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HEIGHT ONE SEPARABLE ALGEBRAS OVER COMMUTATIVE RINGS

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ABSTRACT. In this paper we define an R-algebra S to be height one separable over R (a commutative ring) if S is separable at each localization at a height one prime ideal of R. We prove some general properties of height one separability and give some examples of non-separable, height one separable extensions. It is also shown that if S is an integrally closed domain and R is the fixed subring of G-invariant elements of S, for some finite group G of automorphisms of S, and if each localization of R at a height 1 prime ideal in R is Noetherian, then S is a height one Galois extension (i.e., each localization at a height one prime ideal of R yields a Galois extension) if and only if S is unramified at each minimal prime ideal in S.

Introduction. In [2], Auslander and Buchsbaum characterize separability for a Noetherian ring S over a base ring R in terms of ramification of prime ideals in S. They prove that, with rather general assumptions, S is R-separable if and only if each maximal ideal of S is unramified. If more conditions are put on R and S, namely that R be an integrally closed Noetherian domain and S the integral closure of R in a separable field extension of the quotient field of R, with S projective as an R-module, they achieve the following result: S is R-separable if and only if each prime ideal of height 1 in S is unramified. We will give examples here to show that this result can fail to hold if the Noetherian restriction on R is removed or if S is not R-projective. The setting here is rather closely related to the problem of the purity of the branch locus (see [1]). One of the examples here will show that if the base ring R is a local ring which is not regular, then purity may indeed fail to hold for R.

We will focus our attention here on the prime ideals of the base ring R, and call S *height 1 separable* over R if S is separable at each localization at a height 1 prime ideal of R. We establish some general properties of height 1 separable algebras and give several examples of height 1 separable algebras which are not separable. In §3 we examine the situation where

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R and S are integrally closed domains and G is a finite group of automorphisms of S such that the ring S^G of G-invariant elements of S is precisely R, and such that R_p is Noetherian for each prime ideal p in R of rank 1. We show that S_p is a Galois extension of R_p, for each prime p in R of rank 1, if and only if S is unramified at each minimal prime in S.

All rings in this paper are commutative with identity. The base ring will always be denoted by R, and all unadorned tensors are taken over R.

1. Height 1 separability. For a ring R, let $X'(R)$ denote the set of all prime ideals in R having rank (height) $\leqq 1$. If S is an R-algebra and Q is in $X'(S)$, then by $Q \cap R$ we mean the contraction of Q to R. We remark that if Going Down holds for the R-algebra S (e.g., if S is R-flat), then $Q \cap R$ is in $X'(R)$ if Q is in $X'(S)$. We call this property NBU (for No Blowing Up, as in **[4]**.)

DEFINITION 1.1. An R-algebra S is *height 1 separable* over R if S_p is a separable R_p-algebra for all p in $X'(R)$.

We prove now some general facts about height 1 separability.

PROPOSITION 1.2. *Suppose that R_1 and R_2 are R-algebras and that S_i is a height* 1 *separable R_i-algebra for $i = 1, 2$. Assume that $R_1 \otimes R_2$ satisfies* NBU *over both R_1 and R_2 (where $R_1 \otimes R_2$ is an R_1-algebra by the map $r \mapsto r \otimes 1$ and an R_2-algebra by $r \mapsto 1 \otimes r$). Then $S_1 \otimes S_2$ is a height* 1 *separable $R_1 \otimes R_2$ algebra.*

PROOF. Let p be in $X'(R_1 \otimes R_2)$ and let $p_i = p \cap R_i$. Then $(S_i)_{p_i}$ is separable over $(R_i)_{p_i}$, and so $(S_1)_{p_1} \otimes (S_2)_{p_2}$ is separable over $(R_1)_{p_1} \otimes (R_2)_{p_2}$. Localizing further we have $((S_1)_{p_1} \otimes (S_2)_{p_2})_p$ is separable over $((R_1)_{p_1} \otimes (R_2)_{p_2})_p$. Since $((S_1)_{p_1} \otimes (S_2)_{p_2})_p = (S_1 \otimes S_2)_p$, and likewise for $(R_1 \otimes R_2)_p$, the result follows.

COROLLARY 1.3. *If S_1 and S_2 are height* 1 *separable R-algebras, then so is $S_1 \otimes S_2$.*

PROOF. Let $R_1 = R_2 = R$ in (1.2).

COROLLARY 1.4. *If S is a height* 1 *separable R-algebra and T is an R-algebra satisfying* NBU *over R, then $T \otimes S$ is height* 1 *separable over T.*

PROOF. Let $S_2 = S$, $S_1 = T$, $R_1 = T$ and $R_2 = R$ in (1.2).

COROLLARY 1.5. *If I is an ideal of R such that R/I satisfies NBU over R and if S is a height* 1 *separable R-algebra, then $R/I \otimes S = S/IS$ is height* 1 *separable over R/I.*

PROOF. Follows directly from (1.4).

We remark that the hypothesis that R/I satisfy NBU over R is fairly restrictive. If I contains a prime ideal of height $\geqq 1$ and R/I has a prime ideal P of height 1, then NBU is not satisfied: if P_0, P_1 are primes in R with $P_0 \subsetneqq P_1 \subseteqq I$, then $P_0 \subseteqq P_1 \subsetneqq P \cap R$.

PROPOSITION 1.6. *If I is an ideal of S and S is a height 1 separable R-algebra, then S/I is a height 1 separable R-algebra.*

PROOF. If p is in $X'(R)$, then $(S/I)_p = S_p/I_p$, and the result follows since S_p is R_p-separable.

COROLLARY 1.7. *If S is a height 1 separable R-algebra and $f\colon S \to S'$ is a surjective homomorphism, then S' is a height 1 separable R-algebra.*

PROOF. Immediate, from (1.6).

The next proposition deals with the R/I'-algebra S/I, where I is an ideal of S and I' is an ideal of R contained in $I \cap R$. But first we prove two lemmas.

LEMMA 1.8. *Let I be an ideal of R and $p \supseteqq I$ a prime ideal. Then $R_p/I_p = (R/I)_{p/I}$.*

PROOF. It is straightforward to check that the map from R_p to $(R/I)_{p/I}$ given by $r/s \to (r + I)/(s + I)$, for r in R and s in $R - p$, is a surjection with kernel I_p.

LEMMA 1.9. Let I be any ideal in R and let p be a prime ideal in R/I. Denote $p \cap R$ by Q. Then, if M is any R-module with $IM = 0$, $M_Q = M_p$.

PROOF.

$$\begin{aligned} M_Q &= M \otimes R_Q = M/IM \otimes R_Q = (M \otimes R/I) \otimes R_Q \\ &= M \otimes R_Q/I_Q = M \otimes (R/I)_{Q/I} = M \otimes (R/I)_p = M_p. \end{aligned}$$

PROPOSITION 1.10. *Let S be a height 1 separable R-algebra and I an ideal in S. Let I' be an ideal of R contained in $I \cap R$ such that R/I' satisfies NBU as an R-algebra. Then S/I is a height 1 separable R/I'-algebra.*

PROOF. Let p be in $X'(R/I')$; then $Q = p \cap R$ is in $X'(R)$. By (1.6), S/I is height 1 separable over R, and, hence $(S/I)_Q$ is a projective $(S/I \otimes S/I)_Q$ -module. Then, by (1.9) it follows that $(S/I)_p$ is a projective $(S/I \otimes_{R/I'} S/I)_p$-module, and is therefore a separable R/I'-algebra.

Next, we have a version of transitivity.

PROPOSITION 1.11. *Let T be a height 1 separable S-algebra, finitely generated as an S-module, where S is a height 1 separable extension of R and in-*

tegral over R. Then T is a height 1 separable R-algebra.

PROOF. Let p be in $X'(R)$. We claim first that T_p is a height 1 separable S_p-algebra. For Q in $X'(S_p)$, let Q' be $Q \cap S$. Then Q' is in $X'(S)$, and we have that $T_{Q'}$ is a separable $S_{Q'}$-algebra. Localizing further, we see that $(T_{Q'})_p = (T_{Q'} \otimes R_p)$ is separable over $(S_{Q'})_p$ and the claim follows since $(T_{Q'})_p = (T_p)_Q$ and $(S_{Q'})_p = (S_p)_Q$.

Now suppose M is a maximal ideal of S_p, for p in $X'(R)$. Then $M \cap R_p = pR_p$. Since $ht(M) \leqq ht(M \cap R_p) \leqq 1$, we have that M is in $X'(S_p)$. It follows that T_p is separable over S_p([3], p. 72) and therefore separable over R_p. Hence T is height 1 separable over R.

2. Polynomial extensions.

DEFINITION 2.1. A monic polynomial $f(x)$ in $R[x]$ is a *height* 1 *separable polynomial* if $R[x]/f(x))$ is a height 1 separable R-algebra.

We recall that if R is a commutative ring (with 1), a monic polynomial f in $R[x]$ is separable if and only if there exist g and h in $R[x]$ such that $gf + hf' = 1$, i.e., if and only if the ideal generated by f and f', (f, f'), is precisely $R[x]$. The following proposition is a straightforward generalization to the height 1 case.

PROPOSITION 2.2. *A monic polynomial f in $R[x]$ is height* 1 *separable if and only if $(f_p, f_p') = R_p[x]$ for each p in $X'(R)$, where f_p denotes the polynomial f with its coefficients considered elements in R_p.*

COROLLARY 2.3. *If f is a height* 1 *separable polynomial in $R[x]$ and $S = R[x]/(f)$, then whenever c is a root of f in S, $f'(c)$ is a unit in S_p, for all p in $X'(R)$.*

PROOF. Immediate from (2.2).

Let $f(x)$ be a height 1 separable polynomial in $R[x]$, and let $S = R[x]/(f)$. Let $a = x + (f)$ in S. Then a is a root of f in S, and, since the leading coefficient of $x - a$ is a unit, the Euclidean algorithm is valid here. Hence, $x - a$ divides f in $S[x]$. Let

$$\begin{aligned} f(x) &= (x-a)(b_0 + \cdots + b_{n-2}x^{n-2} + x^{n-1}) \\ &= (b_0x + \cdots + b_{n-2}x^{n-1} + x^n) - (ab_0 + \cdots + ab_{n-2}x^{n-2} + ax^{n-1}) \\ &= x^n + (b_{n-2} - a)x^{n-1} + (b_{n-3} - ab_{n-2})x^{n-2} \\ &\qquad + \cdots + (b_0 - ab_1)x - ab_0. \end{aligned}$$

Since $f(a) = 0$, we have the following.

$$a^n = \sum_{i=0}^{n-1} (ab_i - b_{i-1})a^i, \tag{1}$$

where $b_{n-1} = 1$ and $b_{-1} = 0$. Recall that if T is a separable R-algebra, then the separability idempotent for T is the unique idempotent e in $T \otimes T$ such that e maps to 1 under the multiplication map $T \otimes T \to T$ and $(1 \otimes t)e = (t \otimes 1)e$, for all t in T. We are going to describe the separability idempotent for S_p in terms of a, b_i, and $f'(a)$, for p in $X'(R)$.

PROPOSITION 2.4. *In the setting described above, the element*

$$e = \sum_{i=0}^{n-1} a^i \otimes \frac{b_i}{f'(a)}$$

in $S_p \otimes S_p$, for p in $X'(R)$, is the separability idempotent for S_p.

PROOF. If $m\colon S_p \otimes S_p \to S_p$ is the multiplication map, then we have

$$m(e) = \sum_{i=0}^{n-1} \frac{a^i b_i}{f'(a)} = \frac{f'(a)}{f'(a)} = 1.$$

To show that $(1 \otimes s)e = (s \otimes 1)e$ for all s in S_p, we need only show that $(1 \otimes a)e = (a \otimes 1)e$, since S_p is generated over R_p by a. We have the following:

$$(1 \otimes a)e = \sum_{i=0}^{n-1} a^i \otimes \frac{ab_i}{f'(a)}$$

and

$$(a \otimes 1)e = \sum_{i=0}^{n-1} a^{i+1} \otimes \frac{b_i}{f'(a)}.$$

From (1) we obtain

$$\begin{aligned} a^n \otimes \frac{1}{f'(a)} &= \left[\sum_{i=0}^{n-1} (ab_i - b_{i-1})\, a^i\right] \otimes \frac{1}{f'(a)} \\ &= \sum_{i=0}^{n-1} a^i \otimes \frac{ab_i - b_{i-1}}{f'(a)}, \end{aligned}$$

since the terms $ab_i - b_{i-1}$ are in R. Then,

$$\begin{aligned} (a \otimes 1)e &= \sum_{i=0}^{n-1} \left[a^i \otimes \frac{b_{i-1}}{f'(a)} + a' \otimes \frac{ab_i - b_{i-1}}{f'(a)} \right] \\ &= \sum_{i=0}^{n-1} a^i \otimes \frac{ab_i}{f'(a)} = (1 \otimes a)e. \end{aligned}$$

Thus, e is the separability idempotent.

The key to the above is the fact that if S_p is R_p-separable, then $f'(a)$ is a unit in S_p. If $f'(a)$ is not a unit in S, then S itself cannot be R-separable. If we let

$$e' = \sum_{i=0}^{n-1} a^i \otimes b_i,$$

an element of $S \otimes S$, then the above shows that it is still true that $(1 \otimes s)e' = (s \otimes 1)e'$ for all s in S, but it is no longer true that $\sum a^i b_i = 1$, since $\sum a^i b_i = f'(a)$. However, the element e' acts like a "pre-image" for the separability idempotent of each localization S_p, p in $X'(R)$, in the sense that the ideal generated by $m(e')$ in S_p, namely S_p itself, is the same ideal generated by $m(e)$, where m is the multiplication map, since $m(e') = f'(a)$ is a unit in S_p.

We point out that if the base ring R is an integrally closed Noetherian domain and $S = R[x]/(f)$ is height 1 separable over R, then S is a finite product of finitely generated projective height 1 separable R-algebras, each of which is therefore R-separable, and thus S itself is separable over R.

Next we give an example which is based on the following straightforward exercise in [**6**; #8, p. 114]. Let k be a field and let R' be the ring of all polynomials in $k[x, y]$ having no term in a power of x alone. Note that the ideal (y) is not prime in R', since $(xy)^2$ is in (y) but xy is not in (y). Let M be the prime ideal in R' consisting of all polynomials having constant term zero. Then M is minimal over (y). Further, $\operatorname{rank}(M) \geqq 2$ since M contains the zero ideal and the prime ideal of all polynomials with constant term equal to zero and no term in a power of y alone.

EXAMPLE 2.5. (A height 1 separable R-algebra which is not separable.) Using the notation in the above exercise, let $R = R'_M$. Then, MR_M is minimal over (y), and by localizing we have ensured that the element y cannot be contained in any prime in $X'(R)$, for the existence of such a prime would contradict the minimality of MR_M over (y). Let $f(t) = t^n - y$, where $1/n$ is in R, and let $S = R[t]/(f)$. Then S is a finitely generated projective R-module. Since y is not a unit in R, S is not R-separable [**5**; 2.4]. But, since y is not in any prime p in $X'(R)$, y is a unit in R_p, and so $S_p = R_p[t]/(f)$ is R_p-separable.

We also show now that the domain R in (2.5) is integrally closed. This will follow if we show that R' is integrally closed. Suppose g is in the quotient field of R' and integral over R'. Then since g is also in the quotient field of $k[x, y]$ and integral over $k[x, y]$, g is in $k[x, y]$. Write $g = g(x, y)$ as a polynomial in y:

$$g(x, y) = g_0(x) + g_1(x)y + \cdots + g_n(x)y^n.$$

Since $g_1(x)y + \cdots + g_n(x)y^n$ is in R', we see that $g_0(x)$ must be integral over R'. Thus, g_0 satisfies a monic polynomial in $R'[t]$:

$$g_0(x)^n + a_1(x, y)g_0(x)^{n-1} + \cdots + a_n(x, y) = 0,$$

where each a_i is in R'. Since each $a_i(x, y)$ contains no terms in powers of x alone, we see that $c_i = a_i(x, 0)$ must be an element of k, $i = 1, \ldots, n$. Thus

$$g_0(x)^n + c_1 g_0(x)^{n-1} + \cdots + c_n = 0.$$

This says that $g_0(x)$ is integral over the field k, but this is impossible unless $g_0(x)$ is a constant. Thus we have

$$g(x, y) = \text{constant} + g_1(x)y + \cdots + g_n(x)y^n$$

and we have that $g(x, y)$ is in R'. This shows that R' is integrally closed, and, hence, that $R = R'_M$ is also integrally closed.

We note that in the above example the ring R is not Noetherian, and we have a situation where the principal ideal theorem fails.

3. Height 1 Galois extensions. In this section we discuss height 1 Galois extensions and finish with some more examples of height 1 separable extensions which are not separable.

DEFINITION 3.1. Let S be an R-algebra and let G be a finite group of R-algebra automorphisms of S. Then S is a *height 1 Galois extension of R with group G* if S_p is a Galois extension of R_p for each p in $X'(R)$.

We remark that if g is an R-algebra automorphism of S and p is in $X'(R)$, then the map $g_p\colon S_p \to S_p$ given by $g_p(s/t) = g(s)/t$, for s in S and t in R-p, is an R_p-automorphism of S_p. If G is a finite group of R-algebra automorphisms of S, let G_p denote the set of induced R_p-algebra automorphisms of S_p. Since a finite group of automorphisms is "almost finite" as defined in [7], then ([7], 1.13) shows that $(S^G)_p = S_p^{G_p}$, for p in $X'(R)$.

We will now characterize height 1 Galois extensions in a special setting. Recall that a prime ideal Q in an R-algebra S is said to be unramified if $p = Q \cap R$ satisfies the following:

(1) $pS_Q = QS_Q$; and

(2) S_Q/pS_Q is a separable field extension of R_p/pR_p

Let R and S be integrally closed domains and G a finite group of automorphisms of S such that $S^G = R$, and suppose that, for each p in $X'(R)$, R_p is Noetherian. (This will be the case, for example, if S is a Krull domain.) Then S is integral over R. Since R is integrally closed, Going Down holds for S over R. If L is the quotient field of S then any element in L can be written as a quotient with numerator in S and denominator in R. Letting G act on L, we see that if K is the quotient field of R, then $K = L^G$, and so L is a Galois extension of K.

PROPOSITION 3.2. *With S and R as described above, S is a height* 1

Galois extension of R if and only if S is unramified at all the primes Q in $X'(S)$ with $ht(Q) = 1$.

PROOF. Suppose that S is height 1 Galois over R and let Q be a height 1 prime in S; let $p = Q \cap R$. Note that since $pS_Q \subseteqq QS_Q$ and $pS_Q \neq 0$, then pS_Q will equal QS_Q, since QS_Q is of height 1, provided that pS_Q is a prime ideal in S_Q. This follows from the separability of S_p over R_p, since then we have that S_p/pR_p is separable over the field R_p/pR_p and is therefore a finite product of fields. Then, S_Q/pS_Q is also a product of fields and hence a field itself, being a local ring, since it is just a further localization of S_p/pS_p. Hence, pS_Q is a prime ideal of S_Q and $pS_Q = QS_Q$. By the above we have also shown that S_Q/pS_Q is separable over R_p/pR_p. Thus we have that S is unramified over R at all the height 1 primes in $X'(S)$.

Conversely, suppose that S is unramified at the height 1 primes in $X'(S)$. Let p be in $X'(R)$. Then, since $(S_p)^G = (S^G)_p = R_p$, the quotient field of S_p is a Galois field extension of the quotient field of R_p, and hence a finite separable field extension. Thus S_p is a finitely generated R_p-module. If we can show that S_p/pS_p is separable over R_p/pR_p, it follows that S_p is R_p-separable.

Since R is integrally closed and S is integral over R such that the quotient field of S is a finite separable field extension of the quotient field of R, there are only a finite number of primes in S lying over p; denote these by $Q_1, \ldots, Q_n$. Each Q_i is in $X'(S)$, and we are given that $S_{Q_i}/Q_iS_{Q_i}$ is a separable field extension of R_p/pR_p, for each i. We will show that S_p/pS_p is R_p/pR_p-separable by showing that $S_p/pS_p = \Pi S_{Q_i}/Q_iS_{Q_i}$. For the moment replace R_p by R, pR_p by p, S_p by S, and the ideals Q_iS_p by Q_i. Then we still have $Q_i \cap R = p$, and, since p is maximal in R, Q_i is maximal in S. Further, if Q is any maximal ideal of S, then $Q \cap R = p$, and so Q must be one of the Q_i's. Thus, $Q_1, \ldots, Q_n$ are all the maximal ideals in S. Let $I = Q_1 \cap \cdots \cap Q_n$. Then $pS \subseteqq I$. Now, for each i, we have $(pS)_{Q_i} = (Q_i)_{Q_i} = (Q_1 \cap \cdots \cap Q_n)_{Q_i} = I_{Q_i}$. Hence, $pS = I$. By the Chinese Remainder Theorem we have $S/pS = \Pi S/Q_i$. Since

$$S/Q_i = S/Q_i \otimes_{S_{Q_i}} S_{Q_i} = S/Q_i \otimes_S S_{Q_i} = S_{Q_i}/Q_iS_{Q_i}$$

we see that

$$S/pS = \prod S_{Q_i}/Q_iS_{Q_i}.$$

Going back to our original notation, the above says that

$$S_p/pS_p = \prod S_{Q_i}/Q_iS_{Q_i}$$

(note that $(S_p)_{Q_i} = S_{Q_i}$). As we noted above, this shows that S_p is R_p-separable. Finally, since S_p is connected for each p in $X'(R)$, we see that

S is height 1 Galois over R ([3], p. 81). This completes the proof of the proposition.

We remark that if S and R are as in Example 2.5, then the first part of the proof of 3.2 shows that S is unramified at each height 1 prime ideal of S.

Recall that if R and S are domains with $R \subseteqq S$, and such that the quotient field of S, say L, is a finite extension of the quotient of R, then the complementary module $C(S/R)$ is defined to be the set of all elements x in L such that $\mathrm{tr}(xS) \subseteqq R$, where tr is the trace map, and the Dedekind different $D(S/R)$ is then defined to be the set of all x in L such that $xC(S/R) \subseteqq S$. As in [**4**; Chap. IV], we now consider the situation where S is a Krull domain and G is a finite group of automorphisms of S. Then, letting $R = S^G$, R is also Krull, and so both R and S are integrally closed domains. If L and K are the quotient fields of S and R, respectively, then $K = L^G$, and so L is a Galois extension of K. The following appears as Proposition 1.6.3 in [**4**; p. 84]: In the setting just described, S is unramified over R at p in $X'(S)$ if and only if p does not contain $D(S/R)$. Fossum uses this result in the following test for ramification. Let u be a primitive element for L over K such that u is in S, and let $f(t)$ be its minimal polynomial. Then, one can check that $C(S/R) \subseteqq f'(u)^{-1}S$, and so $f'(u)$ is in $D(S/R)$. Thus, if there are primitive elements $u_1, \ldots, u_n$ in S such the elements $f_1'(u_1), \ldots, f_n'(u_n)$ are not in any height 1 prime ideal of S, then S is unramified at all height 1 primes in $X'(S)$. This result is used in the following two examples of height 1 Galois extensions (which are also examples of height 1 separable extensions that are not separable).

EXAMPLE 3.3. (See [**4**; p. 85] or [**8**; p. 58].) Let k be a field of characteristic p and let n be an integer relatively prime to p, and suppose k contains a primitive n-th root of unity, w. The map on $S = k[x_1, \ldots, x_r]$, $r \geqq 2$, defined by $x_i \to wx_i$ and extending linearly, is a k-automorphism. The cyclic group G generated by this automorphism is the finite group Z/nZ. The fixed subring $R = S^G$ is the subalgebra of S generated by all monomials of degree n. Each x_i is a primitive element with minimal polynomial $f_i(t) = t^n - x_i^n$. It is easy to see that no height 1 prime ideal in S can contain all the elements $f_i'(x_i) = nx_i^{n-1}$, $i = 1, \ldots, r$. Thus, S is unramified at each height 1 prime in $X'(S)$, and by (3.2) is a height 1 Galois extension of R.

In particular, S is height 1 separable over R. We show that S is not R-separable. Let I be the ideal in R generated by all monomials of degree n. Then $R/I = k$, a field. Now, S/IS cannot be separable over the field R/I because S/IS contains the nilpotent elements $x_i + IS$. Thus, S is not separable over R.

We further note that, since R/I is a field, S/IS is not height 1 separable over R/I. However, this does not violate (1.5) because R/I does not satisfy *NBU* over R.

It is well known [**1**] that if R is a regular local ring and S is the localization at a maximal ideal of the integral closure of R in a finite separable field extension of the quotient field of R, then S is unramified over R if each minimal prime ideal of S is unramified (purity of the branch locus). The above example can be modified to show that the regularity condition cannot be dropped.

In the example, let J be the ideal $(x_1, x_2, \ldots, x_r)$ in S, and let $J' = J \cap R$. Both S_J and $R_{J'}$ are integrally closed domains, and, as in the remarks preceding (3.3), the quotient field of S_J is a Galois field extension of the quotient field of $R_{J'}$. It follows that S_J is a local $R_{J'}$-algebra, as in the setting described in [**1**]. Denote $R_{J'}$ by R' and S_J by S'. We claim that the minimal primes of S' are unramified over R', but S' is not unramified, and so R' is not regular. Note that R' and S' are each Krull, and, as argued above, Fossum's test shows that S' is unramified at each height 1 prime ideal. The second half of the proof of (3.2) then shows that S' is height 1 Galois, and hence height 1 separable, over R'. Now, if R' were regular, it would follow by [**1**; 1.4] that S' is unramified over R'. Let J_0 be the ideal $J'R'$. Then S'/J_0S' contains the non-zero nilpotent elements $x_i + J_0S'$, and so $J_0S' \neq JS'$, the maximal ideal of S'. Thus, S' is not unramified over R', and R' is not regular.

The next example is similar and is based on the example found in [**8**; p. 58].

EXAMPLE 3.4. Let k, w, and n be as in (3.3). Let $S = k[x, y]$. The map defined by $x \to wx$, $y \to w^{-1}y$ is a k-automorphism of S. If we let $R = S^G$, where G is the group generated by this automorphism, then we see that $R = k\,[x^n, y^n, xy]$. As in (3.3), the elements x and y are primitive elements for the quotient field of S over that of R, with minimal polynomials $f_1(t) = t^n - x^n$ and $f_2(t) = t^n - y^n$, respectively. Since no height 1 prime ideal in S can contain both nx^{n-1} and ny^{n-1}, we see that S is unramified at the height 1 primes in $X'(S)$ and is therefore a height 1 Galois extension of R. By an argument similar to that in (3.3) we see that S is not R-separable.

We remark that in both (3.3) and (3.4) we have the following situation. R is integrally closed and Noetherian and S is the integral closure of R in a separable field extension of the quotient field of R. The fact that the quotient field of S is separable over the quotient field of R follows from the height 1 separability of S over R (by localizing further). Since S is unramified at each minimal prime ideal in S, but S is not R-separable, S is not R-projective [**2**; 3.7]. Thus, the hypothesis that S be R-projective cannot be removed in [**2**; 3.7].

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