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SOME DISCRETE SUBSPACES OF βm

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ABSTRACT. By considering some discrete subspaces of the Stone-Čech compactification βm of a discrete space, we show that a nondiscrete door space which is not maximal door can be embedded in βm for every infinite discrete space m. This provides a counterexample to the converse of a theorem of Y. Kim. Maximal door spaces are characterized in terms of their embedding in βm.

By a space we shall mean a Hausdorff topological space. An infinite cardinal number m and a discrete space of cardinality m will be denoted by the same symbol, and βm will represent its Stone-Čech compactification. The cardinality of a set A will be denoted by $|A|$, $\mathrm{Cl}_X A$ is the closure of A in X, and N is the set of natural numbers. See [1] for a general reference.

A *door space* is a space in which every subset is either open or closed. A nondiscrete door space is called *maximal door* if the only finer door topology for the set is discrete. Kim [2] characterized nondiscrete door spaces and maximal door spaces as follows. A Hausdorff space X is nondiscrete door (maximal door) if and only if $X = S \cup \{p\}$ where S is an infinite discrete set and p is a point such that the restriction of its neighborhoods to S forms a filter (an ultrafilter) in S. Kim also showed that for every maximal door space X there is a discrete space m such that X can be embedded in βm; and, furthermore m may be taken to be $|X|$. He left open the question of whether every door space which can be embedded in βm for some m must be maximal door. We answer this question in the negative, and supply a stronger condition which does characterize maximal door spaces.

THEOREM 1. *For every infinite cardinal m there exists a nondiscrete door space X with $|X| > m$ so that X can be embedded in βm, but X is not maximal door. In particular, there is a nondiscrete door space of cardinality $2^{\aleph_0}$ which is not maximal door, but can be embedded in $\beta\aleph_0$.*

PROOF. We first construct for each infinite cardinal m a certain discrete subspace of βm which is of cardinal $n > m$. When $m = \aleph_0$, we have $n = 2^{\aleph_0}$. By [1; 12B] every m can be taken to be the union of a

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collection $\{A_\alpha \mid \alpha \in I\}$ where $|A_\alpha| = m$ for each $\alpha \in I$, $|I| > m$, and $|A_\alpha \cap A_\beta| < m$ for $\alpha \neq \beta$. It follows from [1; 6.9(a)] that for each α, $\mathrm{Cl}_{\beta m} A_\alpha = \beta A_\alpha$, which is homeomorphic to βm. By [1; 12I] we may choose $x_\alpha \in \mathrm{Cl}_{\beta m} A_\alpha$ such that every neighborhood of x_α intersects A_α in a set of cardinality m. Now by [1; 6.9(c)] $\mathrm{Cl}_{\beta m} A_\alpha$ is open in βm and is therefore a neighborhood of x_α. Suppose $x_\beta \in \mathrm{Cl}_{\beta m} A_\alpha$ for $\alpha \neq \beta$. Then $x_\beta \in \mathrm{Cl}_{\beta m} A_\alpha \cap \mathrm{Cl}_{\beta m} A_\beta$ which must be a neighborhood of x_β in the relative topology on $\mathrm{Cl}_{\beta m} A_\beta$. But $A_\beta \cap \mathrm{Cl}_{\beta m} A_\alpha \cap \mathrm{Cl}_{\beta m} A_\beta = A_\alpha \cap A_\beta$ is a neighborhood of x_β in A_β which has cardinality less than m, contradicting the choice of x_β. Hence $S = \{x_\alpha \mid \alpha \in I\}$ is a discrete collection. In case $m = \aleph_0$ we may choose $|I| = 2^{\aleph_0}$ by [1; 6Q.1].

For each $p \in \mathrm{Cl}_{\beta m} S \backslash S$ we see that $S \cup \{p\}$ is a nondiscrete door space embedded in βm. We now show that not every such $S \cup \{p\}$ can be maximal door. Suppose $S \cup \{p\}$ is maximal door for each $p \in \mathrm{Cl}_{\beta m} S \backslash S$. Consider the extension $f: \beta S \to \mathrm{Cl}_{\beta m} S$ of the inclusion map of S into $\mathrm{Cl}_{\beta m}$ S. We shall now show that f is one-to-one and onto, and hence a homeomorphism.

If $p \in \mathrm{Cl}_{\beta m} S \backslash S$, then p is a cluster point of the ultrafilter $\mathcal{F}$ of the restrictions of its neighborhoods to S. This ultrafilter is a z-ultrafilter, and so has a unique limit x in βS, and $f(x) = p$.

On the other hand, $\mathcal{F}$ is the only ultrafilter in S of which p is a cluster point. For suppose $\mathcal{G}$ is a filter in S which clusters to p in $S \cup \{p\}$. Then each element of $\mathcal{F}$ intersects G, and G must be in $\mathcal{F}$ for each $G \in \mathcal{G}$. But each $x \in \beta S \backslash S$ is the limit of an ultrafilter in S, so $f(x) = p$ for only one $x \in \beta S \backslash S$.

Thus $\mathrm{Cl}_{\beta m} S = \beta S$. But $|\mathrm{Cl}_{\beta m} S| \leqq |\beta m| < |\beta n| = |\beta S|$. Hence there must exist some $p \in \mathrm{Cl}_{\beta m} S \backslash S$ such that $S \cup \{p\}$ is not maximal door.

A subset of S of βm is said to be *strongly discrete* if for each $s \in S$ there is a neighborhood $U_s \subset \beta m$ of s such that if $s \neq t$, then $U_s \cap U_t \cap m = \emptyset$. This definition is equivalent to that in [3].

THEOREM 2. *A nondiscrete door space* $S \cup \{p\}$ *is maximal door if and only if it can be embedded in some* βm *in such a way that* S *is strongly discrete.*

PROOF. Kim [2] showed that every maximal door space could be embedded in such a way.

Suppose, then, that $S \cup \{p\}$ can be embedded in βm for some m and that S is strongly discrete in βm. Let f be a continuous function from S to [0, 1]. We shall show that f can be extended to βm, so that $\mathrm{Cl}_{\beta m} S = \beta S$. For each $s \in S$ let U_s be a neighborhood of s so that the U_s's illustrate that S is strongly discrete. Extend f to $S \cup m$ by defining $f(x) = f(s)$ if $x \in U_s$, and $f(x) = 0$ otherwise. Now f is a continuous function on m, and hence it has a unique extension F to βm. But m is dense in $S \cup m$, and f and F agree on m. Thus f and F must agree on all of $S \cup m$ and, in particular, on S. Therefore S is C^* embedded in $\mathrm{Cl}_{\beta m}\ S$, so $\mathrm{Cl}_{\beta m}\ S = \beta S$ [1; 6.9]. Now $p \in \beta S \backslash S$ and, since S is discrete, the unique z-ultrafilter $\mathcal{G}$ in S which converges to p is an ultrafilter of open subsets of S. Thus $F \cup \{p\}$ is open in $S \cup \{p\}$ for each $F \in \mathcal{G}$, and $\mathcal{G}$ is exactly the restriction to S of the neighborhoods of p. Hence $S \cup \{p\}$ is maximal door.

In light of Theorem 1 it is natural to ask whether a countable nondiscrete door space which is embedded in βm must be maximal door. Theorem 4 gives an affirmative answer to this question.

LEMMA 3. *Every countable discrete subset of βm is strongly discrete.*

PROOF. Let $S = \{x_i \mid i \in N\}$ be a countable discrete subset of βm, and let U_i be an open set which contains s_i but not s_j for $i \neq j$. By the regularity of βm, for each i there is an open set V_i such that $s_i \in V_i \subset \mathrm{Cl}_{\beta m} V_i \subset U_i$.

We now define a neighborhood W_i for each i so that $\{W_i \mid i \in N\}$ illustrates that S is strongly discrete. Let $W_1 = V_1$, and for each $i > 1$, let $W_i = V_i \backslash \bigcup_{j=1}^{i-1} \mathrm{Cl}_{\beta m} V_j$. It is clear that each W_i is an open neighborhood of x_i, and that $W_i \cap W_j = \emptyset$ when $i \neq j$.

THEOREM 4. *Every countable nondiscrete door space which can be embedded in βm is a maximal door space.*

PROOF. This follows directly from Theorem 2 and Lemma 3.

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