posed if K does not have finite rank. That is, the solution x depends discontinuously on the data g (see, e.g., [6]). We assume that $g \in R(K)$, the range of K, and by solution we mean the minimal norm solution.

This instability of the inversion process can have dire numerical consequences since in practical circumstances the data result from measurements and hence only an approximation g^δ to g is available which satisfies

$$\|g - g^\delta\| \leqq \delta, \tag{2}$$

where δ is a known error level. In order to provide a stable solution method for (1), Tikhonov took as an approximation to x the minimizer $x^\delta(\alpha)$ of the functional $F_\alpha(z; g^\delta) := \|Kz - g^\delta\|^2 + \alpha\|z\|^2$ (we denote the inner product and norm in each of the spaces H_1 and H_2 by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively). Here α is a positive parameter, the regularization parameter, whose role is to affect a trade-off between fidelity and regularity in the approximate solution. Tikhonov [5, 6] showed that if $C_1\delta^2 \leqq \alpha \leqq C_2\delta^2$ for some positive constants C_1 and C_2, then the approximations $\{x^\delta(\alpha)\}$ converge weakly to x as $\delta \to 0$, while if $\delta =$

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$o(\sqrt{\alpha})$ strong convergence results. Weak convergence for a more general class of regularization methods in infinite dimensional Hilbert spaces was established recently by Engl [1]. In particular, Engl shows that the condition $\delta = O(\sqrt{\alpha})$ is sufficient to guarantee the weak convergence of $x^\delta(\alpha)$ to x.

The approximations $x^\delta(\alpha)$ above are minimizers of $F_\alpha(\cdot\,; g^\delta)$ over the infinite dimensional space H_1 and hence are not effectively computable. Our aim in this note is to discuss the convergence of certain computable, i.e., finite dimensional, approximations to x.

2. Results. Suppose that $V_1 \subset V_2 \subset \cdots$ is an expanding sequence of finite dimensional subspaces of H_1 whose union is dense in H_1. We will denote the minimizer of $F_\alpha(\cdot\,; g)$ over H_1 by $x(\alpha)$. The minimizers of $F_\alpha(\cdot\,; g)$ and $F_\alpha(\cdot\,; g^\delta)$ over V_m will be denoted by $x_m(\alpha)$ and $x_m^\delta(\alpha)$, respectively. It is easy to see that $x(\alpha)$ is characterized by

$$(Kx(\alpha) - g, Ky) + \alpha(x(\alpha), y) = 0 \tag{3}$$

for all $y \in H_1$, or equivalently

$$x(\alpha) = (K^*K + \alpha I)^{-1}K^*g \tag{4}$$

where K^* is the adjoint of K. The approximations of $x^\delta(\alpha)$ have the same characterization with g replaced by g^δ. The finite dimensional approximation $x_m(\alpha) \in V_m$ is characterized by the equation in (3) holding for all $y \in V_m$ and $x_m^\delta(\alpha)$ satisfies the corresponding condition with g replaced by g^δ.

The analysis will be intimately connected with the degree to which the subspaces V_m support the operator K, that is, on the number

$$\gamma_m := \|K(I - P_m)\| = \|(I - P_m)K^*\|,$$

where P_m is the orthogonal projector of H_1 onto V_m. Note that the assumptions of $\{V_m\}$ imply that the continuous functions $f_m(z) = \|(I - P_m)z\|$ converge pointwise and monotonically to zero on H_1. Since K is compact, it follows that $\{f_m\}$ converges uniformly to zero on $K^*(B)$, where B is the unit ball in H_2, i.e., $\gamma_m \to 0$ as $m \to \infty$.

We now suppose that α is related to m, say $\alpha = \alpha_m$, in such a way that $\alpha_m \to 0$ as $m \to \infty$. The following result is proved in [2].

LEMMA 1. *If* $\gamma_m = 0(\sqrt{\alpha_m})$, *then* $x_m(\alpha_m) \to x$ *as* $m \to \infty$. *In fact,* $\|x_m(\alpha_m) - x\| = 0(\|(I - P_m)x(\alpha_m)\|)$.

Our next result is a stronger version of a stability estimate which appears in [2].

LEMMA 2. $\|x_m(\alpha) - x_m^\delta(\alpha)\| \leqq \delta/\sqrt{\alpha}$.

PROOF. From (3)

$$(Kx_m(\alpha) - g, Ky) + \alpha(x_m(\alpha), y) = 0$$

and

$$(Kx_m^\delta(\alpha) - g^\delta, Ky) + \alpha(x_m^\delta(\alpha), y) = 0$$

for all $y \in V_m$. We then have

(5) $(Kx_m(\alpha) - Kx_m^\delta(\alpha), Ky) + \alpha(x_m(\alpha) - x_m^\delta(\alpha), y) = (g - g^\delta, Ky)$

for all $y \in V_m$. In particular, setting $y = x_m(\alpha) - x_m^\delta(\alpha)$, we have

$$\|x_m(\alpha) - x_m^\delta(\alpha)\|^2 \leqq (g - g^\delta, K(x_m(\alpha) - x_m^\delta(\alpha)))/\alpha.$$

However,

$$\begin{aligned} K(x_m(\alpha) - x_m^\delta(\alpha)) &= K_m(K_m^*K_m + \alpha I)^{-1}K_m^*(g - g^\delta) \\ &= K_mK_m^*(K_mK_m^* + \alpha I)^{-1}(g - g^\delta) \end{aligned}$$

where K_m is the restriction of K to V_m, and $\|K_mK_m^*(K_mK_m^* + \alpha I)^{-1}\| \leqq 1$. Therefore $\|x_m(\alpha) - x_m^\delta(\alpha)\|^2 \leqq \delta^2/\alpha$.

Hereafter we suppose that the parameters m, α and γ depend on δ in such a way that $m = m_\delta \to \infty$ as $\delta \to 0$ and $\alpha = \alpha_{m_\delta} \to 0$ as $\delta \to 0$. For simplicity of notation we will write α and γ, respectively, for α_{m_δ} and γ_{m_δ}. From Lemmas 1 and 2 we immediately obtain the following.

THEOREM 1. *If* $\gamma = 0(\sqrt{\alpha})$ *and* $\delta = o(\sqrt{\alpha})$, *then* $x_m^\delta(\alpha) \to x$ as $\delta \to 0$.

It is well known (see, e.g., [**4**]) that if $x \in R(K^*)$ then $\|x - x(\alpha)\| = 0(\sqrt{\alpha})$. Also if $x = K^*w$, then $x(\alpha_m) = (K^*K + \alpha_m I)^{-1}K^*KK^*w = K^*(KK^* + \alpha_m I)^{-1}KK^*w$ and hence $\|(I - P_m)x(\alpha_m)\| \leqq \gamma_m \|w\|$.

Combining this with Lemmas 1 and 2 we have the following corollary.

COROLLARY 1. *If* $x \in R(K^*)$, $\alpha = C\delta$, and $\gamma = O(\sqrt{\alpha})$, then $\|x - x_m^\delta(\alpha)\| = O(\sqrt{\alpha})$.

In a similar way, using the fact that $x \in R(K^*K)$ implies $\|x - x(\alpha)\| = O(\alpha)$ we obtain the next corrollary.

COROLLARY 2. *If* $x \in R(K^*K)$, $\alpha = C\delta^{2/3}$ *and* $\gamma = 0(\delta^{2/3})$, *then* $\|x - x_m^\delta(\alpha)\| = O(\delta^{2/3})$.

We now investigate weak convergence of the approximations, beginning with some preliminary results.

LEMMA 3. *If* $y \in V_k$ *for some* k, *then* $(x_m(\alpha) - x_m^\delta(\alpha), K^*Ky) \to 0$ *as* $\delta \to 0$.

PROOF. Since $V_k \subset V_m$ for $m \geqq k$, we have by (5) and Lemma 2

$$|(x_m(\alpha) - x_m^\delta(\alpha), K^*Ky)| \leqq \alpha|(x_m^\delta(\alpha) - x_m(\alpha), y)| + |(g - g^\delta, Ky)|$$
$$\leqq \delta(\sqrt{\delta} + \|K\|)\|y\|.$$

LEMMA 4. *If* $\gamma = O(\sqrt{\alpha})$, $\delta = O(\sqrt{\alpha})$ *and* $z \in N(K)$ *(the nullspace of* K*), then* $(x_m(\alpha) - x_m^\delta(\alpha), z) \to 0$ *as* $\delta \to 0$.

PROOF. Note that, as in (4), $x_m(\alpha)$ has the representation. $x_m(\alpha) = (K_m^*K_m + \alpha I)^{-1}K_m^*g$ where K_m is the restriction of K to V_m and $x_m^\delta(\alpha)$ has a similar representation with g replaced by g^δ. Since $(v, K^*y) = (Kv, y)$ for $v \in H_1$ and $y \in H_2$ and $(v, K_m^*y) = (K_m v, y)$ for $v \in V_m$ and $y \in H_2$ we have $(v, K^*y - K_m^*y) = 0$ for all $y \in H_2$ and $v \in V_m$, that is, $K_m^* = P_m K^*$ where P_m is the orthogonal projection of H_1 onto V_m. We therefore have

$$(x_m(\alpha) - x_m^\delta(\alpha), z) = (P_m K^*(K_m K_m^* + \alpha I)^{-1}(g - g^\delta), z)$$
$$= ((I - P_m)K^*(K_m K_m^* + \alpha I)^{-1}(g^\delta - g), z)$$

since $z \in N(K)$. But,

$$|((I - P_m)K^*(K_m K_m^* + \alpha I)^{-1}(g^\delta - g), z)|$$
$$= |((K_m K_m^* + \alpha I)^{-1}(g^\delta - g), K(I - P_m)z)|$$
$$\leqq \delta/\alpha\ \|K(I - P_m)^2 z\|$$
$$\leqq (\delta\gamma/\alpha)\ \|(I - P_m)z\| \to 0 \text{ as } m = m_\delta \to \infty.$$

Consider now the situation of a sequence $\{g^{\delta_n}\}$ of increasingly accurate data satisfying $\|g - g^{\delta_n}\| \leqq \delta_n \to 0$ as $n \to \infty$. For each n we suppose that $F_\alpha(\cdot\,; g^{\delta_n})$ is minimized over a subspace V_m where $m = m(\delta_n) \to \infty$ and $\alpha = \alpha_{m(\delta_n)} \to 0$ as $n \to \infty$. For simplicity of notation we will dispense with some of the subscripts and seek conditions which insure that

$$x_m^{\delta_n}(\alpha) := x_{m(\delta_n)}^{\delta_n}(\alpha_{m(\delta_n)}) \xrightarrow{w} x \text{ as } n \to \infty.$$

THEOREM 2. *If* $\gamma = O(\sqrt{\alpha})$ *and* $\delta = O(\sqrt{\alpha})$, *then* $x_m^{\delta_n}(\alpha)$ *converges weakly to* x *as* $n \to \infty$.

PROOF. In light of Lemma 1, it is enough to show that the sequence $\{x_m(\alpha) - x_m^{\delta_n}(\alpha)\}$ converges weakly to zero as $n \to \infty$.

Since $\delta = O(\sqrt{\alpha})$, we see from Lemma 2 that the sequence $\{x_m(\alpha) - x_m^{\delta_n}(\alpha)\}$ is uniformly bounded, say $\|x_m(\alpha) - x_m^{\delta_n}(\alpha)\| \leqq M$. Therefore by the Banach-Steinhaus Theorem it is sufficient to show that

$$(x_m(\alpha) - x_m^{\delta_n}(\alpha), z) \to 0 \text{ as } n \to \infty \tag{6}$$

for all z in a dense subspace of H_1. From Lemma 4 we see that (6) certainly holds for all $z \in N(K) = N(K^*K) = R(K^*K)^\perp$.

Suppose then that $z = K^*Kw$. Given $\varepsilon > 0$ there is a k and a $y \in Y_k$ such that $M\|K\|^2\ \|y - w\| < \varepsilon$. Therefore, for $m \geqq k$,

$$\begin{aligned} |(x_m(\alpha) - x_m^{\delta_n}(\alpha), z)| &\leqq |(x_m(\alpha) - x_m^{\delta_n}(\alpha), z - K^*Ky)| \\ &\quad + |(x_m(\alpha) - x_m^{\delta_n}(\alpha), K^*Ky)| \\ &< \varepsilon + |(x_m(\alpha) - x_m^{\delta_n}(\alpha), K^*Ky)|. \end{aligned}$$

But $(x_m(\alpha) - x_m^{\delta_n}(\alpha), K^*Ky) \to 0$ by Lemma 3. Therefore we find that $(x_m(\alpha) - x_m^{\delta_n}(\alpha), z) \to 0$ for each z in the dense subspace $R(K^*K) + R(K^*K)^{\perp}$, and hence $x_m^{\delta_n}(\alpha) \xrightarrow{w} x$.

3. Remarks. A weaker version of Lemma 2, with correspondingly weaker convergence results, is proved in [**2**]. We note that the main results of this paper, namely Theorems 1 and 2 are the exact analogues of the classical infinite dimensional results on Tikhonov regularization. The new feature is the extra condition $\gamma = O(\sqrt{\alpha})$ relating the regularization parameter to the degree of support of the operator by the finite dimensional subspaces. It should also be pointed out that the asymptotic convergence rate of $O(\delta^{2/3})$ established in Corollary 2 is in fact optimal even for the classical infinite dimensional Tikhonov approximations (see [**3**]).

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