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SOME SUBORDINATION RELATIONS

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ABSTRACT. If $P_n(z) = \sum_{k=1}^{n} a_k z^k$, $a_1 = 1$, $P_{n+1}(z) = P_n(z) + a_{n+1}z^{n+1}$, and if $P_{n+1}(z)$ is univalent for $|z| < 1$, then $P_n(z/2) \prec P_{n+1}(z)$, $n \geqq 1$, and the constant 1/2 is best possible. If $f(z) = \sum_{k=1}^{\infty} a_k z^k$, $a_1 = 1$, is analytic and univalent for $|z| < 1$, $s_n(z) = \sum_{k=1}^{n} a_k z^k$, then $s_n(z/8) \prec s_{n+1}(z/4) \prec f(z)$, $n \geqq 1$ (and the constant 1/8 is best possible), and $s_{n+1}(z/8) \prec s_n(z/4) \prec f(z)$.

Let γ denote the disc $|z| < 1$ and let S denote the class of functions $f(z)$ analytic and univalent in γ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$. For a function $g(z)$ analytic in γ, if $g(0) = 0$ and $g(z)$ is subordinate to $f(z)$, we write $g(z) \prec f(z)$. Let $P_n(z) = \sum_{k=1}^{n} a_k z^k$, $a_1 = 1$, and let $P_{n+1}(z) = P_n(z) + a_{n+1}z^{n+1}$.

THEOREM 1. *If* $P_{n+1}(z) \in S$, *then*

$$P_n(z/2) \prec P_{n+1}(z), \qquad n \geqq 1, \tag{1}$$

and the constant 1/2 *is best possible.*

The fact that the constant 1/2 is best possible is shown by the function $P_2(z) = z + (1/2)z^2 \in S$. We deduce Theorem 1 from the following more precise form.

THEOREM 2. *If* $P_{n+1}(z) \in S$ *then*

$$P_n(z/2) \prec P_{n+1}(z), \quad n = 1, 2,$$
$$P_n(\alpha_n z) \prec P_{n+1}(z), \quad n \geqq 3,$$

where α_n *is the root of the equation*

$$\frac{\alpha^{n+1}}{n+1} - \frac{1}{4}\left(\frac{1-\alpha}{1+\alpha}\right)^2 = 0$$

in the interval (0, 1). $\alpha_n > 1/2$ *for all* $n \geqq 3$, α_n *increases with* n *and* $\lim_{n\to\infty}\alpha_n = 1$.

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To prove this theorem we require two well-known inequalities which we state as lemmas.

LEMMA 1. *If $f(z) \in S$, then, for all real θ,*

$$|f'(re^{i\theta})| \geqq \frac{1-r}{(1+r)^3}, \quad 0 \leqq r < 1.$$

LEMMA 2. *If $\sum_{k=1}^{n} b_k z^k \in S$, then $|b_n| \leqq 1/n$.*

For Lemma 1 see, for example, [2]. Lemma 2 follows from the fact that, with the given hypothesis, all the zeros of the derivative $\sum_{k=1}^{n} kb_k z^{k-1}$ lie outside γ.

PROOF OF THEOREM 2. With $n = 1$, since $|a_2| \leqq 1/2$ by Lemma 2, for $|z| = 1$ we have $|z + a_2z^2| \geqq 1 - |a_2| \geqq 1/2$, which implies that $(1/2)z \prec z + a_2z^2$.

In the case $n = 2$, let λ_1, λ_2 be the zeros of $P_3'(z)$. Then $P_3'(z) = 3a_3 (z - \lambda_1)(z - \lambda_2)$, $|\lambda_1| \geqq 1$, $|\lambda_2| \geqq 1$, and for $0 \leqq r < 1$ and all real θ,

$$(2) \qquad |P_3'(re^{i\theta})| \geqq 3\,|a_3|(1 - r)^2.$$

Let Δ now denote the image of γ under the mapping $w = P_3(z/2)$, let D denote the image of γ under the mapping $w = P_3(z)$, and let d be the distance of the boundary of Δ from the boundary of D. Then by (2),

$$(3) \qquad d \geqq \int_{1/2}^{1} \min_{\theta} |P_3'(re^{i\theta})|\, dr \geqq |a_3|/2^3.$$

If $a_3 = 0$ then the consequence $P_2(z/2) \prec P_3(z)$ is trivial. If $a_3 \neq 0$ then it follows from (3) and $|a_3(z/2)^3| < |a_3|/2^3$.

Let

$$h_n(\alpha) = \frac{\alpha^{n+1}}{n+1} - \frac{1}{4}\left(\frac{1-\alpha}{1+\alpha}\right)^2.$$

Then $h_n(0) = -1/4$, $h_n(1) = 1/(n + 1)$, and $h_n(\alpha)$ increases with α. It follows that there is exactly one solution $\alpha = \alpha_n$ of the equation $h_n(\alpha) = 0$. Also, since $h_n(\alpha)$ is a decreasing function of n for fixed $\alpha (0 < \alpha < 1)$, it is clear that α_n increases with n and $\lim_{n\to\infty} \alpha_n = 1$. In the case $n \geqq 3$, let Δ, D denote the images of γ under the mappings $w = P_{n+1}(\alpha_n z)$, $w = P_{n+1}(z)$, respectively, and let d be the distance of the boundary of Δ from the boundary of D. Then by Lemma 1,

$$(4) \qquad d \geqq \int_{\alpha_n}^{1} \min_{\theta} |P_{n+1}'(re^{i\theta})|\, dr \geqq \frac{1}{4}\left(\frac{1-\alpha_n}{1+\alpha_n}\right)^2.$$

By Lemma 2 we have

$$|a_{n+1}(\alpha_n z)^{n+1}| < \frac{\alpha_n^{n+1}}{n+1} = \frac{1}{4}\left(\frac{1-\alpha_n}{1+\alpha_n}\right)^2,$$

and the rest of the theorem now follows from (4).

The relation (1) of Theorem 1 results from the fact that $\alpha_n > 1/2, n \geqq 3$.

THEOREM 3. *If* $P_n(z) \in S$ *and* $|a_{n+1}| \leqq 2/9$, *then* $P_{n+1}(z/2) \prec P_n(z)$, $n \geqq 1$.

PROOF. The case $n = 1$ is trivial. By an argument similar to that used in the proof of Theorem 2 for the case $n \geqq 3$, it is sufficient to note that, for $n \geqq 2$,

$$|a_{n+1}|/2^{n+1} \leqq |a_{n+1}|/8 \leqq 1/36.$$

We remark that we have not attempted to prove a more precise form of this result, but it is clear that for a conclusion of the form $P_{n+1}(\beta z) \prec P_n(z)$ some restriction on the size of $|a_{n+1}|$ is necessary.

Our last theorem indicates a reciprocal subordination relation between the successive partial sums of the Taylor series of a univalent function.

THEOREM 4. *If* $f(z) = \sum_{k=1}^{\infty} a_k z^k \in S$ *and* $s_n(z) = \sum_{k=1}^{n} a_k z^k$, *then*

$$s_n(z/8) \prec s_{n+1}(z/4) \prec f(z), \tag{5}$$

$$s_{n+1}(z/8) \prec s_n(z/4) \prec f(z), \tag{6}$$

for $n \geqq 1$. *The constants* $1/4$ *in* (6) *and* $1/8$ *in* (5) *are best possible.*

PROOF. It is known that if $f(z) \in S$, then $s_n(z/4)$ is univalent [4], $s_n(z/4) \prec f(z)$ for all n and the constant $1/4$ is sharp. [3] Statement (5) now follows from Theorem 1. The case $f(z) = z(1 - z)^{-2}$, $n = 1$, shows that the constant $1/8$ cannot be increased. To prove (6), we note first that, since $|a_2| \leqq 2$ (see, for example, [2]), $|s_2(z/8)| \leqq 5/32 < 1/4$ in γ. Next, by Theorem 3, for $n \geqq 2$, it is sufficient to show that

$$|a_{n+1}|/4^n \leqq 2/9. \tag{7}$$

For $n = 2$ and 3, (7) follows from the inequalities $|a_3| \leqq 3$ (see, for example, [2]), $|a_4| \leqq 4$ (see, for example, [1]). Finally, by Lemma 2, since $4s_{n+1}(z/4) \in S$, (7) for the case $n \geqq 4$ follows from the inequality $4|a_{n+1}|/4^{n+1} \leqq 1/(n + 1)$.

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