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REFLEXIVE ALGEBRAS OF MATRICES

GEORGE PHILLIP BARKER AND JOYCE JABEN CONKLIN

ABSTRACT. We study some sufficient properties for an algebra of matrices to be reflexive. In particular we show that a semismple algebra is reflexive. Commutative algebras are then considered, and it is seen that a commutative algebra of 3×3 matrices is reflexive if either it can be diagonalized or it is of dimension 2. Finally we show that the algebra of all operators which leave invariant every element of a complemented lattice of subspaces forms a semisimple algebra. This is related to a result by Harrison and Longstaff on reflexive lattices of subspaces.

1. Introduction. Let V be a vector space of finite dimension n over the complex number $\mathbf{C}$. The algebra of all linear operators an V is denoted by Hom V. The algebra of $n \times n$ matrices over $\mathbf{C}$ is denoted by M_n.

The set of all subspaces of V is a modular lattice under the operations intersection (meet) and sum (join) of two subspaces. Further, any sublattice of this lattice is again modular.

Let $\mathscr{L}$ be a lattice of subspaces of V and $\mathscr{A}$ a subalgebra of Hom (V). We define the operations Alg and Lat as follows. Alg $\mathscr{L}$ is the set (necessarily an algebra) of all $A \in$ Hom V which leave invariant every subspace $W \in \mathscr{L}$. Similarly Lat $\mathscr{A}$ is the lattice of all subsapaces of V which are left invariant by every element of $\mathscr{A}$. $\mathscr{L}$ (respectively $\mathscr{A}$) is called reflexive iff Lat Alg $\mathscr{L} = \mathscr{L}$, (Alg Lat $\mathscr{A} = \mathscr{A}$ respectively). The classification of reflexive algebras and reflexive lattices is far from complete even in finite dimensional spaces, although some progress has been made (cf. **[1, 3, 5, 6, 11]**). It is worth noting, however, that every finite dimensional algebra is isomorphic to a reflexive one (cf. Brenner and Bulter, J. London Math. Soc. **40** (1965), 183–187). In this paper we shall study reflexivity and give a more algebraic proof of a result due to Harrison and Longstaff **[7]**. We shall also study some particular types of algebras such as commutative algebras of matrices. We close with a discussion of subspaces lattices which may be useful in generating examples.

In what follows all lattices will contain $\{0\}$ and V, and all algebras will contain the identity, I, except for certain subalgebras of nilpotent matrices.

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2. Reflexive Algebras. We begin with some lemmas concerning direct sums.

LEMMA 1. *If $\mathscr{L}_1$ and $\mathscr{L}_2$ are subspace lattices of V_1 and V_2 respectively, then* $\text{Alg}\,(\mathscr{L}_1 \oplus \mathscr{L}_2) = \text{Alg}\,\mathscr{L}_1 \oplus \text{Alg}\,\mathscr{L}_2$.

The proof is direct and is omitted as is the proof of lemma 2.

LEMMA 2. *If $\mathscr{A}_1$ and $\mathscr{A}_2$ are algebras of operators in* Hom V_1 *and* Hom V_2 *respectively, where for $i = 1, 2$ dim $V_i = n_i$, then*

$$\text{Lat}\,(\mathscr{A}_1 \oplus \mathscr{A}_2) = \text{Lat}\,\mathscr{A}_1 \oplus \text{Lat}\,\mathscr{A}_2.$$

Notation. We use $\mathbf{C}[x]$ and $\mathbf{C}[A]$ to denote all polynomials with complex coefficients in x and A.

REMARKS. This result is not to be confused with Lemma 1 of [3]. In that paper Brickman and Fillmore are looking at Lat $(A_1 \oplus A_2)$ where A_1 and A_2 are linear transformations. Thus in [3] the algebra under consideration is $\mathbf{C}[A_1 \oplus A_2]$, while in Lemma 2 we are considering the generally larger algebra $\mathbf{C}[A_1] \oplus \mathbf{C}[A_2]$.

Let $\mathscr{A}$ be a subalgebra of Hom V, and let $T \in$ Hom V be nonsingular. If $TAT^{-1} = B$, then it is easily shown that $T(\text{Lat}\,\mathscr{A}) = \text{Lat}\,B$. We shall make implicit use of this observation subsequently.

LEMMA 3. *Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are subalgebras of* Hom V_1 *and* Hom V_2, *respectively. Then $\mathscr{A}_1 \oplus \mathscr{A}_2$ is a reflexive subalgebra of* $\text{Hom}(V_1 \oplus V_2)$ *if and only if $\mathscr{A}_1$ and $\mathscr{A}_2$ are reflexive.*

This is an immediate consequence of Lemmas 1 and 2.

PROPOSITION 4. *If $\mathscr{A}$ is a semisimple subalgebra of* Hom V, *then $\mathscr{A}$ is reflexive.*

PROOF. It is known (cf. [2, 96]) that by a similarity we may assume

$$\mathscr{A} = \mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_p$$

where each $\mathscr{A}_j$ consists of block diagonal matrices $A_j = \{\text{diag}\,(A, \ldots, A) | A \in M_{nj}\}$. Since $I \in \mathscr{A}$, then $\mathscr{A}$ has no kerne!. By proposition 2.2 of [1], each such $\mathscr{A}_j$ is reflexive. Then, by Lemma 3 (extended in the obvious way), $\mathscr{A}$ is a slso reflexive.

Let $\mathscr{A}$ be a subalgebra of Hom V. Then $\mathscr{A}$ is a *-subalgebra of Hom V if and only if, for each $A \in \mathscr{A}$, we have $A^* \in \mathscr{A}$ as well. [A^* is the conjugate transpose matrix.)

The result cited in the preceding proof has been known for sometime. For instance, it was known to T. Molien in 1893 (cf. Math. Ann. **41**

(1893), 83–156). The more recent reference is given for the convenience of the reader.

COROLLARY 5. *Let* $\mathscr{A}$ *be a *-subalgebra of* Hom V. *Then* $\mathscr{A}$ *is refiexive.*

PROOF. By Lemma 1 of [2] a *-subalgebra of Hom V is semisimple. Therefore by proposition, 4, $\mathscr{A}$ is reflexive.

This is of course the finite dimensional case of a known result on von Neumann algebras (cf. [5]).

If A is a linear transformation then we call A reflexive provided $\mathbf{C}[A]$ is reflexive. If $\mathscr{A}$ is a commutative reflexive algebra then (cf. [5]) each $A \in \mathscr{A}$ is a reflexive operator. There are examples ([1] and [5]) which show that the converse is false. However, we have a simple criterion for the converse.

Notation. If $\mathscr{A} = \mathbf{C}[A]$ we write Lat A for Lat $\mathbf{C}[A]$. Also if $\mathscr{A}_1, \ldots, \mathscr{A}_p$ are subalgebras of Hom V, then $v_{j=1}^{p} \mathscr{A}_j$ denotes the algebra generated by $\mathscr{A}_1, \ldots, \mathscr{A}_p$.

PPOPOSITION 6. *Let* $\mathscr{A}$ *be an algebra for which* $\mathscr{A} \supset$ Alg Lat A *for all* $A \in \mathscr{A}$. *Then* $\mathscr{A}$ *is reflexive if and only if for same basis* $\{T_1, \ldots, T_m\}$ *of* $\mathscr{A}$ *we have*

$$(*) \qquad \operatorname{Alg}\,(\bigcap_{j=1}^{m} \operatorname{Lat} T_j) = v_{j=1}^{m} \operatorname{Alg} \operatorname{Lat} T_j.$$

PROOF. Note that $\bigcap$ Lat $T_j \supset$ Lat $\mathscr{A}$ for any subset $\{T_1, \ldots, T_m\} \subset \mathscr{A}$. If the set is in fact a basis, then for any $A \in \mathscr{A}$ we have $A = \sum \alpha_j T_j$ for suitable α_j, whence $\bigcap$ Lat $T_j =$ Lat $\mathscr{A}$.

First assume $\mathscr{A}$ is reflexive (in which case $\mathscr{A} \supset$ Alg Lat A for all $A \in \mathscr{A}$). Then $\mathscr{A} =$ Alg Lat $\mathscr{A}$. Since $\bigcap_{j=1}^{m}$ Lat $T_j \subset$ Lat T_k, for all k, then Alg $(\bigcap_{j=1}^{m}$ Lat $T_j) \supset$ Alg Lat T_k, for all k. Hence Alg $(\bigcap_{j=1}^{m}$ Lat $T_j)$ $\supset v_{j=1}^{m}$ Alg Lat T_j and we have

$$\mathscr{A} = \operatorname{Alg} \operatorname{Lat} \mathscr{A} = \operatorname{Alg}(\bigcap_{j=1}^{m} \operatorname{Lat} T_j) \supset v_{j=1}^{m} \operatorname{Alg} \operatorname{Lat} T_j = v_{j=1}^{m} \mathbf{C}[T_j] = \mathscr{A}$$

since $\{T_1, \ldots, T_m\}$ is a basis. Therefore equality holds.

Conversely, suppose (*) holds for the basis $\{T_1, \ldots, T_m\}$. Then

$$\mathscr{A} \subset \operatorname{Alg} \operatorname{Lat} \mathscr{A} = \operatorname{Alg}\,(\bigcap_{j=1}^{m} \operatorname{Lat} T_j) = v_{j=1}^{m} \operatorname{Alg} \operatorname{Lat} T_j \subset \mathscr{A}$$

since Alg Lat $T_j \subset \mathscr{A}$ for each j. Thus $\mathscr{A}$ is reflexive.

REMARK. If $\{T_1, \ldots, T_m\}$ is a set of generators for $\mathscr{A}$ rather than a basis (and $\mathscr{A} \supset$ Alg Lat A for all A), then the result remains true because we still have $\bigcap$ Lat T_j = Lat $\mathscr{A}$.

2. Polynomial algebras. We shall consider the case of an algebra generated by a single operator A which is a polynomial in another operator N.

LEMMA 7. *Let $p(x) \in \mathbf{C}[x]$ with $p'(0) \neq 0$, and let $N \in \mathrm{Hom}\ V$ be nilpotent. If $A = p(N)$, then $\mathbf{C}[A] = \mathbf{C}[N]$.*

PROOF. This is an immediate consequence of the inverse function theorem (cf. **[15]**) and the fact that for a differentiable function g there is a polynomial q such that $g(A) = q(A)$ (**[10]**, 169).

THEOREM 8. *Let $N \in \mathrm{Hom}\ V$ be nilpotent, and let $A = p(N)$ for some $p(x) \in \mathbf{C}[x]$. Then A is reflexive if and only if either N is reflexive or $p'(0) = 0$.*

PROOF. As we noted earlier, if N is reflexive, then so is A. So suppose $p'(0) = 0$. Let the elementary divisors of N be of orders $n_1 \geqq n_2 \geqq \cdots \geqq n_t$. We have two cases depending upon the last value of k for which $p^{(k)}(0) \neq 0$.

Case 1. Suppose $p'(0) = \cdots = p^{(k-1)}(0) = 0$ and $p^{(k)}(0) \neq 0$ with $1 < k < n_1$. Then (cf. **[12**, p. 126**]**) A has elemntary divisors $(\lambda - p(0))^{m_{ij}}$, $i = 1, \ldots, t, j = 1, \ldots, k$, where $m_{ij} = [(n_i - j)/k] + 1$, and $[x]$ is the greatest integer function.

Since $n_1 \geqq n_2 \geqq \cdots \geqq n_t$ we have $m_{i1} \geqq m_{i2} \geqq \cdots \geqq m_{ik}$ and $m_{11} \geqq m_{21} \geqq \cdots \geqq m_{t1}$. Since either m_{12} or m_{21} is the degree of the second largest elementary divisor of A, then by the result of Deddens and Fillmore **[5]** A is reflexive if $m_{11} - m_{12} \leqq 1$. But $k > 1$ so that

$$\begin{aligned} 0 \leqq m_{11} - m_{12} &= \left[\frac{n_1 - 1}{k}\right] + 1 - \left[\frac{n_1 - 2}{k}\right] - 1 \\ &\leqq \left(\frac{n_1 - 1}{k}\right) - \left(\frac{n_1 - 2}{k}\right) + 1 \leqq 1 + 1/k. \end{aligned}$$

Since $m_{11} - m_{12}$ is an integer we have the desired inequality.

Case 2. If $p'(0) = \cdots = p^{(k-1)}(0) = 0$ and $p^{(k)}(0) \neq 0$ for $k \geqq n$, then $A = p(N) = 0$ which is reflexive.

For the converse, if A is reflexive and $p'(0) \neq 0$, then by Lemma 7 we have $N \in \mathbf{C}[A]$, and so N is reflexive.

It is now straightforward to translate this result to the case $A = p(C)$ where C is a general linear transformation. Essentially one considers the decomposition of C into its primary summands as in **[3]** or **[5]** and the induced decomposition of A. This we leave to the reader.

3. Commutative algebras. There are examples (see **[1**, p. 825**]**) of commutative algebras $\mathscr{A}$ each element of which is reflexive but such that $\mathscr{A}$ is not reflexive. The first such example occurs for $n = 4$, so we shall examine the cases $n = 2$ and $n = 3$ as the case $n = 1$ is trivial.

Let $\mathscr{A}$ be a commutative algebra of 2×2 matrices each element of which is reflexive. Then (cf. **[14]**) $\mathscr{A}$ can be simultaneously triangularized. If each $A \in \mathscr{A}$ has a single eigenvalue, then since each A is reflexive, we must have $A = \alpha I$ for some $\alpha \in C$. Therefore $\mathscr{A} = \{\alpha I | \alpha \in C\}$ and is reflexive. The other possibility is that some $A \in \mathscr{A}$ has (is similar to) the form $\left[\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right]$ with $\alpha \neq \beta$.
Then

$$A - \alpha I = \begin{bmatrix} 0 & 0 \\ 0 & \beta - \alpha \end{bmatrix} \in \mathscr{A},$$

whence $\left[\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right] \in \mathscr{A}$. Similarly, $\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right] \in \mathscr{A}$. In this case, then,

$$\mathscr{A} = \left[\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} : \alpha, \beta \in C\right].$$

So dim $\mathscr{A} = 2$. There are clearly $2 = n^2 - \dim \mathscr{A}$ linearly independent rank 1 annihilators of $\mathscr{A}$, so by Longstaff's criterion (**[11]**) $\mathscr{A}$ is reflexive. Thus any algebra of 2×2 matrices each element of which is reflexive is a reflexive algebra.

THEOREM 9. *Let $\mathscr{A}$ be a commutative algebra of* 3×3 *matrices. Assume each $A \in \mathscr{A}$ is reflexive. Then $\mathscr{A}$ is reflexive iff and only if all $A \in \mathscr{A}$ can be simultaneously diagonalized or* dim $\mathscr{A} = 2$.

PROOF. Recall that we are assuming $I \in \mathscr{A}$. Thus if dim $\mathscr{A} = 2$ there is an $A \in \mathscr{A}$ such that $\{I, A\}$ is a basis for $\mathscr{A}$. Then $\mathscr{A} = \mathbf{C}[A]$ is reflexive.

Suppose $\mathscr{A}$ can be diagonalized. We assume all $A \in \mathscr{A}$ to be diagonal and note that dim $\mathscr{A} \leqq 3$. The cases dim $\mathscr{A} = 1$ or 2 are trivial, so we assume dim $\mathscr{A} = 3$. Then there are $6 = 3^2 - \dim \mathscr{A}$ linearly independent rank 1 annihilators of $\mathscr{A}$, whence $\mathscr{A}$ is reflexive.

Conversely suppose $\mathscr{A}$ is reflexive. Since $\mathscr{A}$ is commutative, the algebra can be simultaneously triangularized and block diagonalized with a single eigenvalue in each block [**14**, p. 4]. Assume $\mathscr{A}$ is in this form. We shall argue on the number of distinct eigenvalues possible for an $A \in \mathscr{A}$. If some $A \in \mathscr{A}$ has three distinct eigenvalues, then it has three diagonal blocks and so must every other matrix in $\mathscr{A}$. That is, $\mathscr{A}$ is in diagonal form. If no element of $\mathscr{A}$ has three distinct eigenvalues but same $A \in \mathscr{A}$ has two, then A must be of the form

$$\begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \oplus [\gamma],\ [\alpha] \oplus \begin{bmatrix} \gamma & \beta \\ 0 & \gamma \end{bmatrix}, \text{ or diag } (\alpha, \alpha, \gamma)$$

where $C \oplus D$ is the direct sum of matrices $C \oplus D = \left[\begin{smallmatrix} C & 0 \\ 0 & D \end{smallmatrix}\right]$. But each $A \in \mathscr{A}$ is reflexive so $\beta = 0$ and $\mathscr{A}$ is again in diagonal form.

Finally suppose each $A \in \mathscr{A}$ has only one eigenvalue. Since $\mathscr{A}$ is triangular, $\dim \mathscr{A} \leqq 4$. If $\dim \mathscr{A} = 1$, then $\mathscr{A} = \{\alpha I \mid \alpha \in C\}$, and if $\dim \mathscr{A} = 2$ the conclusion is satisfied. We shall show that dimensions 3 and 4 are not possible. Let

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathscr{A} \subseteq \operatorname{span}\{I, E_1, E_2, E_3\}$. If $\dim \mathscr{A} = 4$, equality holds and $E_1 + E_3 \in \mathscr{A}$. But $E_1 + E_3$ is not reflexive. Now suppose $\dim \mathscr{A} = 3$. Then any $A \in \mathscr{A}$ is of the form

$$(*) \qquad A = \begin{bmatrix} \lambda & \alpha & \beta \\ 0 & \lambda & \gamma \\ 0 & 0 & \lambda \end{bmatrix}.$$

I $\alpha \gamma \neq 0$, then $(A - \lambda I)^2 \neq 0$, so $\mathscr{A}$ is spanned by I, B, and E_2 where

$$B = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix}.$$

In fact, $\mathscr{A} = \mathbf{C}[B]$. But the elementary divisor of B is of degree 3, whence B is not reflexive, a contradiction. Thus either $\alpha = 0$ or $\gamma = 0$. Without loss of generality we assume $\gamma = 0$. Since $3 = \dim \mathscr{A} = \dim \operatorname{span}\{I, E_1, E_2\}$, then $\mathscr{A} = \operatorname{span}\{I, E_1, E_2\}$. (It is easily checked that the set of all matrices of the form (*) with $\gamma = 0$ is closed under addition and multiplication.) Since for any $A \in \mathscr{A}$, $A - \lambda I$ with λ the eigenvalue of A is nilpotent of order two, then the elementary divisors of A are of degrees 2 and 1. Thus each $A \in \mathscr{A}$ is reflexive. We establish that $\mathscr{A}$ is not reflexive by showing it does not have $6 = n^2 - \dim \mathscr{A}$ linearly independent rank one annihilators. We can immediately find five. Let E_{ij} be the matrix with a 1 in position (i, j) and zeros elsewhere and let

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then $\mathscr{R} = \{E_{21}, E_{23}, E_{31}, E_{32}, E\}$ is a set of linearly independent rank 1 annihilators of $\mathscr{A}$ under the inner product $\langle A, B\rangle = \operatorname{tr} A^*B$. Suppose there were a sixth, say $R = [r_{ij}]$. If we use the facts that rank $R = 1$ and that $0 = \langle E_1, R\rangle = \langle E_2, R\rangle = \langle I, R\rangle$, then a direct computation shows that R is a linear combination of the elements of $\mathscr{R}$. Thus $\mathscr{A}$ is not refiexive and the result is established.

If $\mathscr{A}$ is an algebra of n x n matrices, the commutant (or centralizer) of $\mathscr{A}$ is the algebra

$$\mathscr{A}' = \{B \mid AB = BA \text{ for all } A \in \mathscr{A}\}.$$

If $\mathscr{A} = \mathscr{A}'$ we call $\mathscr{A}$ a maximal commutative algebra. In order for a maximal commutative algebra to be reflexive it must in some sense be relatively spare. To make this precise we first introduce some terms. If $\mathscr{A}$ is a commutative subalgebra of Hom V, then there is a decomposition of V into $\mathscr{A}$ indecomposable subspaces $V = V_1 \oplus \cdots \oplus V_r$ of dimensions $n_1, \ldots, n_r$, respectively, with respect to which $\mathscr{A}$ has a representation $\mathscr{A} \cong \mathscr{T}_1 \oplus \cdots \oplus \mathscr{T}_r$ where the elements $T_j \in \mathscr{T}_j$ are $n_j \times n_j$ upper triangular matrices with constant diagonal. We shall call the ideals $\mathscr{T}_j$ the indecomposable components of $\mathscr{A}$. If N is a nilpotent matrix we say that N is nilpotent of class p if $N^p = 0$ but $N^{p-1} \neq 0$. If $\mathscr{N}$ is an algebra of nilpotent matrices we say $\mathscr{N}$ is nilpotent of class p if p is the largest nilpotency class of the elements of $\mathscr{N}$. If $\mathscr{T}$ is an algebra of upper triangular matrices (as in the preceding) of the form $\mathbf{C}I + \mathscr{N}$ where $\mathscr{N}$ is the subalgebra of $\mathscr{T}$ of nilpotent matrices, we call p the nilpotency class of $\mathscr{T}$ if $\mathscr{N}$ is of class p.

THEOREM 10. *Let $\mathscr{A}$ be a maximal commutative algebra of $n \times n$ matrices with indecomposable $\mathscr{T}_1, \ldots, \mathscr{T}_r$ of orders $n_1, \ldots, n_r$ respectively. If some $\mathscr{T}_j$ is nilpotent of class n_j, where $n_j > 1$, $\mathscr{A}$ is not reflexive. Further, if some $\mathscr{T}_k$ is nilpotent of class $n_k - 1$ with $n_k > 3$, then $\mathscr{A}$ is not reflexive.*

PROOF. If $\mathscr{A} = \mathscr{T}_1 \oplus \cdots \oplus \mathscr{T}_r$, then $\mathscr{A}$ is reflexive iff and only if each $\mathscr{T}_k$ is reflexive. Further if $\mathscr{T} = \mathscr{A}$, then, for each k, $\mathscr{T}_k = \mathscr{T}_k'$ (in M_{n_k}). So it suffices to show that if some $\mathscr{T}_j$ is of nilpotency class $n_j > 1$ or $n_j - 1 > 2$, then $\mathscr{T}_j$ contains a nonreflexive element.

First suppose $\mathscr{T}_j$ is of nilpotency class $n_j > 1$. Then $\mathscr{T}_j$ is similar to the algebra $\mathbf{C}I + \mathscr{N}$ where $\mathscr{N}$ is generated by

$$U = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

(cf. [**16**], Theorem 8, p. 51). But the elementary divisor of U is of degree $n_j > 1$, so U is not reflexive.

Next suppose $\mathscr{T}_j$ is of class $n_j - 1 > 2$. Again from [**16**, Theorem 1, p. 100] the nilpotent sublagebra of $\mathscr{T}_j$ contains the matrix $N = E_{21} + \cdots + E_{n_j-1 n_j-2}$, which is not reflexive.

For an algebra $\mathscr{A}$ we let $\mathscr{J}$ denote the (Jacobson) radical of $\mathscr{A}$. When

$\mathscr{A}$ is commutative we draw attention to this assumption by using $\mathscr{W}$ to denote the radical which consists of all nilpotent elements of $\mathscr{A}$. For any algebra $\mathscr{A}$ we have $\mathscr{A} = \mathscr{J} \oplus \mathscr{S}$ where $\mathscr{S}$ is a semisimple subalgebra and the sum is a vector space sum. It is easy to see that Let $\mathscr{A}$ = Lat $\mathscr{J} \cap$ Lat $\mathscr{S}$. So if $\mathscr{A}$ is reflexive, then

$$\mathscr{A} = \text{Alg Lat } \mathscr{A} \supseteq \text{Alg Lat } \mathscr{J} \vee \text{Alg Lat } \mathscr{S} = \text{Alg Lat } \mathscr{J} \vee \mathscr{S} \supseteq \mathscr{A}$$

Thus $\mathscr{A} \supseteq$ Alg Lat $\mathscr{J}$. An open question is whether this condition is also sufficient for reflexivity. In the commutative case it is.

PROPOSITION 11. *Let $\mathscr{A}$ be a commutative algebra with radical $\mathscr{W}$, and suppose that $\mathscr{A} \supseteq$ Alg Lat $\mathscr{W}$. Then $\mathscr{A}$ is reflexive.*

PROOF. Since $\mathscr{A}$ is commutative, we can simultaneously block triangularize $\mathscr{A}$ so that $\mathscr{A} = \mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_p$ where, for each j, we have

$$\mathscr{A}_j = \left\{ \begin{bmatrix} \rho & & * \\ & \ddots & \\ 0 & & \rho \end{bmatrix} : \rho \in \mathbf{C} \right\}$$

(cf. [**8**] page 134 or [**14**]). Let $\mathscr{S}_j$ and $\mathscr{W}_j$ be the semisimple subalgebra and the radical of $\mathscr{A}_j$ respectively. If $\mathscr{A} \supseteq$ Alg Lat $\mathscr{W}$, then $\mathscr{A}_j \supseteq$ Alg Lat $\mathscr{W}_j$. Clearly Lat $\mathscr{W}_j \cap$ Lat $\mathscr{S}_j$ = Lat $\mathscr{W}_j$, so we have

$$\text{Alg Lat } \mathscr{A}_j = \text{Alg}(\text{Lat } \mathscr{W}_j \cap \text{Lat } \mathscr{S}_j) = \text{Alg Lat } \mathscr{W}_j \subseteq \mathscr{A}_j.$$

Thus each $\mathscr{A}_j$ is reflexive and the proof is finished by Lemma 3.

4. Subspace Lattices. In [7] Harrison and Longstaff showed that a complemented subspace lattice which satisfies an additional condition is reflexive. Their proof involves continuous geometries. Here we consider the algebraic implications of the assumption on the lattice.

THEOREM 12. *Let $\mathscr{L}$ be a complemented lattice of subspaces. Then $\mathscr{A}$ = Alg $\mathscr{L}$ is semisimple.*

PROOF. Since every complemented modular lattice is relatively complemented ([**4**], 31], then $\mathscr{L}$ is both relatively complemented and atomic. Thus we can find atoms $H_1, \ldots, H_m$ of $\mathscr{L}$ such that no one of them is contained in the sum of the remaining and such that if $V_j = H_1 \oplus \cdots \oplus H_j$, then $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_m = V$ is a composition series for V as a (right) $\mathscr{A}$ module where $\mathscr{A}$ = Alg. $\mathscr{L}$. Since each V_j has an invariant complement, we obtain a block diagonal form for $\mathscr{A}$: if $A \in \mathscr{A}$, then $A = A_1 \oplus \cdots \oplus A_m$. Let $\mathscr{A}_j$ be the restriction of $\mathscr{A}$ to H_j. By Burnisde's theorem (e.g., [**8**] p. 276) $\mathscr{A}_i$ is isomorphic to M_{n_i} where $n_i = \dim H_i$. Using standard techniques (cf. [**17** pp. 4–5]) we can write $\mathscr{A} = \mathscr{B}_1 \oplus$

$\cdots \oplus \mathscr{B}_t$ where each $\mathscr{B}_j$ is of the form $\mathscr{B}_j = \{B_j \oplus \cdots \oplus B_j: B_j \in M_{n_j}\}$. Thus $\mathscr{A}$ is semisimple.

REMARK. By Lemmas 1 and 2

$$\text{Lat Alg } \mathscr{L} = \text{Lat } \mathscr{B}_1 \oplus \cdots \oplus \text{Lat } \mathscr{B}_t.$$

Thus $\mathscr{L}$ is reflexive if it contains the direct sum of the lattices of the $\mathscr{B}_j$. However, the sets Lat $\mathscr{B}_j$ are well known. Their atoms are the graph subspaces of the corresponding vector spaces (cf. [6 pp. 261–2] or [7 p. 1023]). The result of Harrison and Longstaff [7] now follows by a direct argument from the representation of Alg $\mathscr{L}$ obtained in Theorem 12.

BIBLIOGRAPHY

1. E.A. Azoff, *K-Reflexivity in finite dimensional spaces*, Duke Math. J. **40** (1973), 821–830.

2. G.P. Barker, L.Q. Eifler, and T.P. Kezlan, *A non-commutative spectral theorem*, Linear Algebra and its Appl. **20** (1978), 95–100.

3. L. Brickman and P.A. Fillmore, *The invariant subspace lattice of a linear transformation*, Canad. J. Math. **19** (1967), 810–822.

4. P. Crawley and R.P. Dilworth. *Algebraic Theory of Lattices*. Prentice Hall, Inc. Inglewood Cliffs, N.J., 1973.

5. J.A. Deddens and P.A. Fillmore, *Reflexive linear transformations*, Linear Algebra and its Appl. **10** (1975), 89–93.

6. P.R. Halmos, *Reflexive lattices of subspaces*, J. London Math. Soc., **4** (1971), 257–263.

7. K.J. Harrison and W.E. Longstaff, *Reflexive subspace lattices in finite dimensional Hilbert spaces*, Indiana U. Math. J. **26** (1977), 1010–1025.

8. N. Jacobson. *Lectures in Abstract Algebra*, Vol. II. D. Van Nostrand Co., Inc. Princeton, N.J., 1953.

9. R.E. Johnson, *Distinguished rings of linear transformations*, Trans. Amer. Math. Soc. III, (1964), 400–412.

10. P. Lancaster. *Theory of Matrices*. Academic Press, New York, 1969.

11. W.E. Longstaff, *On the operation Alg Lat in finite dimensions*, Linear Algebra and its Appl. **27** (1979), 27–29.

12. A.I. Mal'cev. *Foundations of Linear Algebra*. W.H. Freeman and Company, New York, 1963.

13. M. Marcus and H. Minc. *Introduction to Linear Algebra*. Macmillan Company, New York, 1965.

14. M. Rosenlicht, *Initial results in the theory of linear algebraic groups, in M.A.A. Studies in Mathematics. vol.* 20: *Studies in Algebraic Geometry*, Washington, D.C., 1980.

15. W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill Book Company, New York, 1964.

16. D.A. Suprunenko and R.I. Tyshkevich. *Commutative Matrices*. Academic Press, New York, 1968.

17. J.F. Watters, *Block triangularization of algebras of matrices*, Linear Algebra and its Appl., **32** (1980), 3–7.

University of Missouri-K.C., Kansas City, Missouri 64110 and North Carolina State University, Raleigh, North Carolina 27650
University of Missouri-K.C., Kansas City, Missouri 64110