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SOME EXTENSIONS OF THE ARITHMETIC-MEAN THEORY OF ROBIN'S INTEGRAL EQUATION FOR BODIES WITH VERTICES

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ABSTRACT. A form of Neumann's method of the arithmetic mean was used previously to obtain an existence theorem for the homogeneous case of Robin's integral equation in E^2 for a convex closed curve with the vertex. In the present paper, this theory is developed to provide two further results, one of them uniqueness, and it is then shown how both existence and uniqueness follow for the non-homogeneous case (relevant to the Neumann problem). Finally, it is shown how a parallel theory can be given, embracing both the homogeneous and non-homogeneous cases, for E^3, that is to say, for Robin's integral equation for a convex closed surface with a vertex.

1. Introduction. Robin's integral equation in E^2 for a function σ on a simple closed curve C, is a Fredholm equation whose homogeneous case is

$$\sigma(A) = \frac{1}{\pi} \oint \sigma(A') \frac{\cos x}{r} ds' \ (A, A' \in C, r = \overrightarrow{A}' A), \tag{1}$$

C being conveniently parametrized by arc-length s, and where x is the angle between the r-direction and the outward-normal one to C at A.

Along with (1), we consider the integral equation of the first kind,

$$K + 2 \oint \sigma(A') \log r ds' = 0, \tag{2}$$

where K is a constant.

Physically, in the context of electrostatics, (2) represents the fact that the potential due to the charge density σ on a cylindrical conductor with cross section C, takes the constant value K on the cylinder, and for a continuous solution, (1) follows from the constancy of the potential throughout the interior of C.

The theory of these equations is well known when C is globally twice continuously differentiable. For example, existence for (1) follows from the Fredholm theory [**6**] and, for solutions within the class of continuous functions, (1) and (2) are shown to be equivalent. Recently, however [**3**], an existence theorem has been given for (1) when C fails to be twice continuously differentiable through having one exceptional point which is a vertex. Under these conditions, the Fredholm theory, even in its extension to weakly singular kernels, is inapplicable, and the method used was a sequence one with an adaptation of Neuman's method of the arithmetic mean. As always with Neumann's method, the additional condition is required that C be convex, and within this restriction, the theorem can be generalized for any finite number of vertices.

Nowadays it would be presumptuous to describe the result as "new", in view of the very powerful and general methods of functional analysis that are available. However, it often happens that methods belonging to classical "hard" analysis yield more. In the present case, we were able to majorize the growth rate of the solution with approach to the vertex, findings, remarkably, conforming with the Schwarz-Christoffel theory of complex analysis [**8**].

The purpose of this paper is threefold: (a) to prove uniqueness for the homogeneous case and equivalence of (1) and (2); (b) to extend existence and uniqueness to the general non-homogeneous case; (c) to show how a parallel theory (if a little less complete) can be given for the homogeneous and non-homogeneous cases in E^3, that is, for a convex simple closed surface with a vertex (or finite number of vertices).

As regards (c), it was not seen at the time of the first paper [**3**] how to obtain the three-dimensional existence theorem, certain points in the theory for E^3 being distinctly more difficult. These difficulties are now largely resolved. Again, we can majorize the growth rate of the solution, and the result in this case might be quite new inasmuch as there is no three-dimensional analogue of the Schwarz-Christoffel result with which to compare it.

2. Preliminary considerations. We take as equations for $C, x = x(s), y = y(s)$, $s \in [0,c]$, (arc-length), where $x(s), y(s)$ are twice continuously differentiable on $[0,c]$, and the end-points 0 and c of the interval give the same point $P \in C$, the only double point, which is the vertex. The approach of the existence proof for (1) is, starting with any positive continuous function $f_0(s)$, to define a sequence $\{f_n(s)\}$ by

$$f_{n+1}(S) = \frac{1}{\pi}\Omega_z \oint f_n(S')\frac{\cos c}{r}ds', \tag{3}$$

where, for any positive $z < \frac{1}{2}x, \Omega_z$ is the linear operator defined by

$$\begin{aligned}(\Omega_z \oint)(S) &= \begin{cases}\phi(S) + a, & S \in [z, c-z],\\ a, & S \in [0, z) \cup (c - z, c],\end{cases}\\ a &= \frac{1}{c}\Big(\int_0^z + \int_{c-z}^c\Big)\phi(S)ds.\end{aligned} \tag{4}$$

The f_n are all positive, bounded and, except at z and $c - z$, continuous, and their integrals on C have all the same value $Q > 0$.

By a quasi-solution of (1) we mean a function $\tau(s)$ which satisfies

$$\tau(s) = \frac{1}{\pi}\Omega_z \oint \tau(S')\frac{\cos x}{r}ds'. \tag{5}$$

The existence proof is in two main parts. The first is where the adaptation of the arithmetic-mean argument to simple-layer kernels occurs, being used in the process of showing that a subsequence of $\{f_n\}$ converges uniformly to a quasi-solution τ. Thereafter, a sequence $\{z_m\}$ of values of z is taken which converges monotonically to zero, and the corresponding sequence $\{\tau_m\}$ of quasi-solutions. The second part of the proof consists in showing that a subsequence of $\{\tau_m\}$ converges pointwise to a solution σ of (1). We refer the reader to the paper concerned [**3**] for the details of the proof, details to which we shall have to make frequent reference in what follows.

REMARK 1. This definition is a slightly tighter one than that given previously [**3**], but it is easy to show that the quasi-solutions actually found satisfy the present condition.

Two fundamental lemmas uniformly majorize the f_n, then in turn the τ_m, and have as their final consequence the majorizing condition on σ. This can be expressed as

$$\sigma = O(\tau_p^{-\beta}), \quad \beta > \beta_0 = \frac{\pi - 2\alpha}{2\pi - 2\alpha}, \tag{6}$$

where r_p is the distance from $(x, y) \in S$ to P and 2α is the angle of the vertex (the angle made at P by the limiting half-tangents having C between them). In (6), β is any number greater than β_0, and since β_0 is certainly less than $\frac{1}{2}$, we can take, instead of (6), the blunter but simpler condition

$$\sigma = O(r_p^{-\frac{1}{2}}), \tag{7}$$

holding independently of α.

It will abbreviate discussion to say that a function $f(s), s \in [0, c]$, which is continuous on $(0, c)$ and $O(r_p^{-\beta})$, is of class $C_p^{-\beta}$. It is a tidy convention to take $\cos x/r$ as zero when A in (1) is the vertex and the symbolism is of itself meaningless since C has no normal. Thus (1) has the solution 0 there, so that any solution $\sigma(s)$is defined on the whole of $[0, c]$. We always understand Lebesgue-integrability of functions, so the "integrable" implies absolute integrability.

3. Two lemmas.

The theorems of the two sections that follow depend upon, in addition to the existence theorem [**3**], the following simple but fundamental results.

LEMMA 1. *If a sequence $\{\phi_n(s)\}$ is defined according to* (3), *starting with any positive integrable $\phi_0(s)$ whose integral on C is Q, it converges uniformly to the quasi-solution $\tau(s)$ of the existence theorem.*

For the argument, which in the existence theorem is applied to $\{f_n/f_{n+1}\}$ [**3**], holds equally for $\{\phi_n/\tau\}(n = 1, 2, \ldots)$ showing now that $\phi_n/\tau \to 1$ uniformly, and since τ is bounded, the result follows.

LEMMA 2. *Any integrable solution σ of* (1) *is either non-negative or non-positive.*

It follows by (1) and the continuity of $\cos x/r$ (away from P), that σ is continuous on $(0, c)$, and assuming that for the σ we take, the proposition is untrue, we study the inequality

$$|\sigma(A)| \leq \frac{1}{\pi} \oint |\sigma(A')| \left|\frac{\cos x}{r}\right| ds'. \tag{8}$$

Then

(i) By convexity, $|\cos x/r| = \cos x/r$;

(ii) By the continuity of σ (on $(0, c)$), and convexity again, there are points $A \in C$ at which the inequality is strict;

(iii) By the continuity again, strict inequality holds when we integrate (8) on C; and

(iv) The integral of $\cos x/r$ with respect to A, is π, except if A' is P. It follows, making a legitimate change of order of integration, that

$$\begin{aligned}\oint |\sigma(s)|ds &< \frac{1}{\pi}\oint ds \oint |\sigma(s')|\frac{\cos x}{r}ds' \\ &= \frac{1}{\pi}\oint |\sigma(s')|ds' \oint \frac{\cos x}{r}ds = \oint |\sigma(s')|ds',\end{aligned}$$

an absurdity which proves the point.

For any real number $\tilde{Q}$, there is a solution of (1) whose integral on C is $\tilde{Q}$. For one is $\tilde{\sigma} = \tilde{Q}\sigma/Q$, where σ is the solution by the existence theorem. By this remark and Lemma 2, we shall find it sufficient, where we need to be specific, to work only with non-negative solutions of (1).

4. Further theorems on the homogeneous equation in E^2.

THEOREM 1. (uniqueness). *A solution of* (1) *whose integral on C is Q, is the only one in the class of integrable functions whose integral has this value.*

As we have said, an integrable solution is continuous. Let $\hat{\sigma}$ be one whose integral on C has the same value as that of σ given by the existence theorem. Then $\sigma - \hat{\sigma}$ is a solution whose integral is zero, and by Lemma 2 and continuity, this is only possible if σ and $\hat{\sigma}$ coincide.

THEOREM 2. (equivalence of the integral equations). *A solution of* (2) *of class $C_p^{-\beta}$ $(\beta < 1)$ and zero at P, is a solution of Robin's equation* (1)*, and any solution of* (1) *is a solution of* (2).

The equivalence is imperfect, but perfection is impossible since (2), as an equation of the first kind, does not have uniqueness. For example, the second statement asserts that a solution exists, but any function differing from it, say only on a non-empty set of measure zero, is also a solution but is not a solution of (1).

We shall prove the theorem in some detail, for we consider the result to be of significance, while the analysis is delicate and does not extend to give the equivalence in the other cases (non-homogeneous in E^2, either in E^3).

If $\sigma(s)$ is a solution of (2) of class $C_p^{-\beta}(\beta < 1)$, minus the integral in (2), A being now understood as any point of the plane, defines a potential $V(x,y)$ which is everywhere continuous. The justification of this statement is identical with the classical one for a smooth curve (Kellogg [5]) except at the vertex itself, while at the vertex, the fact of σ being $O(r_p^{-\beta})$ does not, since $\beta < 1$, spoil the argument.

Having that V takes the constant value K on C, the well-known classical derivation of Robin's equation (constancy of V throughout the interior of C and the limiting value of the normal derivative, Kellogg [5]), gives us the first part of the theorem, except at the vertex itself, where it is taken care of by the convention $\cos x/r = 0$ (§2).

Turning to the converse, which is more difficult, the existence and uniqueness theorems tell us that any solution of (1) will be of class $C_p^{-\beta}$ where, by (7), β can be taken as 1/2. Thus just as before, σ provides a simple-layer potential $V(x,y)$ defined and continuous on the whole of E^2. Clearly, we shall suffer no loss of generality by assuming that σ is non-negative.

Inside C, we take a simple closed curve with vertex, C' (Figure 1(a)). We require that it be continuously differentiable except at the vertex, and be any one of a family of curves whose interiors are a nested sequence approximating to the interior of C. These conditions are realized very easily under the convexity of C by taking the origin of coordinates inside the curve, and C' as any one of the geometrically similar curves $x = \gamma x(s), y = \gamma y(s), s \in [0,c], \gamma \in (0,1)$.

We divide C' into two parts by a line L (Figure 1(a)), imagining L as any one of a family of parallel lines approaching the vertex $P \in C$. Each of the two arcs of C', we join by the segment T of L inside, to form two simple closed curves, C, towards P and C, away.

We next enclose the part of C above L in a "box" B whose base contains T (Figure 1(b)), and study the contribution V^* to V coming from on the arc of C inside B. If q is the integral of σ on this arc (the "total charge" inside B), we have by Gauss' flux theorem,

$$\oint_B \frac{\partial V^*}{\partial \nu} ds = 2\pi q \tag{9}$$

(inward-normal derivative). Since the arc is arbitrarily small according to the closeness of L to P, we can arrange that

$$\left| \oint_B \frac{\partial V^*}{\partial \nu} ds \right| < \frac{1}{3}\varepsilon, \tag{10}$$

say. But since σ is non-negative in the arc, $\partial V^*/\partial \nu$ is around B, so that the integral in (10) is non-negative and dominates on just the segment T. In fact,

$$0 < \int_T \frac{\partial V^*}{\partial \nu} ds < \frac{1}{3}\varepsilon. \tag{11}$$

On the whole of C_2, Gauss' theorem gives

$$\oint_{C_2} \frac{\partial V}{\partial \nu} ds = \int_T \frac{\partial V}{\partial \nu} ds + \int_{C_2 \setminus T} \frac{\partial V}{\partial \nu} ds = 0, \tag{12}$$

all the "charge" being outside. With L fixed to satisfy (11) (a prescription which is independent of C'), we find, by taking C' close enough to C, that the last integral in (12) is arbitrarily small. For, Robin's equation being satisfied, the limiting value of $\partial V/\partial \nu$ is 0, and below L, this limit is approached uniformly (Kellogg [5]). Thus if $V^+ = V - V^*$ is the potential due to σ (non-negative) on the arc of C outside B, so that $\partial V^+/\partial \nu$ on T (ν reckoned into B) is non-positive, the right-hand equality in (12), with (11), gives us

$$\int_T \left| \frac{\partial V^+}{\partial \nu} \right| ds \leq \left| \int_T \frac{\partial V}{\partial \nu} ds \right| + \int_T \frac{\partial V^*}{\partial \nu} ds < \frac{2}{3}\varepsilon. \tag{13}$$

We could easily have shown that the total flux across T is arbitrarily small, but this is not enough for our purposes, and what the analysis above effectively affirms is that the positive and negative contributions are separately arbitrarily small.

Suppose that there is a point inside C at which the gradient of V is non-zero. Then since it is continuous, we shall have for some number $e > 0$, and C' and L respectively sufficiently close to C and P,

$$I(C_2) = \iint_R \left\{ \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 \right\} dxdy > e, \tag{14}$$

where R is the closed interior of C_2. By the harmonicity of V inside C, the integrand here is $\operatorname{div}(V \operatorname{grad} V)$, and transforming by Green's theorem,

$$I(C_2) = \oint_{C_2} V \frac{\partial V}{\partial \nu} ds = \int_T V \frac{\partial V}{\partial \nu} ds + \int_{C_2 \setminus t} V \frac{\partial V}{\partial \nu} ds > e. \tag{15}$$

Through its continuity, $|V|$ has a positive upper bound M on C and its interior. Hence, using again the fact that, below $L, \partial V/\partial \nu$ tends uniformly to zero as C' approaches C, we find from (15) that if this approach is close enough,

$$\int_T \left|\frac{\partial V}{\partial \nu}\right| ds > \frac{e}{M}. \tag{16}$$

But if $\varepsilon \leq e/M$ and the closeness of L and C' is chosen for the satisfaction of (11) and (13), the integral here is less than e/M.

Thus the supposition before (14) is wrong, whence V has a constant value, let us say K, throughout C and its interior. Since it is expressed on C by minus the integral in (2), the theorem is proved.

5. Theorems for the non-homogeneous equation in E^2.

The general nonhomogeneous form Robin's equation in E^2 is

$$\sigma(A) = \frac{1}{\pi} \oint \sigma(A') \frac{\cos x}{r} ds' - \frac{1}{2\pi} \left(\frac{\partial U}{\partial \nu}\right)_A, \tag{17}$$

where $U(x, y)$ is a function harmonic on a region containing C and its interior (an "applied field" in the physical context), and the derivative is the outward-normal one to C at A (zero conventionally if A is P).

This derivative satisfies the condition

$$\oint \frac{\partial U}{\partial \nu} ds = 0, \tag{18}$$

the only way the fact that U is given and harmonic on a set larger than C enters. If, instead of $\partial U/\partial \nu$, we have any continuous function $\phi(s)$ whose integral on C is zero, the theory of (17) will be the same and will provide a solution of the interior Neumann problem for the curve C with vertex and boundary data $\phi(s)$, for (17) will be the integral-equation formulation of this problem.

We extend the existence theorem for (1) to (17), noting that, because of the non-homogeneity and the possibility of a solution having non-uniform sign, arithmetic-mean analysis is not applicable directly.

We define a sequence $\{f_n\}$ by (3), except that Ω_z operates additionally on the last term of (17). By virtue of (18), the integral of each f_n on C is, again, Q, the value of the integral of f_0. We now define $\{g_n\}$ by $g_n = f_{n+1} - f_n$. The terms of this sequence satisfy (3) precisely, and the integral of each of them on C is zero.

For the same $z < \frac{1}{2}c$, we take the quasi-solution $\tilde{\tau}$ given by the first part of the homogeneous existence theorem [**3**] through a sequence $\{\tilde{f}_n\}$ with positive first function $\tilde{f}_0$. Then $\tilde{\tau} > 0$ on $[0, c]$, and we choose a number M large enough so that $M\tilde{\tau} + g_0$ also is positive. But $M\tilde{\tau}$ is the quasi-solution for $\{M\tilde{f}_n\}$, while $\{M\tilde{\tau} + g_n\}$ also satisfies (3), with the terms of both sequences having on C the same integral $M\tilde{Q}$. It follows by Lemma 1 (§3) that $\{M\tilde{\tau} + g_n\}$ converges uniformly to $M\tilde{\tau}$, whence $\{g_n\}$ converges uniformly to zero.

We now define another sequence, $\{h_n\}$, by $h_n = M\tilde{\tau} + f_n$, so that, like the relationship of f_{n+1} to f_n,

$$h_{n+1}(s) = \Omega_z\left\{\frac{1}{\pi}\oint h_n(S')\frac{\cos x}{r}ds' - \frac{1}{2\pi}\frac{\partial U}{\partial \nu}\right\}, \tag{19}$$

and we choose M' large enough so that both h_0 and h_1 are positive. The proof in the homogeneous case [**3**] has (in different order) the steps (i) that $\{f_n\}$ is uniformly bounded away from zero, and (ii) that $\{f_n\}$is uniformly bounded and equicontinuous. Now, with a possible further increase of M', a modified form of the proof of (i) shows easily that the whole of $\{h_n\}$ is positive, and with this knowledge, the extension of (ii) is immediate. Hence, by the Ascoli-Arzelà theorem, a subsequence $\{h_{n_p}\}$ converges uniformly to a limit function $\lambda(s)$. But (19) with f instead of h and attaching the suffix n_p, can be written as

$$f_{n_p}(s) = \Omega_z\left\{\frac{1}{\pi}\oint f_{n_p}(s')\frac{\cos x}{r}ds' - \frac{1}{2\pi}\frac{\partial U}{\partial \nu}\right\} - g_{n_p}(s),$$

whence, since $g_{n_p} \to 0, \lambda(s)$ is a quasi-solution of (17) by the definition for the non-homogeneous case that corresponds to (5).

The fundamental lemmas for the homogeneous case [**3**] are easily verified to hold equally for the non-homogeneous case, whereby the proof of the theorem proceeds in the same way in this case to show that a sequence $\{\lambda_m(s)\}$ of quasi-solutions defines a solution σ. We obtain the same majorizing condition (6) upon its growth rate.

Theorem 1 (uniqueness) can be extended at once. In fact, since $\sigma - \hat{\sigma}$, where $\hat{\sigma}$ is a second solution with same integral on C, satisfies the homogeneous equation, the proof is identical.

6. Robin's equation in E^3; geometrical preliminaries. Robin himself [**7**] appears to have been the first to apply arithmetic-mean analysis to the equation bearing his name, showing in the case of the homogeneous equation for a convex smooth surface in E^3, that an interatively-defined sequence of the usual kind converges to the solution, which is assumed to exist. The idea of using this type of analysis to prove existence itself was proposed by the present author [**2**]. Again, proof was for a surface both convex and smooth, and interest could hardly have been said to extend beyond the method as such, in view of freedom from the convexity and single-surface restrictions which is enjoyed with the Fredholm method. Much more interesting is its application to situations without the smoothness condition, as discussed in §1, but, while this has been done for E^2, it is only now that we are able to do it for E^3.

Robin's equation in the homogeneous case for a simple closed surface S, the three-dimensional analogue of (1), is

$$\sigma(A) = \frac{1}{2\pi}\int_S \sigma(A')\frac{\cos x}{r^2}dS', \tag{20}$$

and there is an analogue of (2) which we shall not write down as we shall bot be needing it. We suppose that S is a regular surface (Kellogg [5]), convex, and twice continuously differentiable except at a point P which is to be the one and only vertex.

We take P as the origin of a Cartesian coordinate frame with positive z-axis directed into S. By simple but tedious considerations of surface theory which we bypass here, S in the neighbourhood of P can be shown to have a representative of the form $x = f(u,\phi)\cos\phi, y = f(u,\phi)\sin\phi, z = u\cos[\gamma,(\phi)], u \in [0,\delta], \phi \in [0,2\pi]$. We write this as

$$x = \{u\sin[\gamma(\phi)] + \psi(u,\phi)\}\cos\phi, \quad y = \{u\sin[\gamma(\phi)] + \psi(u,\phi)\}\sin\phi,$$

$$z = u\cos[\gamma(\phi)], \quad u \in [0,\delta], \quad \phi \in [0,2\pi], \tag{21}$$

and the continuity of S requires that $\psi(u,\phi) \to \psi(0,\phi) = 0$ as $u \to 0, \forall\phi$. But additionally, we shall assume for partial derivatives of ψ,

$$\lim_{u\to 0}\psi_u(u,\phi) = \lim_{u\to 0}\psi_\phi(u,\phi) = 0, \tag{22a}$$

$$|\psi_{uu}(u,\phi)| < M, \quad |\psi_{u\phi}(u,\phi)| < uM' \quad (M, M' \text{ constants}). \tag{22b}$$

In fact, (22a) simply assures that S have a tangent cone at the vertex P, a representation of this being (21) with ψ replaced by 0, while (22b) provides that this cone have the "first-order contact" property characteristic of the tangent definitions of differential geometry. The assumptions should not be seen as restricting the generality of the theory; they merely formalize the geometrical quality that we intuitively associate with a vertex, something which, typically, is achieved much more easily in the two-dimensional theory.

Consider any point $(u_0,\phi_0) \in (o,\delta] \times [0,2\pi]$, and the generator L of the tangent cone which passes through the corresponding point (x_0, y_0, z_0) of the cone. For a number $\tilde{\gamma}$ not too large, there is a right circular cone which does not pass outside the tangent cone and shares the same generator $L, \tilde{\gamma}$ being its half-angle (angle between L and its axis) which is independent of the choice of (u_0,ϕ_0). Clearly, the two cones are mutually tangent along L (v, Figure 2). With the same origin P, we may take the axis of the new cone as new positive z-axis, and (21), (22a,b) as relative to a new Cartesian frame with this new z-axis, measuring ϕ from the plane through L so that (u, ϕ_0) becomes $(u, 0)$, and $\gamma(0), \tilde{\gamma}$. With respect to this set of transformations, corresponding to all initial choices of the old ϕ_c, the 5 M and M' in (22b) can be chosen so as to hold uniformly. The justification of these assertions is again a matter of tedious surface theory; they are adding no new postulate.

The set of possible values of $\tilde{\gamma}$ has a greatest, η, and $\tilde{\gamma}$ will be taken as η henceforth. As regards (21) and (22a,b) in any of the new representations, we shall have the inequality

$$\gamma(\phi) \geq \gamma(0) = \eta, \tag{23}$$

which is vital for our purposes. In fact, if the tangent cone itself is right-circular (as, for example, if S is a surface of revolution), η is the half-angle of this cone; with the positive z-axis the axis of the cone, the original representation (21) and all the new ones coincide, with $\gamma(\phi) = \eta, \phi \in [0, 2\pi]$.

For any of the representations discussed, we write the element of area as $dS = j(u, \phi) du d\phi$, and call the function j the area factor of the representation. For each of the new representations, we consider the ratio $j(u, \phi)/j(u, 0)$, preferring to write it as $j(u, \phi : \phi_0)/j(u, 0 : \phi_0)$ to indicate that it is for the representation defined for the angle ϕ_0 of the original representation (21). Although the function j changes with ϕ_0, nevertheless we can think of the ratio as a function of three variables, and we call

$$\Lambda = \limsup_{u \to 0} \frac{j(u, \phi : \phi_0)}{j(u, 0 : \phi_0)} - 1 \tag{24}$$

the index of asymmetry of the vertex P. It is unique and non-negative, and is zero for a surface of revolution, or any surface which tends to be so in the limiting sense that its tangent cone is right-circular.

7. Majorizing lemmas for the homogeneous existence problem in E^3. We mentioned before (6) two fundamental lemmas used for the existence theorem in E^2 [**3**], and one of two obstacles hitherto to giving an existence theorem for E^3 was to obtain analogues of these lemmas. We show now how this is done.

We consider S to be represented by parameters (u, v) on a closed set T which is the union of a finite family of closed regular regions, and by a function ϕ on S, we mean $\phi(u, v)$, $(u, v) \in T$. One of the closed regions is the rectangle $[0,, \delta] \times [0, 2\pi]$ of the last section, where we denoted (u, v) by (u, ϕ). The area factor j is given by $(j_x^2 + j_x^2 + j_z^2)^{1/2}$, where j_x, j_y, j_z are the Jacobians of x, y, z with respect to u and v. We denote by r_p the distance from P to the point $(x, y, z) \in S$ corresponding to $(u, v) \in T$. Then we have

LEMMA A. *If $\phi(u, v)$ is integrable on S and for some constant K satisfies*

$$(25) \qquad |\phi(u, v)| < \frac{K}{r_p}, \; r_p > 0,$$

then there is a constant K' such that $\Psi(u, v)$, defined by

$$(26) \qquad \Psi(u, v) = \frac{1}{2\pi} \int_S \Phi(u'u') \frac{\cos x}{r^2} dS',$$

satisfies

$$(27) \qquad |\Psi(u, v) < \frac{K'}{r_p}, \; r_p > 0.$$

To indicate in outline how this is proved, we refer to (21) and (22a,b) as they were before the subsequent discussion. The lemma will clearly be true if we can prove it for the integral

$$(28) \qquad I(u_0, \phi_0) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_x^{\delta} \Phi(u, \phi) \frac{\cos x}{r^2} j(u, \phi) du,$$

$$u_0 \in (0, \delta'], \;\; 0 < \delta' < \delta.$$

Let us now change to the representation described in that subsequent discussion, in which (u_0, ϕ_0) becomes $(u_0, 0)$. We evaluate $\cos x / r^2$ in terms of the representation (21), as now in terms of the new ϕ and $\gamma(\phi)$. Then, with the use of (22b) (in which, we recall, M and M' hold uniformly), we find that the integral changes in value as little as we please, uniformly with respect to how we had chosen (u_0, ϕ_0), if we take, instead of $(x, y, z) \in S$, (x, y, z) as the point of the tangent cone for the same (u, ϕ). This can be summed up by our writing

$$(29) \qquad |I(u_0, 0)| < \frac{G_1}{2\pi} \int_0^{2\pi} d\phi \int_0^{\delta} |\Phi(u, \phi)| \frac{\cos x}{r62} j(u, \phi) du,$$

where G_1 is as close as we please to 1 (uniformly with respect to the previous $(u_,, \phi_0)$), with the smallness of δ, and $\cos x/r^2$ and $j(u, \phi)$ are evaluated from (21) with $\psi = 0$.

The latter is found to be

$$(30) \qquad j(u, \phi) = u\{\sin^2[\gamma(\phi)] + [\gamma'(\phi)]^2\}^{\frac{1}{2}},$$

and using a bound B for j/u, we obtain with the evaluation of $\cos x/r^2$,
(31)

$$|I(u_0, 0)| < \frac{BG_1}{2\pi} \int_0^{2\pi} \{\sin\eta \cos[\gamma(\phi)] - \cos\eta \sin[\gamma(\phi)] \cos\phi\} d\phi \int_0^{S} |\Phi(u, \phi)| \frac{u^2}{D(u, \phi)} du,$$

$$D(u, \phi) = \{u_0^2 + u^2 - 2u_0 u[\sin\eta \sin[\gamma(\phi)] \cos\phi + \cos\eta \cos[\gamma(\phi)]]\}^{3/2}.$$

We now use our condition (25), being able to replace r_p by u if we replace G_1 by another constant G_2 as close as we please to 1 with the smallness of δ. Instead of $r_p \to u$, however, let us assume for a moment, with a purpose, that we have $r_p^\beta \to u^\beta$, corresponding to the same generality as with the lemma in E^2 [**3**]. Then with the change of variable $u/u_0 \to x$ and replacement of the resulting inner integral by the dominant one on $[0, \infty)$, the inner integral that we study is

$$(32) \qquad \frac{1}{u_0^\beta} \int_0^\infty \frac{x^{2-\beta} dx}{\{1 - 2xa(\phi) + x^2\}^{3/2}},$$

where we have written $a(\phi)$for the square-bracketed function in (31). But we cannot evaluate this integral unless $\beta = 1$ or 2, in which cases it is elementary. When $\beta = 1$, its value is $1/\{1 - a(\phi)\}$.

REMARK 2. A value in the general case is obtainable by specializing a still more general result given by Erdélyi et al [**4**] (p. 310, formula 22). This result appears to be wrong, however, as it does not reduce correctly in our simple elementary case. Anyhow, it is too complicated to be of use for present purposes.

Now, with this value and some manipulation of the resulting function of ϕ, we reach the stage

(33)
$$|I(u_0, 0)| < \frac{BKG_2}{2\pi u_0}\Big\{ \cot\eta \int_0^{2\pi} d\phi + \operatorname{cosec}\eta \int_0^{2\pi} \frac{\cos[\gamma(\phi)] - \cos\eta}{1 - \sin\eta \sin[\gamma(\phi)]\cos\phi - \cos\eta\cos[\gamma(\phi)]} d\phi \Big\}.$$

But because of (23), the second integral here is non-positive. Therefore, when we replace u_0 by u, then u by r_p and G_2 by a new positive constant G_3 arbitrarily close to 1 with the smallness of δ, we obtain

$$|I(u, 0)| < \frac{BKG_3 \cot\eta}{r_p}, \quad u \in (0, \delta'], \tag{34}$$

which is sufficient to prove the lemma.

The second lemma involves the index of asymmetry Λ, defined by (24).

LEMMA B. *If we are concerned with Ψ only in an arbitrarily small neighbourhood of P, and*

$$\Lambda < \sec\eta - 1, \tag{35}$$

we can, for a neighbourhood which is sufficiently small, take the constant K' of Lemma A as νK, where $\nu < 1$.

For, with $j(u, \phi)$ given by (30) and taking for the upper bound B of $j(u, \phi)/u$, the least upper bound (supremum), it is easy to see that

$$\lim_{\delta \to 0} \frac{B}{\sin\eta} = 1 + \Lambda. \tag{36}$$

Thus if, firstly, δ is small enough, we shall have, from (34) and (35),

$$|I(u,0) < \frac{KG_4}{r_p}, \quad u \in (0,\delta'], \tag{37}$$

G_4 replacing G_3 and although larger, less than 1. Secondly, if we take δ' small enough relative to δ to take care of the bounded contribution to (26) coming from S outside the portion given by(21), we shall have (27) with $K' = \nu K$, where ν is greater than G_4 but still less than 1.

The condition (35), without counterpart in E^2, restricts the generality of the theory in E^3. It certainly holds for a surface of revolution (Λ being zero), and much more broadly, but that it is a real restriction (at least as regards Lemma B) is shown by specific examples on surfaces departing widely from Axial symmetry about the vertex.

8. Existence for the homogeneous equation in E^3. The remaining obstacle to proving existence for (20) is, when we define a positive sequence $\{f_n(u,v)\}$ by appropriate analogues of (3) and (4), to prove the first step, namely, that this sequence is uniformly bounded and equicontinuous.

We now make it the second step, the first, proved independently and as easily as in E^2, being that $\{L_n\}$ and $\{U_n\}$, where $L_n = \inf(f_n/f_{n+1}), U_n = \sup(f_n/f_{n+1})$, are, respectively, monotonically non-decreasing and non-increasing. In place of the set $[0,z) \cup (c-z,c]$in E^2, we have the rectangle $R_\varsigma = [0,\varsigma) \times [0,2\pi]$, the remaining set on which the functions on (u,v) representing S are specified being the union T_ξ of a finite family of closed regular regions.

Let Z be the area of S. By the analogues of (3) and (4), $f_n(u,v) \leq Q/Z, (u,v) \in R_\varsigma, Q$ being the surface integral of each f_n on S. So, if $V_n = \sup f_n(u,v) > Q/Z$, it is taken at some $(u_n,v_n) \in T_\varsigma$ to which corresponds $A_n \in S$. We choose a spatial neighbourhood $N(\varepsilon;A_n)$ so small that, within, $\cos x/r$ has a supremum X which is independent of the location of (u_n,v_n) in T_ξ (if $N(\varepsilon;A_n)$ contains $P, \cos x/r$ is unbounded).

Now, with $M = S \cap N(\varepsilon;A_n)$, and remembering the definition of

$\{U_n\}$,

(38)

$$
\begin{aligned}
V_n = f_n(u_n, v_n) &= \frac{1}{2\pi}\Omega_\xi \int_X f_{n-1}(u', v')\frac{\cos x}{r^2}dS' \\
&= \frac{1}{2\pi}\Omega_\xi \int_M f_{n-1}(u', v')\frac{\cos x}{r^2}dS' + \frac{1}{2\pi}\Omega_\xi \int_{S\setminus M} f_{n-1}(u', v')\frac{\cos x}{r^2}dS' \\
&\leq \frac{U_{n-1}V_n X}{2\pi}\Omega_\xi \int_M \frac{dS'}{r} + \frac{X}{2\pi\varepsilon}\Omega_\xi \int_{S\setminus M} f_{n-1}(u', v')dS'.
\end{aligned}
$$

The results of Ω_ς operating onto the last two integrals are, respectively, bounded by $H\varepsilon$ and $Q(1+\alpha/Z)$, where for the former, H is independent of the location of (u_n, v_n), and for the latter, we first extended the integral over the whole of S, and use α to denote the area of the subsurface corresponding to R_ξ. Hence, since $U_n \leq U_0$, we have from (38) that

$$
V_n\Big(1 - \frac{U_0 H X\varepsilon}{2\pi}\Big) < \frac{XQ}{2\pi\varepsilon}\Big(1 + \frac{\alpha}{Z}\Big), \tag{39}
$$

and with ε small enough for the left member to be positive, the uniform boundedness is proved. Having this, equicontinuity on T_ς, although a little harder than for the theorem in E^2, presents no essential difficulty and is proved as in the case of a smooth surface [**2**].

In the latter part of the existence proof, we use a sequence $\{\xi_m\}$ decreasing monotonically to zero, and correspondingly, a sequence of sequences, $\{f_{nm}(u, v)\}$, $m = 1, 2, \ldots$. We need to know that, for any sufficiently small ξ', these sequences are equi-uniformly bounded on $T_{\varsigma'}$.

We choose any ξ, ξ'' such that $\xi'' < \xi' < \delta'$, where δ' is a number by which Lemma B is satisfied; clearly, if we obtain the result for such a ς', it will be true for any $\varsigma' \geq \delta'$. If δ' is small enough, the said lemma applies recursively to show that $f_{nm} \leq K/\varsigma'' + Q/Z$ for $\varsigma'' < u < \varsigma'$. Hence if, for $T_{\varsigma''} V_{nm} = \sup f_{nm} > K/\varsigma'' + Q/Z$, it is taken at a point $(u_n, v_n) \in T_{\varsigma'}$ and not anywhere in $T_{\varsigma''}\setminus T_{\varsigma'}$. We choose $N(\varepsilon; A_n)$ so that independently of where (u_n, v_n) is in $T_{\varsigma'}$, it excludes the subsurface of S corresponding to the rectangle $R_{\varsigma''}$. Then, with the easily-shown fact that $\{U_{1m}\}$ (the sequence of values of U_1 corresponding to $\{\varsigma_m\}$) is bounded, the argument leading to (39) (U_{1m} now replacing U_0) holds again to establish the point in question. With this information, we are able to complete the existence proof in analogy with the proof in E^2.

REMARK 3. In fact, the present argument bears close relationship to the one previously given for a smooth surface [**2**], but is more refined as (39) gives an explicit bound for $\{V_n\}$. The previous method showed by contradiction that $\{V_n\}$ could not be unbounded, and even under its extension to the present situation, would not suffice, as it would not show that, on $T_{\varsigma'}$, the double sequence $\{V_{nm}\}$ is bounded.

9. Discussion and further results for E^3. The assertion (27) of Lemma A with the refinement of Lemma B, follows through to the end of the existence proof to give us the majorizing condition on the solution σ:

$$\sigma = 0(r_p^{-1}). \tag{40}$$

This corresponds to the coarse result (7) of the theory in E^2; our inability to treat the integral (32) for general β has precluded our obtaining an analogue of the fine inequality (6). This failure has the non-trivial consequence of leaving us unable to obtain an analogue of Theorem 2; ironically, the same failure in E^2 would not have had that consequence.

The homogeneous existence theorem can be extended to the non-homogeneous case exactly as in E^2 (§5), giving the same majorizing condition (40).

The essential difficulty of E^3 is the unboundedness of the kernel $\cos x/r^2$, generally, not just about the vertex, and this has more implications than we have encountered. Another is that, while we can prove uniqueness just as in E^2 (§4 and §5), we can only do so within a less general framework. If, by the class $\mathcal{F}(r)$, we mean functions $f(A)$ such that $rf(A)$, where r is the distance from arbitrary A_0 to A, is integrable, we prove uniqueness within the class $\mathcal{F}(r)$. We know that a solution of (20) tends absolutely to infinity as $A \to P$ [**1**], and the existence and uniqueness theorems now tell us that, if it is of class $\mathcal{F}(r)$, it is nevertheless subject to (40).

Aside from our having no majorizing condition more refined than (40), the other disappointing feature of the theory in E^3 is of course the restriction (35). Whether either or both would be obviated by a more sophisticated treatment of (32), and hence of (31), is a matter of speculation at this time.

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