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FREE RESOLUTIONS OF SOME EDGE IDEALS OF SIMPLE GRAPHS

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ABSTRACT. The goal of this paper is to study the structure of the minimal free resolutions associated to a class of squarefree monomial ideals by using the one-to-one correspondence between squarefree quadratic monomial ideals and the set of all simple graphs. In [**6**], Hà and Van Tuyl demonstrated an inductive procedure to construct the minimal free resolution of certain classes of edge ideals. We will provide a simplified and more constructive proof of this result for the class of simple graphs containing a vertex of degree 1. Furthermore, by using the graphical structure of a tree, we provide a comprehensive description of the Betti numbers associated to the corresponding edge ideal along with providing an implementation of this graphical method coded in Python for use in SAGE. Furthermore, for specific subclasses of trees, we will generate more precise information including explicit formulas for the projective dimensions and Castelnuovo-Mumford regularity corresponding to the associated edge ideals. Although the methods discussed to study the edge ideals of simple graphs rely on the graph having a vertex of degree 1, we show how these methods and results can be used to gain information about the edge ideals of graphs that do not have a vertex of degree 1 by studying the class of edge ideals associated to cycles.

1. Introduction. By a *graph* G, we mean a vertex set $V_G = \{x_0, \dots, x_n\}$ along with a set of edges $E_G \subset V_G \times V_G$. Moreover, if $\{x_i, x_j\} \in E_G$ we will say x_i and x_j are *connected by an edge.* A graph is called *simple* if it is undirected and contains no loops or multiple edges. In this paper, we will restrict ourselves to the class of simple graphs, thus enabling a one-to-one correspondence between the set of simple graphs and the set of square-free quadratic monomial ideals.

$$\left\{\begin{array}{c}\text{Square-free quadratic}\\ \text{monomial ideals}\\ I \subset S = k[x_0, \dots, x_n]\end{array}\right\} \stackrel{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\text{Simple graphs } G\\ \text{on } n+1 \text{ vertices}\end{array}\right\}.$$

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For a given graph G and the corresponding edge ideal I_G, we consider the relationship between the graphical structure of G and the algebraic properties of the minimal free resolution of I_G.

1.1. Edge ideals. Given a simple graph G on the vertex set $\{x_0, \ldots, x_n\}$, we would like to study the ideal whose generators are formed by the edges of the graph. This ideal will reside in the polynomial ring with variable set corresponding to the vertex set of the graph, namely, $k[x_0, x_1, \ldots, x_n]$ where k is an arbitrary field.

1.1.1. Definition. For a graph G with vertex set $\{x_0, \ldots, x_n\}$, the *edge ideal of* G is the ideal

$$I_G := (x_i x_j \mid \{x_i, x_j\} \in E_G) \subset S := k[x_0, \ldots, x_n].$$

Edge ideals were first introduced by Villarreal in [**12**] and are a current topic of study in algebraic combinatorics. Connections between the algebraic properties of the edge ideal, I_G, and the combinatorial data associated to the planar graph, G, are an area of active research (see [**1–4, 6, 7, 9, 10, 12, 13**]).

1.2. Graded free resolutions. It is often beneficial to record more detailed information in a module's free resolution by considering graded modules and graded resolutions. Letting $A = (A, +)$ denote an abelian group, we will often use the suspension notation, $M(a)$, to denote the A-graded translate of a free R-module M that satisfies $[M(a)]_b = [M]_{a+b}$ for all $a, b \in A$. The following example illustrates the two gradings considered in this paper.

1.2.1. Example. Let us consider the polynomial ring $S = k[x_0, \ldots, x_n]$.

(1) Let $A = \mathbf{Z}, +)$. Then the *standard grading* (or *coarse grading*) of S is defined by $\deg(\mathbf{x}^{\mathbf{a}}) = a_0 + a_1 + \cdots + a_n$ for each $\mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbf{Z}^{n+1}$. If we let $S = k[x_0, x_1, x_2, x_3]$, then $\deg(x_0^2 x_2 x_3^3) = 2+0+1+3 = 6$ in the standard grading.

(2) Let $A = (\mathbf{Z}^{n+1}, +)$. Then the *fine grading* of S is defined by $\deg(\mathbf{x}^{\mathbf{a}}) = \mathbf{a}$ for each $\mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbf{Z}^{n+1}$. Thus for $S = k[x_0, x_1, x_2, x_3]$, $\deg(x_0^2 x_2 x_3^3) = (2, 0, 1, 3)$ in the fine grading.

The previous example illustrates that the fine grading carries information of the standard grading along with a detailed description of the variables contributing to the overall degree of the monomial. In general, we would like to consider monomials in the fine grading, but it often takes considerably more work to keep track of all the degree shifts. For this reason, in the cases where we do not need all of this extra information the standard grading will be used.

We would further like to consider maps between graded modules; and, in particular, graded free resolutions of graded modules. To do this we must first make the following definition.

1.2.2. Definition. Let M, N be A-graded modules with $a \in A$. Then an *A-graded homomorphism of degree a* is a homomorphism $\phi : M \to N$ such that for all homogeneous $m \in M$

$$\deg(\phi(m)) = \deg(m) + a.$$

If $a = 0$, then ϕ is called *degree-preserving*.

Furthermore, a *graded free resolution* for a finitely generated graded module $M \subset S$, is a free resolution of M

$$0 \longrightarrow F_r \xrightarrow{\phi_r} F_{r-1} \xrightarrow{\phi_{r-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

in which each map, ϕ_i, is degree preserving.

Section 2 will focus on the minimal free resolutions of edge ideals associated to *trees*, i.e., simple, connected graphs containing no cycles. When considering trees as a subclass of simple graphs, we notice their relatively simplistic structure. Transferring to the study of the edge ideals of trees, we would expect that the corresponding minimal free resolutions would be relatively simple. However, in [**9**], Nagel and Reiner show that for the class of edge ideals associated to trees, the Betti numbers corresponding to these edge ideals can be as complicated as desired. For certain classes of ideals associated to simple graphs, Hà and Van Tuyl introduced in [**6**] an inductive procedure to compute the minimal free resolution of such ideals. In Theorem 2.1.1 we will restrict to the class of trees and show a simplified development of this inductive procedure for this case. Furthermore, we will prove in Theorem 2.2.7 the following description of the Betti numbers for the edge ideal of a given tree.

Theorem. *Given a tree T on the vertex set $\{x_0, \dots, x_n\}$ and a vector $\mathbf{a} \in \mathbf{N}^{n+1}$, the following are equivalent.*

(1) $\beta_{i,\mathbf{a}}(S/I_T) = 1$ *for some i.*

(2) *The subforest of T defined by $\mathbf{a}$ is maximal.*

When the above theorem is combined with Theorem 2.2.2, a comprehensive description of the multi-graded Betti numbers for the edge ideals of trees is obtained. It should be noted that in the above theorem, the property of maximality is an algebraic property of the corresponding edge ideal I_T that will be introduced in Definition 2.2.3. The proof of this theorem leads to an implementation in the open-source mathematics software SAGE [**11**]. The code for this implementation is written in Python and provided at the end of Section 2.

Due to the complexity of minimal free resolutions of the edge ideals of trees, the edge ideals of paths and a class of graphs that occur as a natural extension of the class of paths is considered in Section 3. For these special cases we will generate more specific results concerning the minimal free resolutions of the corresponding quotient rings to the edge ideals. Specifically in Proposition 3.1.1, the following results concerning the minimal free resolutions of the edge ideals of paths are proven. It should be noted that Zheng also studied the Castelnrovo-Mumford regularity of the edge ideals of trees in [**13**] from a different viewpoint by considering the pairwise disconnected edges of the graph.

Proposition. *Let P_n denote an n-length path. Then*

(1) *the length of the minimal free resolution for S/I_{P_n} is $\lceil (2n)/3 \rceil$.*

(2) *the Castelnuovo-Mumford regularity of S/I_{P_n} is $\lceil n/3 \rceil$.*

In particular, this proposition shows that even in the case of the extremely simplistic graphical structure of paths, the minimal free resolutions of the corresponding edge ideals are relatively complicated. Taking Hilbert's syzygy theorem into consideration, they have roughly two-thirds of the maximum length of a minimal free resolution of an ideal in the same polynomial ring. Furthermore, in Corollary 3.1.3 the following result concerning the rank of the last module in the minimal free resolution of an edge ideal for a path is obtained.

Corollary. *For a path of length n,*

$$\beta_{\lceil (2n)/3 \rceil}(S/I_{P_n}) = \begin{cases} 1 & \text{if } 3 \nmid n \\ (n/3)+1 & \text{if } 3 \mid n. \end{cases}$$

Both of the above results concerning paths clearly demonstrate the integral role that divisibility of a path's length by 3 plays in the algebraic structure of the minimal free resolution of the edge ideal. Since trees can be inductively constructed from paths, Theorem 2.1.1 shows that divisibility by 3 also plays an important role in the algebraic structure of the edge ideals of trees.

In Section 4 the minimal primary decomposition of the edge ideal of an arbitrary simple graph is considered as it relates to the set of all minimal vertex covers of the planar graph. In particular, the following one-to-one correspondence is used to determine an explicit formula for the number of associated prime ideals corresponding to the edge ideal of a path.

$$\left\{ \begin{array}{c} \text{Minimal vertex covers of} \\ \text{a simple graph } G \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Associated prime} \\ \text{ideals of } I_G \end{array} \right\}.$$

This one-to-one correspondence enables us to easily determine whether a given prime ideal is indeed an associated prime ideal of the corresponding edge ideal.

The results that we obtain for the edge ideals of trees will be used in Section 5 to generate information concerning the minimal free resolution of cycles and more general graphs. In Proposition 5.0.6, we will provide an explicit formula for the length of the minimal free resolution corresponding to the edge ideal of a cycle. In particular, we will see that it is very closely related to the length of the minimal free resolution corresponding to a path. It should be noted that Theorem 2.1.1 does not apply to the class of edge ideals of cycles, because the procedure is dependent upon the existence of a vertex of degree one.

2. Edge ideals of trees. Hà and Van Tuyl describe in [**6**] a method to decompose edge ideals of particular simple graphs to generate information on the minimal free resolutions of the corresponding quotient rings S/I_G. The decomposition that is used is based upon the concept of splittable monomial ideals which were originally defined by Eliahou and Kervaire in [**1**].

2.0.3. Definition. Let I be a monomial ideal in $S = k[x_0, \dots, x_n]$, and let $\mathcal{G}(I)$ denote the minimal set of monomial generators of I. Then I is *splittable* if I is the sum of two nonzero monomial ideals J and K, i.e., $I = J + K$, such that

(1) $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$; and

(2) there is a *splitting function*

$$\begin{aligned} \mathcal{G}(J \cap K) &\longrightarrow \mathcal{G}(J) \times \mathcal{G}(K) \\ w &\longmapsto (\phi(w), \psi(w)) \end{aligned}$$

satisfying

(a) for all $w \in \mathcal{G}(J \cap K)$, $w = \mathrm{lcm}\,(\phi(w), \psi(w))$; and

(b) for every subset $S \subset \mathcal{G}(J \cap K)$, both $\mathrm{lcm}\,(\phi(S))$ and $\mathrm{lcm}\,(\psi(S))$ strictly divide $\mathrm{lcm}\,(S)$.

If J and K satisfy the above conditions, then we say that $I = J + K$ is a *splitting* of I.

In this paper, we will avoid this complex definition and provide a more constructive approach to the development of minimal free resolutions corresponding to the edge ideals of certain classes of simple graphs by using the mapping cone construction for a short exact sequence.

2.1. Mapping cone decomposition. In [**6**], Hà and Van Tuyl remarked that if the simple graph G has a vertex of degree 1, say x_k, then the edge formed by x_k and its neighbor, x_n, form a splitting edge of I_G, i.e., $I_G = (x_k x_n) + I_{G\setminus\{x_k,x_n\}}$ forms a splitting of I_G. In this case, we can also recover the inductive result concerning the minimal free resolutions of the corresponding quotient rings as proved by Hà and Van Tuyl (see [**6**]) using more of a constructive approach.

2.1.1. Theorem. *Let G be a simple graph with vertex set $V_G = \{x_0, \dots, x_n\}$ and the added restriction that G has a vertex of degree 1, say x_n. Furthermore, let x_{n-1} be the neighbor of x_n, and set $(-\mathbf{2}) = (0, \dots, 0, -1, -1)$. Then the mapping cone procedure applied to the sequence*

$$0 \to (S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n))(-\mathbf{2}) \xrightarrow{x_{n-1}x_n} S/I_{G\setminus\{x_{n-1},x_n\}} \to S/I_G \to 0$$

provides a minimal free resolution of S/I_G, where

$$I_{G\setminus\{x_{n-1},x_n\}} := (x_i x_j \mid x_i x_j \text{ is a generator of } I_G \text{ and } x_i x_j \neq x_{n-1}x_n),$$

i.e.,

$$\begin{aligned}\beta_{i,\mathbf{a}}(S/I_G) = \beta_{i,\mathbf{a}}&(S/I_{G\setminus\{x_{n-1},x_n\}})\\ &+ \beta_{i-1,\mathbf{a}}(S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)(-\mathbf{2}))\end{aligned}$$

for all $\mathbf{a} \in \mathbf{N}^{n+1}$.

Proof. Since x_n does not divide a minimal generator of $I_{G\setminus\{x_{n-1},x_n\}}$,

$$I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n) = I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}).$$

However, this implies that the exact sequence

$$0 \to (S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n))(-\mathbf{2}) \to S/I_{G\setminus\{x_{n-1},x_n\}} \to S/I_G \to 0$$

factors as

$$\begin{tikzcd}
0 \arrow[r] & (S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n))(-\mathbf{2}) \arrow[r, "x_{n-1}x_n"] \arrow[d, "x_n"] & S/I_{G\setminus\{x_{n-1},x_n\}} \arrow[r] & S/I_G \arrow[r] & 0. \\
& (S/I_{G\setminus\{x_{n-1},x_n\}} : x_{n-1})(0, \dots, -1, 0) \arrow[ur, "x_{n-1}"] & & &
\end{tikzcd}$$

Furthermore, let

$$\mathcal{F} : 0 \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow S \longrightarrow S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n) \longrightarrow 0$$

be a minimal free resolution of

$$S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n) = S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}),$$

and let

$$\mathcal{G} : 0 \longrightarrow G_t \longrightarrow G_{t-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow S \longrightarrow S/I_{G\setminus\{x_{n-1},x_n\}} \longrightarrow 0$$

be a minimal free resolution of $I_{G\setminus\{x_{n-1},x_n\}}$. Then the matrices representing the maps $\delta_i : F_i \to G_i$ cannot contain a unit entry; and consequently, the mapping cone construction applied to the sequence

$$0 \longrightarrow (S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n))(-\mathbf{2}) \longrightarrow S/I_{G\setminus\{x_{n-1},x_n\}} \longrightarrow S/I_G \longrightarrow 0$$

provides a minimal free resolution of S/I_G. Therefore,

$$\beta_{i,\mathbf{a}}(S/I_G) = \beta_{i,\mathbf{a}}(S/I_{G\setminus\{x_{n-1},x_n\}}) + \beta_{i-1,\mathbf{a}}(S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)(-\mathbf{2}))$$

for all $\mathbf{a} \in \mathbf{N}^{n+1}$. $\square$

Examining $S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)$ more closely, we can see that

$$I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n) = \mathfrak{a} + (x_0, \ldots, x_s)$$

where $\{x_n, x_0, \ldots, x_s\}$ are the neighbors of x_{n-1} and the generators of $\mathfrak{a}$ are square-free quadrics in $k[x_{s+1}, \ldots, x_{n-2}]$. This shows that $I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)$ can be realized as a subgraph of G.

If we further restrict ourselves to a subclass of simple graphs where each graph in the class has at least one vertex of degree 1, then we will obtain an inductive construction for the minimal free resolution of the corresponding quotient rings.

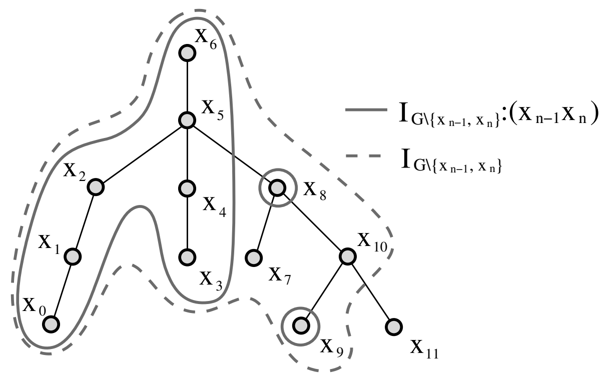

FIGURE 1.

2.2. Multi-graded Betti numbers of the edge ideals of trees. By definition, every tree has a leaf. Hence, in the case of the edge ideals of trees, Theorem 2.1.1 provides a comprehensive description of the corresponding minimal free resolutions, because the quotient ideal $I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)$ can be realized graphically as a *subforest* of I_G, i.e., a disjoint union of trees. Figure 1 illustrates the relationship between I_G, $I_{G\setminus\{x_{n-1},x_n\}}$ and $I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)$.

Furthermore, the minimal free resolution of the edge ideal of a forest can be obtained by tensoring together the minimal free resolutions of the edge ideals of each component tree in the forest.

We would also like to take a closer look at the finely graded Betti numbers associated to the edge ideal of a tree. Hochster proved the following result concerning the possible degree shifts for Betti numbers in a minimal free resolution corresponding to a monomial ideal. A proof of this result can be found in the book by Miller and Sturmfels (see [**8**]).

2.2.1. Proposition (Hochster's formula (dual version)). *The non-zero Betti numbers of a squarefree monomial ideal $I \subset S$ lie only in square-free degrees, i.e.,*

$$\beta_{i+1,\mathbf{a}}(S/I) = 0 \text{ if } a_i \geq 2 \text{ for some } i \in \{0, \dots, n\}$$

where $\mathbf{a} = (a_0, \dots, a_n) \in \mathbf{N}^{n+1}$.

Using Theorem 2.1.1 and Hochster's formula, further limits can be placed on the Betti numbers corresponding to the quotient rings of edge ideals of trees.

2.2.2. Theorem. *Let T_n denote a tree with n edges. Then the finely graded Betti numbers*

$$\beta_{i,\mathbf{a}}(S/I_{T_n}) \in \{0,1\}$$

for all $\mathbf{a} \in \mathbf{N}^{n+1}$.

Proof. Proceed by induction on the number of edges n. If $n = 1$, then $I_{T_1} = (x_0x_1)$ and the claim is clearly true. Consider a tree T_n with n edges; then by removing a leaf, say x_n, the remaining subtree is a tree with only $n-1$ edges. Denote it by T_{n-1}. Without loss of generality, assume x_{n-1} is the neighbor of x_n. Set $(-\mathbf{2}) = (0,\dots,0,-1,-1)$. Then we have the exact sequence

$$0 \longrightarrow S/(I_{T_{n-1}} : (x_{n-1}x_n))(-\mathbf{2}) \longrightarrow S/I_{T_{n-1}} \longrightarrow S/I_{T_n} \longrightarrow 0.$$

Moreover,

$$I_{T_{n-1}} : (x_{n-1}x_n) = (x_0,\dots,x_s) + I_{T_{n_0}} + \cdots + I_{T_{n_k}}$$

where $\{x_0,\dots,x_s\}$ is the set of neighbors of x_{n-1} and $\{T_{n_0}+\cdots+T_{n_k}\}$ is the set of subtrees of T_n occurring in the graphical representation of the quotient ideal $I_{T_{n-1}} : (x_{n-1}x_n)$ as a subforest of T_n. By the induction hypothesis

$$\beta_{i,\mathbf{a}}(S/I_{T_{n_j}}) \in \{0,1\} \text{ for } j = 0,\dots,k.$$

$\beta_{i,\mathbf{a}}(S/(x_0,\dots,x_s)) \in \{0,1\}$ since $S/(x_0,\dots,x_s)$ is resolved by the Koszul complex. Moreover, since the generators of $(x_0,\dots,x_s)$ and $I_{T_{n_j}}$ are disjoint for $j = 0,\dots,k$, the minimal free resolution of $S/((x_0,\dots,x_s) + I_{T_{n_0}} + \cdots + I_{T_{n_k}})$ is resolved by the tensor product of the minimal free resolutions of $S/(x_0,\dots,x_s)$ and $S/I_{T_{n_j}}$ for $j = 0,\dots,k$. Hence,

$$\beta_{i,\mathbf{a}}\left(S/((x_0,\dots,x_s) + I_{T_{n_0}} + \cdots + I_{T_{n_k}})\right) \in \{0,1\}.$$

Thus the mapping cone provides

$$\begin{aligned}&\beta_{i,\mathbf{a}}(S/I_{T_n})\\&= B_{i,\mathbf{a}}(S/I_{T_{n-1}}) + \beta_{i-1,\mathbf{a}}(S/((x_0,\dots,x_s)+I_{T_{n_0}}+\cdots+I_{T_{n_k}})(-\mathbf{2})).\end{aligned}$$

Assume to the contrary that $\beta_{i,\mathbf{a}}(S/I_{T_n}) = 2$, i.e.,

$$\beta_{i,\mathbf{a}}(S/I_{T_{n-1}}) = \beta_{i-1,\mathbf{a}}(S/((x_0,\dots,x_s)+I_{T_{n_0}}+\cdots+I_{T_{n_k}})(-\mathbf{2})) = 1.$$

Then Hochster's formula (2.2.1) implies that $\mathbf{a} = (a_1,\dots,a_{n+1}) \in \{0,1\}^{n+1}$. Thus we obtain the following two cases based upon the value of a_{n+1}.

Case (i). Let $\mathbf{a} = (\dots,1)$. Then $\beta_{i,\mathbf{a}}(S/I_{T_{n-1}}) = 0$, because x_n does not divide any generator of $I_{T_{n-1}}$.

Case (ii). Let $\mathbf{a} = (\dots,0)$. Then $\beta_{i-1,\mathbf{a}}(S/((x_0,\dots,x_s)+I_{T_{n_0}}+\cdots+I_{T_{n_k}})(-\mathbf{2})) = 0$, because the shift of $(-\mathbf{2})$ says that any contribution from the minimal free resolution of $(S/((x_0,\dots,x_s)+I_{T_{n_0}}+\cdots+I_{T_{n_k}}))(-\mathbf{2})$ will be in a shift with last two entries $(\dots,1,1)$.

Therefore $\beta_{i,\mathbf{a}}(S/I_{T_n}) \in \{0,1\}$. □

We would like to provide a comprehensive description of the Betti numbers that occur in the minimal free resolution corresponding to the quotient ring of an edge ideal of a tree. To do this, we make the following definition.

2.2.3. Definition. Let T be a tree. Then T is called *maximal* if

$$\beta_{\mathrm{pd}\,(S/I_T),\mathbf{d}}(S/I_T) = 1 \text{ where } \mathbf{d} = (1,1,\dots,1),$$

i.e., if the minimal free resolution of S/I_T has the maximal shift.

From the definition, the property of maximality is purely an algebraic property dealing with the leftmost Betti number of the minimal free resolution for S/I_T.

2.2.4. Example. Consider the path of length 2, P_2. Then $I_{P_2} = (x_0x_1, x_1x_2)$ and a finely graded minimal free resolution for S/I_{P_2}

is given by

$$0 \longrightarrow S(-1,-1,-1) \longrightarrow \begin{array}{c} S(-1,-1,0) \\ \oplus \\ S(0,-1,-1) \end{array} \longrightarrow S \longrightarrow S/I_{P_2} \longrightarrow 0$$

and P_2 is a maximal graph. Now, let us consider the path of length 3, P_3. Then $I_{P_3} = (x_0x_1, x_1x_2, x_2x_3)$ and a minimal free resolution for S/I_{P_3} is given by

$$0 \longrightarrow \begin{array}{c} S(-1,-1,-1,0) \\ \oplus \\ S(0,-1,-1,-1) \end{array} \longrightarrow \begin{array}{c} S(-1,-1,0,0) \\ \oplus \\ S(0,-1,-1,0) \\ \oplus \\ S(0,0,-1,-1) \end{array} \longrightarrow S \longrightarrow S/I_{P_3} \longrightarrow 0.$$

and hence P_3 is not maximal.

When considering the above example, we start to see that the length of a path affects its maximality. In particular, we will see in Section 3 that a path is maximal if and only if its length is not divisible by 3. Additionally in Section 3, we will describe how to determine maximality by decomposing the planar graph T into smaller subgraphs.

2.2.5. Remark. The above definition of maximality also applies to a forest F. Specifically, a forest is maximal when all of its component trees are maximal.

For a given tree T on the vertex set $\{x_0, \dots, x_n\}$ we can talk about the *subforest of T defined by a vector* $\mathbf{a} \in \mathbf{N}^{n+1}$. This subforest is obtained from T by removing all vertices x_i that have a 0 in the ith-entry of $\mathbf{a}$.

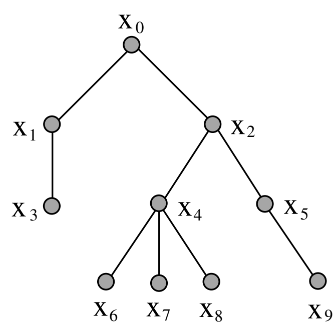

FIGURE 2.

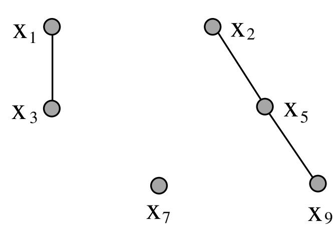

FIGURE 3.

2.2.6. Example. Consider the tree T depicted in Figure 2. Then the subforest of T defined by $(0, 1, 1, 1, 0, 1, 0, 1, 0, 1)$ is given by Figure 3.

Using this idea of a subforest of a tree defined by a vector, we can determine when a particular Betti number will occur in the minimal free resolution corresponding to the quotient ring of the edge ideal of the tree.

2.2.7. Theorem. *Given a tree T on the vertex set $\{x_0, \dots, x_n\}$ and a vector $\mathbf{a} \in \mathbf{N}^{n+1}$, the following are equivalent.*

(1) $\beta_{i,\mathbf{a}}(S/I_T) = 1$ *for some i.*

(2) *The subforest of T defined by* $\mathbf{a}$ *is maximal.*

Proof. Induct on n, the number of edges in the tree. For $n = 1$, $S/I_{T_1} = S/(x_0x_1)$, and the claim clearly holds. Let T_n denote a tree with n edges. Without loss of generality, we will assume that x_n is a leaf of the tree T_n with neighbor x_{n-1}.

Assume $\beta_{i,\mathbf{a}}(S/I_T) = 1$. Consider the subforest of T defined by $\mathbf{a}$, denote it by $F_{\mathbf{a}}$. Notice that starting from $F_{\mathbf{a}}$ we can add vertices one at a time as leaves to reconstruct the original tree T. Then Theorem 2.1.1 provides that

$$\beta_{i,\mathbf{a}}(S/I_T) = 1 \iff \beta_{\mathrm{pd}\,(S/I_{F_{\mathbf{a}}}),\mathbf{d}_{\mathbf{a}}}(S/I_{F_{\mathbf{a}}}) = 1, \tag{1}$$

where $\mathbf{d}_{\mathbf{a}} = (1, 1, \dots, 1)$ and has entries corresponding to $\mathbf{a}$. Furthermore, (1) implies that $F_{\mathbf{a}}$ is maximal. Hence, $\mathbf{a}$ defines a maximal subforest of T.

Conversely, let us assume that $\mathbf{a}$ defines a maximal subforest F of T_n. Theorem 2.1.1 provides that $\mathbf{a} \neq (\dots, 0, 1)$.

Case (i). If $\mathbf{a} = (\dots, 1, 0)$, then $\mathbf{a}$ defines a maximal subforest of the subtree of T corresponding to $T \setminus \{x_{n-1}, x_n\}$. It follows from the induction hypothesis that $\beta_{i,\mathbf{a}}(S/I_{T\setminus\{x_{n-1},x_n\}}) = 1$. Furthermore, the mapping cone procedure and Theorem 2.1.1 imply that $\beta_{i,\mathbf{a}}(S/I_T) = 1$.

Case (ii). If $\mathbf{a} = (\dots, 1, 1)$, then from the definition of maximality, we see that a tree T is maximal if and only if the subforest of T defined by $I_{T\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)$ is maximal. Hence if $\mathbf{a} = (\dots, 1, 1)$ defines a maximal subforest of T, then $\mathbf{a}'|_F = [\mathbf{a} - (0, \dots, 0, 1, 1)]_F$ defines a maximal subforest of F where

$$I_{T\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n) = (x_0, \dots, x_s) + F.$$

Furthermore, the induction hypothesis implies that for some j,

$$\beta_{j,\mathbf{a}'|_{\mathbf{F}}}(S/I_F) = 1.$$

Hence,

$$\beta_{i,\mathbf{a}'}(S/I_{T\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)) = 1,$$

which implies that

$$\beta_{i,\mathbf{a}}(S/I_{T\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)(0,\dots,0,-1,-1)) = 1.$$

The mapping cone procedure and Theorem 2.1.1 then imply that

$$\beta_{i+1,\mathbf{a}}(S/I_T) = 1. \qquad \square$$

The above proof leads to an algorithm for determining when a given Betti number occurs in the minimal free resolution of the associated quotient ring that can be implemented in Python for use in the open-source math software SAGE (see [**11**]) via the following code where the inputs are the tree T, stored as a dictionary of lists; and the maximum number of times the process should run, *depth*.

```
def MaxTest2(T,depth):
    if depth<0:
        raise ValueError
    for l in T:
        if len(T[l])==1:    #Find the first leaf of the tree
            n=T[l][0]       #Let n be the neighbor of the leaf
            break
    try:
        T.pop(l)    #Remove the leaf
    except NameError:    #Return error if no leaf is found,
                         #i.e. if the graph was not a tree
```

```
            raise TypeError , "Graph contains a cycle--Not a tree"
        e=T[n]    #Neighbors of n
        T.pop(n)
        e.remove(l) #Remove the leaf from the list of neighbors of n
        for v in e:
            temp1=T[v]
            temp1.remove(n)   #Remove the neighbor from the lists of
                              #its neighbors
            T.pop(v)   #Remove the neighbors' neighbor from the list
            for w in temp1:
                temp2=T[w]
                temp2.remove(v)
                if len(temp2)>0:  #Remove the vertex v from w's list
                    T[w]=temp2
                else:
                    return False #Removing the neighbor, n, left a
                                 #floating vertex (path of length 0)
        if len(T)==4:   #Forest is either a 3-path, K_{1,3}, or two
                        #disjoint 1-paths
            for x in T:
                if len(T[x])==3:  #K_{1,3}
                    return True
                if len(T[x])==2:  #3-Path
                    return False
            return True
        if len(T)<4:
            return True
        return MaxTest2(T,depth-1)
```

It should be noted that the combinations of Theorems 2.2.2 and 2.2.7 provide a comprehensive description of the Betti numbers occurring in a minimal free resolution of the corresponding quotient ring to an edge ideal of a tree. In particular, Theorem 2.2.7 tells us when a particular shift occurs in the minimal free resolution, and Theorem 2.2.2 tells us that if the shift occurs the corresponding Betti number must be 1.

3. Specific classes of edge ideals of trees. In this section we will look at the subclass of trees known as paths as well as look at a graphical approach to the comprehensive description of the multi-graded Betti numbers provided in Section 2.

3.1. Minimal free resolutions of the edge ideals of paths. By a *path* of length n on the vertex set $V_G = \{x_0, \dots, x_n\}$, we mean a graph with edge set given by $E_G = \{\{x_0, x_1\}, \{x_1, x_2\}, \dots, \{x_{n-1}, x_n\}\}$, and the corresponding edge ideal is given by $I_{P_n} = (x_0x_1, x_1x_2, \dots, x_{n-1}x_n)$. By restricting to the class of paths, we are able to explicitly write down the projective dimension and regularity of the corresponding quotient ring in terms of the path's length.

3.1.1. Proposition. *Let P_n denote an n-length path. Then*
(1) $\operatorname{pd}(S/I_{P_n}) = \lceil (2n)/3 \rceil$.
(2) $\operatorname{reg}(S/I_{P_n}) = \lceil n/3 \rceil$.

Proof. Proceed by induction on length of the path, n. For $n = 1$ the associated edge ideal is $I_{P_1} = (x_0x_1)$, and it is clear that $\operatorname{pd}(S/I_{P_1}) = 1 = \lceil (2(1))/3 \rceil$ and $\operatorname{reg}(S/I_{P_1}) = 1 = \lceil 1/3 \rceil$. Consider I_{P_n} and the following short exact sequence

$$0 \longrightarrow \big(S/I_{P_{n-1}} : (x_{n-1}x_n)\big)(-2) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0.$$

From the mapping cone construction and Theorem 2.1.1 we obtain
(2)
$$\operatorname{pd}(S/I_{P_n}) = \max\big\{\operatorname{pd}(S/I_{P_{n-1}}), \qquad \operatorname{pd}(S/I_{P_{n-1}} : (x_{n-1}x_n)) + 1\big\}$$
and
(3)
$$\operatorname{reg}(S/I_{P_n}) = \max\big\{\operatorname{reg}(S/I_{P_{n-1}}),\ \operatorname{reg}(S/I_{P_{n-1}} : (x_{n-1}x_n)(-2)) - 1\big\}.$$
Furthermore, we notice that $I_{P_{n-1}} : (x_{n-1}x_n) = (x_{n-2}) + I_{P_{n-3}}$. Then the induction hypothesis provides the following information

$$\operatorname{pd}(S/I_{P_{n-1}}) = \left\lceil \tfrac{2(n-1)}{3} \right\rceil \quad \operatorname{pd}\big((S/I_{P_{n-1}} : (x_{n-1}x_n)(-2))\big) = \left\lceil \tfrac{2n}{3} \right\rceil - 1$$
$$\operatorname{reg}(S/I_{P_{n-1}}) = \left\lceil \tfrac{n-1}{3} \right\rceil \quad \operatorname{reg}\big((S/I_{P_{n-1}} : (x_{n-1}x_n)(-2))\big) = \left\lceil \tfrac{n-3}{3} \right\rceil + 2.$$
Therefore, from (2) we conclude that

$$\operatorname{pd} S/I_{P_n} = \max\left\{\left\lceil \frac{2(n-1)}{3} \right\rceil, \left\lceil \frac{2n}{3} \right\rceil\right\} = \left\lceil \frac{2n}{3} \right\rceil$$

and from (3) we conclude that

$$\begin{aligned}\operatorname{reg}(S/I_{P_n}) &= \max\left\{\left\lceil\frac{n-3}{3}\right\rceil + 2 - 1,\ \left\lceil\frac{n-1}{3}\right\rceil\right\}\\ &= \max\left\{\left\lceil\frac{n}{3}\right\rceil,\ \left\lceil\frac{n-1}{3}\right\rceil\right\} = \left\lceil\frac{n}{3}\right\rceil. \qquad \square\end{aligned}$$

Hilbert's syzygy theorem states that the longest a minimal free resolution for S/I_{P_n} could be is $n+1$. However, we see that even though a path appears to be rather simple, the projective dimension is already $\lceil 2n/3\rceil$. Examining the previous theorem more closely, we see that divisibility of the path's length by 3 has an effect on both the projective dimension and the regularity of S/I_{P_n}. Furthermore, since trees can be constructed inductively from paths by the addition of the appropriate leaves, we see that divisibility by 3 also plays an important role in the minimal free resolution of the edge ideal of a tree as shown in the following example.

3.1.2. Example. Let us consider the addition of one leaf to the path of length 7, P_7 by first adding a leaf to the third vertex of P_7 (Figure 4). Then the corresponding edge ideal is

$$I = (x_0x_1x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_2x_8),$$

and the minimal free resolution for S/I is given by

$$0 \to S^3 \to S^{15} \to S^{26} \to S^{21} \to S^8 \to S \to S/I \to 0.$$

Now let us consider the addition of a leaf to the fourth vertex of P_7 (Figure 5). Then the corresponding edge ideal is

$$J = (x_0x_1x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_3x_8),$$

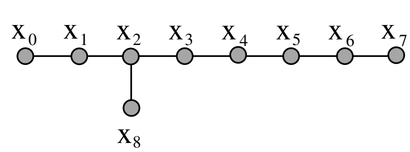

FIGURE 4.

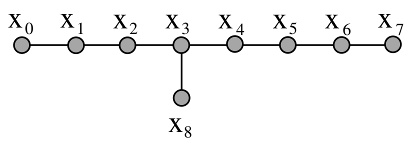

FIGURE 5.

and the minimal free resolution for S/J is given by

$$0 \to S \to S^6 \to S^{18} \to S^{27} \to S^{21} \to S^8 \to S \to S/J \to 0.$$

Let us now take a closer look at the minimal free resolution of S/I_{P_n} and provide more detailed information about the last module in the minimal free resolution courtesy of Theorem 2.1.1.

3.1.3. Corollary. *For a path of length n,*

$$\beta_{\lceil 2n/3 \rceil}(S/I_{P_n}) = \begin{cases} 1 & \textit{if } 3 \nmid n \\ (n/3)+1 & \textit{if } 3 \mid n. \end{cases}$$

Proof. Induct on the length of the path n. For $n = 1$, $I_{P_1} = (x_0x_1)$ and $\beta_1(S/I_{P_1}) = 1$. Consider the coarsely graded short exact sequence

$$0 \longrightarrow \left(S/I_{P_{n-1}} : (x_{n-1}x_n)\right)(-2) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0. \tag{4}$$

However, $I_{P_{n-1}} : (x_{n-1}x_n) = I_{P_{n-3}} + (x_{n-2})$, and hence (4) becomes

$$0 \longrightarrow \left(S/(I_{P_{n-3}} + (x_{n-2}))\right)(-2) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0. \tag{5}$$

Since the generators of $I_{P_{n-3}}$ and (x_{n-2}) are disjoint, the minimal free resolution for $S/(I_{P_{n-3}} + (x_{n-2}))$ is formed by the tensor product of minimal free resolutions for $S/I_{P_{n-3}}$ and $S/(x_{n-2})$. Then by considering the mapping cone construction and applying Theorem 2.1.1 to (5), we obtain

$$\begin{aligned} \beta_{\lceil 2n/3 \rceil}(S/I_{P_n}) = B_{\lceil 2n/3 \rceil}(S/I_{P_{n-1}}) &+ B_{\lceil 2n/3 \rceil - 1}(S/I_{P_{n-3}}) \\ &+ B_{\lceil 2n/3 \rceil - 2}(S/I_{P_{n-3}}). \end{aligned} \tag{6}$$

Moreover, the induction hypothesis along with Proposition 3.1.1 (1) provides

$$\beta_{\lceil 2n/3 \rceil}(S/I_{P_{n-1}}) = \begin{cases} 0 & \text{if } 3 \nmid n \\ B_{\lceil 2(n-1)/3 \rceil}(S/I_{P_{n-1}}) & \text{if } 3 \mid n \end{cases} = \begin{cases} 0 & \text{if } 3 \nmid n \\ 1 & \text{if } 3 \mid n. \end{cases}$$

Furthermore, the induction hypothesis coupled with Proposition 3.1.1 (1) also provides that

$$\beta_{\lceil 2n/3 \rceil - 1}(S/I_{P_{n-3}}) = \beta_{\lceil 2(n-3)/3 \rceil + 1}(S/I_{P_{n-3}}) = 0$$

and

$$\begin{aligned} \beta_{\lceil 2n/3 \rceil - 2}(S/I_{P_{n-3}}) &= \beta_{\lceil 2(n-3)/3 \rceil}(S/I_{P_{n-3}}) \\ &= \begin{cases} 1 & \text{if } 3 \nmid n \\ (n-3)/3 + 1 & \text{if } 3 \mid n. \end{cases} \end{aligned}$$

Then if follows from (6) that

$$\begin{aligned} \beta_{\lceil 2n/3 \rceil}(S/I_{P_n}) &= \begin{cases} 0 + 0 + 1 & \text{if } 3 \nmid n \\ 1 + 0 + (n-3)/3 + 1 & \text{if } 3 \mid n \end{cases} \\ &= \begin{cases} 1 & \text{if } 3 \nmid n \\ (n/3) + 1 & \text{if } 3 \mid n. \end{cases} \quad \square \end{aligned}$$

Furthermore, if we would like to consider when an n-length path is maximal, we can consider the finely graded Betti numbers. Then Theorem 2.1.1 provides the following result.

3.1.4. Corollary. *For a path of length n,*

$$\beta_{\lceil 2n/3 \rceil, \mathbf{d}}(S/I_{P_n}) = \begin{cases} 0 & \textit{if } 3 \mid n \\ 1 & \textit{if } 3 \nmid n \end{cases}$$

where $\mathbf{d} = (1, 1, \ldots, 1)$.

Proof. Induct on the length of the path, n. For $n = 1$, $I_{P_1} = (x_0 x_1)$, and $\beta_{1,(1,1)}(S/I_{P_1}) = 1$. Set $(-\mathbf{2}) = (0, \ldots, 0, -1, -1)$, and consider the exact sequence

$$0 \longrightarrow (S/I_{P_{n-1}} : (x_{n-1}x_n))(-\mathbf{2}) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0. \tag{7}$$

However, $I_{P_{n-1}} : (x_{n-1}x_n) = I_{P_{n-3}} + (x_{n-2})$, and hence (7) becomes

$$0 \longrightarrow \big(S/I_{P_{n-3}} + (x_{n-2})\big)\,(-\mathbf{2}) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0. \tag{8}$$

Considering the leftmost module in the minimal free resolution for S/I_{P_n} and the degree shift $(\mathbf{d}) = (1,\dots,1)$, the mapping cone construction and Theorem 2.1.1 applied to the short exact sequence (8) provide the following relationship among Betti numbers.

$$\begin{aligned}\beta_{\lceil 2n/3\rceil,\mathbf{d}}(S/I_{P_n}) = \beta_{\lceil 2n/3\rceil,\mathbf{d}}&(S/I_{P_{n-1}})\\ &+ \beta_{\lceil 2n/3\rceil,\mathbf{d}}\big(S/I_{P_{n-3}} + (x_{n-2})\big)\,(-\mathbf{2}).\end{aligned} \tag{9}$$

Furthermore, since no minimal generator of $I_{P_{n-1}}$ is divisible by x_n,

$$\beta_{\lceil 2n/3\rceil,\mathbf{d}}(S/I_{P_{n-1}}) = 0$$

and (9) becomes

$$\beta_{\lceil 2n/3\rceil,\mathbf{d}}(S/I_{P_n}) = \beta_{\lceil 2n/3\rceil,\mathbf{d}}\big(S/I_{P_{n-3}} + (x_{n-2})\big)\,(-\mathbf{2}). \tag{10}$$

At this point we break into two distinct cases based upon the divisibility of the paths length, n, by 3.

Case (i). If $3 \mid n$, then $3 \mid (n-3)$. Applying the induction hypothesis to (10) provides

$$\beta_{\lceil 2n/3\rceil,\mathbf{d}}(S/I_{P_n}) = 0.$$

Case (ii). If $3 \nmid n$, then $3 \nmid (n-3)$. Applying the induction hypothesis to (10) provides

$$\beta_{\lceil 2n/3\rceil,\mathbf{d}}(S/I_{P_n}) = 1. \qquad \square$$

In particular, the previous theorem states that paths are maximal precisely when their length is not divisible by 3. Even though we have not previously considered a path of length 0, i.e., the graph consisting of a single vertex, such a graph will be considered not maximal by requiring that the polynomial ring S have at least 2 variables.

The algorithm presented at the end of Section 2 is based upon the decomposition of the original graph into smaller known graphs, in particular into paths and complete bipartite graphs $K_{1,m}$. Then using that $K_{1,m}$ is maximal for all $m \geq 1$ and paths are maximal when $3 \nmid n$, we are able to deduce when an arbitrary tree is maximal. This idea was used in the development of the algorithm shown at the end of Section 2.

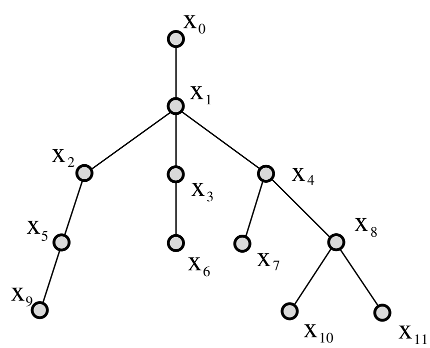

FIGURE 6.

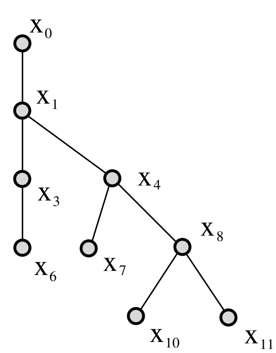

FIGURE 7.

3.1.5. Example. We want to determine whether the tree, T, is maximal (Figure 6). Select a leaf of T, say x_9; and then recall that T is maximal precisely when the subforest defined by $I_{T\setminus\{x_5,x_9\}} : (x_5x_9)$ is maximal. Call this subforest F_1 and consider its maximality (Figure 7). Now select a leaf of F_1, say x_6; and notice that for F_1 to be maximal,

$I_{F_1\setminus\{x_3,x_6\}} : (x_3x_6)$ must be maximal. Then consider the subforest of F_1 defined by $I_{F_1\setminus\{x_3,x_6\}} : (x_3x_6)$, and denote it by F_2 (Figure 8).

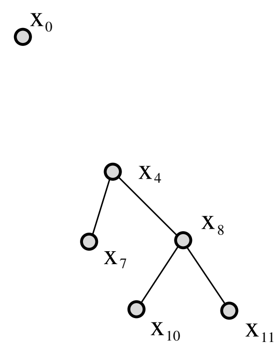

FIGURE 8.

Recall that, for a forest to be maximal, each component tree of the forest must be maximal. However, the vertex x_0 forms a component tree of F_2 that is not maximal. Therefore, the original tree T is not maximal. To verify this, consider the following Betti diagram for S/I_T obtained from Macaulay 2 (see [**5**]).

Total:	1	11	38	68	70	42	14	2
0:	1	-	-	-	-	-	-	-
1:	-	11	15	6	1	-	-	-
2:	-	-	23	50	37	11	1	-
3:	-	-	-	12	32	31	13	2

From this Betti diagram we see that the leftmost module in a minimal free resolution for S/I_T is of rank 2 and has both copies of S in the coarsely graded shift $3 + 7 = 10$. However, for T to be maximal, this leftmost module must have a shift of 11.

In the previous example, we notice that there can be a great advantage in this algorithm by choosing to remove the leaf whose neighbor has the highest degree. However, this is not always the best choice. For

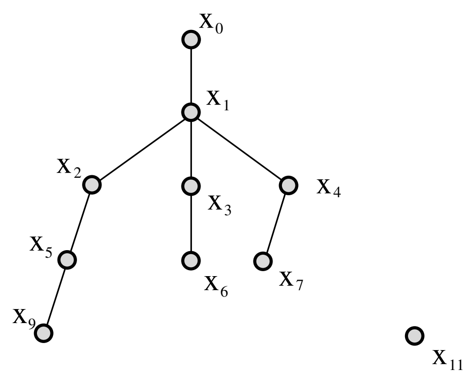

FIGURE 9.

instance, in the previous example x_0 would be the leaf whose neighbor has the highest degree, namely 4, but removing x_0 would still result in more than one step to determine whether or not T is maximal. However, if we were to first remove vertex x_{10}, the subforest of T defined by $I_{T\setminus\{x_8,x_{10}\}} : (x_8x_{10})$ is depicted in Figure 9. After this one step, we can already see that the original tree T is not maximal, because x_{11} constitutes a path of length 0 and hence is not maximal.

We would next like to consider when a path is *level*, i.e., when the last module in the minimal free resolution for S/I_{P_n} has only one degree shift.

3.1.6. Proposition. *Let P_n be a path of length n. Then the corresponding edge ideal, I_{P_n}, is level with level shift given by*

$$\begin{cases} n+1 & \text{if } 3 \nmid n \\ n & \text{if } 3 \mid n. \end{cases}$$

Proof. We will proceed by induction on the path's length, n. For $n = 1$ and $n = 2$, the claim follows from Corollaries 3.1.4 and 3.1.3. For $n = 3$, $I_{P_3} = (x_0x_1, x_1x_2, x_2x_3)$ is level with level shift 3. For $n \geq 4$, we have the following exact sequence as in Theorem 2.1.1.

$$0 \longrightarrow S/(I_{P_{n-3}} + (x_{n-2}))(-2) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0.$$

Moreover, Proposition 3.1.1 provides that

$$\operatorname{pd}(S/(I_{P_{n-3}} + (x_{n-2}))) = \left\lceil \frac{2n}{3} \right\rceil - 1$$

and

$$\operatorname{pd}(S/I_{P_{n-1}}) = \left\lceil \frac{2n-2}{3} \right\rceil.$$

Then we have the following two cases based upon the divisibility of the path's length by 3.

Case (i). If $3 \nmid n$, then $\operatorname{pd}(S/(I_{P_{n-3}} + (x_{n-2}))) = \operatorname{pd}(S/I_{P_{n-1}})$. Furthermore, by the induction hypothesis $I_{P_{n-3}} + (x_{n-2})$ is level. Hence S/I_{P_n} is also level with level shift $(n-3)+1+1+2 = n+1$.

Case (ii). If $3 \mid n$, then $\operatorname{pd}(S/I_{P_{n-1}}) = \operatorname{pd}(S/(I_{P_{n-3}} + (x_{n-2}))) + 1$. Moreover, the induction hypothesis provides that both $I_{P_{n-3}} + (x_{n-2})$ and $I_{P_{n-1}}$ are level with

$$\begin{aligned} \text{level shift of } S/(I_{P_{n-3}} + (x_{n-2}))(-2) &= (n-3)+1+2 = n \\ \text{level shift of } S/I_{P_{n-1}} &= (n-1)+1 = n. \end{aligned}$$

Therefore, S/I_{P_n} is also level with level shift n. □

3.2. Minimal free resolutions of the edge ideals of 3-spiders. It seems natural to extend from paths to the class of graphs resembling

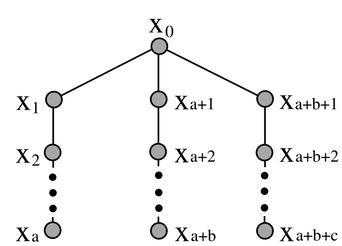

FIGURE 10.

Since this graph resembles a spider with 3 legs, we will call it a 3-spider. This is a natural extension from the class of paths, because if we delete the rightmost leg we would be left with a path of length $a+b$.

Using Proposition 3.1.1 and the mapping cone construction presented in Theorem 2.1.1, we can write an explicit formula for the length of a minimal free resolution corresponding to the quotient ring of a 3-spider.

3.2.1. Proposition. *Let G be the graph of a 3-spider. Then for*

$$\begin{aligned}
c = 1 : \operatorname{pd}(S/I_G) &= \left\lceil \tfrac{2a-1}{3} \right\rceil + \left\lceil \tfrac{2b-1}{3} \right\rceil + 1 \\
c = 2 : \operatorname{pd}(S/I_G) &= \begin{cases} \left\lceil \frac{2a-1}{3} \right\rceil + \left\lceil \frac{2b-1}{3} \right\rceil + 1 & \textit{if } a, b \equiv 1 \bmod 3 \\ \left\lceil \frac{2(a-1)}{3} \right\rceil + \left\lceil \frac{2(b-1)}{3} \right\rceil + 2 & \textit{else} \end{cases} \\
c \geq 3 : \operatorname{pd}(S/I_G) &= \left\lceil \frac{2(a+b+c)}{3} \right\rceil + (-1)^r r \\
\textit{where } r &= \min\{a \bmod 3, b \bmod 3, c \bmod 3\}.
\end{aligned}$$

Proof. Consider the following short exact sequence

$$\begin{aligned}
0 \longrightarrow \left(S/I_{G \setminus \{x_{a+b+c-1}, x_{a+b+c}\}} : (x_{a+b+c-1}x_{a+b+c})\right)(-2) \\
\longrightarrow S/I_{G \setminus \{x_{a+b+c-1}, x_{a+b+c}\}} \longrightarrow S/I_G \longrightarrow 0.
\end{aligned}$$

Then the mapping cone construction and Theorem 2.1.1 imply that

$$\begin{aligned}
\operatorname{pd}(S/I_G) = \max\{&\operatorname{pd}(S/I_{G \setminus \{x_{a+b+c-1}, x_{a+b+c}\}}), \\
&\operatorname{pd}(S/I_{G \setminus \{x_{a+b+c-1}, x_{a+b+c}\}} : (x_{a+b+c-1}x_{a+b+c})) + 1\}.
\end{aligned}$$

However, we can consider $I_{G \setminus \{x_{a+b+c-1}, x_{a+b+c}\}} : (x_{a+b+c-1}x_{a+b+c})$ graphically as follows.

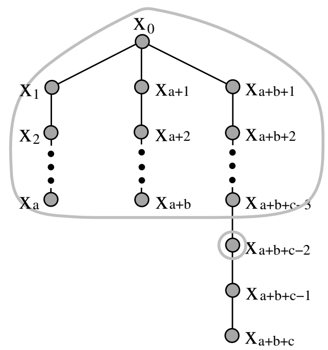

FIGURE 11.

We will proceed by induction on the length of the third leg, c.

$$c = 1 : \operatorname{pd}(S/I_G) = \max\left\{\left\lceil\frac{2(a+b)}{3}\right\rceil, \left\lceil\frac{2(a-2)}{3}\right\rceil + \left\lceil\frac{2(b-2)}{3}\right\rceil + 3\right\}$$
$$= \left\lceil\frac{2a-1}{3}\right\rceil + \left\lceil\frac{2b-1}{3}\right\rceil + 1$$
$$c = 2 : \operatorname{pd}(S/I_G) = \max\left\{\begin{array}{r}\left\lceil\frac{2a-1}{3}\right\rceil + \left\lceil\frac{2b-1}{3}\right\rceil + 1, \\ \left\lceil\frac{2(a-1)}{3}\right\rceil + \left\lceil\frac{2(b-1)}{3}\right\rceil + 2\end{array}\right\}$$
$$= \begin{cases}\left\lceil\frac{2a-1}{3}\right\rceil + \left\lceil\frac{2b-1}{3}\right\rceil + 1 & \text{if } a, b \equiv 1 \bmod 3 \\ \left\lceil\frac{2(a-1)}{3}\right\rceil + \left\lceil\frac{2(b-1)}{3}\right\rceil + 2 & \text{else}\end{cases}$$
$$c = 3 : \operatorname{pd}(S/I_G) = \begin{cases}\max\left\{\left\lceil\frac{2a-1}{3}\right\rceil + \left\lceil\frac{2b-1}{3}\right\rceil + 1, \left\lceil\frac{2(a+b)}{3}\right\rceil + 2\right\} \\ \qquad \text{if } a, b \equiv 1 \bmod 3 \\ \max\left\{\left\lceil\frac{2(a-1)}{3}\right\rceil + \left\lceil\frac{2(b-1)}{3}\right\rceil + 2, \left\lceil\frac{2(a+b)}{3}\right\rceil + 2\right\} \\ \qquad \text{else}\end{cases}$$
$$= \left\lceil\frac{2(a+b+c)}{3}\right\rceil.$$

Assume the statement is true for the third length having length $c-1$. Then

$$\operatorname{pd}(S/I_G) = \max\left\{\left\lceil\frac{2(a+b+c-1)}{3}\right\rceil + (-1)^{r'}r', \left\lceil\frac{2(a+b+c)}{3}\right\rceil + (-1)^{r}r\right\},$$

where $r' = \min\{a \bmod 3, b \bmod 3, (c-1) \bmod 3\}$.

Case (i). If $c \equiv 0 \bmod 3$, then $(c-1) \equiv 2 \bmod 3$ and $r' = \min\{a \bmod 3, b \bmod 3\}$. However, regardless of the value of r', we see that

$$\mathrm{pd}\,(S/I_G) = \left\lceil \frac{2(a+b+c)}{3} \right\rceil.$$

Case (ii). If $c \equiv 1 \bmod 3$, then $(c-1) \equiv 0 \bmod 3$ and $r' = 0$. Furthermore, since $(c-1) \equiv 0 \bmod 3$ implies that $\lceil (2(a+b+c-1))/3 \rceil = \lceil (2(a+b+c))/3 \rceil - 1$, we obtain

$$\mathrm{pd}\,(S/I_G) = \left\lceil \frac{2(a+b+c)}{3} \right\rceil + (-1)^r r.$$

Case (iii). If $c \equiv 2 \bmod 3$, then $(c-1) \equiv 1 \bmod 3$ and $r' \in \{0,1\}$. However, for either value of r' we see that

$$\mathrm{pd}\,(S/I_G) = \left\lceil \frac{2(a+b+c)}{3} \right\rceil + (-1)^r r. \qquad \square$$

4. Minimal primary decompositions. For a simple graph G, consider the minimal primary decomposition of I_G. The prime ideals occurring in the decomposition of I_G can be realized as minimal vertex covers of the planar graph G via the following one-to-one correspondence.

$$\left\{ \begin{array}{c} \text{Minimal vertex covers of} \\ \text{a simple graph } G \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Associated prime} \\ \text{ideals of } I_G \end{array} \right\}.$$

4.0.2. Definition. Let G be a simple graph on the vertex set $V_G = \{x_0, \dots, x_n\}$ such that G possesses no isolated vertex, i.e., for each vertex x_i there is an edge e of G with $x_i \in e$. A *vertex cover* of G is a subset $C \subset V_G$ such that, for each edge $\{x_i, x_j\}$ of G, one has either $x_i \in C$ or $x_j \in C$. Such a vertex cover C is called *minimal* if no subset $C' \subset C$ with $C' \neq C$ is a vertex cover of G.

The above one-to-one correspondence provides a way to quickly determine and verify the prime ideals present in the minimal primary decomposition of I_G using the graphical concept of a minimal vertex cover. Furthermore, if we restrict to the class of paths, the number of minimal vertex covers of P_n (and correspondingly the number of associated prime ideals of I_{P_n}) can be determined by the following recursive formula.

4.0.3. Proposition. *Let $P(n)$ represent the number of minimal vertex covers of P_n. Then*

$$P(n) = P(n-2) + P(n-3).$$

Proof. Proceed by induction of the length of the path, n. The following table illustrates the base case.

n	Minimal Vertex Covers	$P(n)$
0	$\{x_0\}$	1
1	$\{x_0\}$,$\{x_1\}$	2
2	$\{x_1\}, \{x_0, x_2\}$	2
3	$\{x_0, x_2\}, \{x_1, x_2\}, \{x_1, x_3\}$	3

Thus, $P(3) = P(1) + P(0)$. Assume the claim is true for P_{n-1}. We make the following definition.

$$q(n) := |\{\text{Minimal vertex covers of } P_n \text{ that include the vertex } x_n\}|\,.$$

Notice that the above definition for $q(n)$ also means that x_{n-1} is not chosen. Furthermore, we have the following equality.

$$P(n) = q(n) + P(n-2). \tag{11}$$

Moreover,

$$\begin{aligned} P(n) &= \begin{pmatrix} \text{Choose } x_{n-1} \\ \text{Can't choose } x_n \end{pmatrix} + \begin{pmatrix} \text{Choose } x_n \\ \text{Can't choose } x_{n-1} \\ \text{Have to choose } x_{n-2} \\ \text{Don't choose } x_{n-3} \end{pmatrix} + \begin{pmatrix} \text{Choose } x_n \\ \text{Can't choose } x_{n-1} \\ \text{Have to choose } x_{n-2} \\ \text{Choose } x_{n-3} \end{pmatrix} \\ &= \qquad P(n-2) \qquad + \qquad q(n-2) \qquad + \qquad q(n-3) \end{aligned}$$

Using (11) we obtain

$$P(n) = P(n-2) + [P(n-2) - P(n-4)] + [P(n-3) - P(n-5)].$$

Finally, applying the induction hypothesis to $P(n-2)$ provides the claim, namely that

$$\begin{aligned} P(n) &= P(n-2) + P(n-4) + P(n-5) - P(n-4) \\ &\quad + P(n-3) - P(n-5) \\ &= P(n-2) + P(n-3). \qquad \square \end{aligned}$$

This recursive formula leads to the following explicit formula for the number of prime ideals in the minimal primary decomposition of I_{P_n}.

4.0.4. Corollary. *The number of associated prime ideals for* I_{P_n} *is given by*

$$P(n) = \sum_{i=1}^{3} \frac{(r_i+1)^2}{r_i^n(r_i^3+2)}$$

where r_1, r_2, *and* r_3 *represent the* 3 *distinct roots of* $x^3 + x^2 - 1$.

Proof. The proof follows from Proposition 4.0.3 and uses standard techniques of ordinary differential equations. $\square$

5. Edge ideals of cycles. In the previous sections, simple graphs that contained a vertex of degree 1 were considered. It is natural to ask when the previous results can be extended to generate information about the edge ideals of simple graphs that do not contain a vertex of degree 1. The simplest of these is a cycle of length n which can be depicted as in Figure 12.

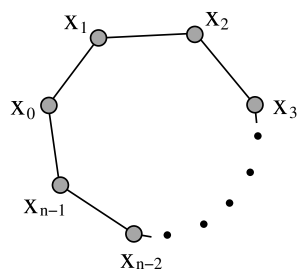

FIGURE 12.

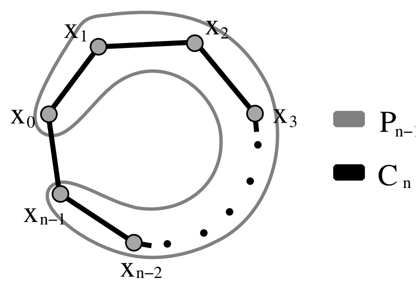

FIGURE 13.

Comparing an n-cycle, denoted C_n, to a path of length $n-1$, we see the relationships among both the graphs and the corresponding edge ideals (Figure 13).

It is tempting to assume that the mapping cone construction used to obtain the inductive construction for the minimal free resolutions corresponding to paths also applies to cycles. However, the following example illustrates that the mapping cone construction does not necessarily produce the minimal free resolution in the case of cycles.

5.0.5. Example. Consider the edge ideal of the cycle of length 5,

$$I_{C_5} = (x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_0x_4) \subset S = k[x_0, \ldots, x_4].$$

Set $J = (x_0x_1, x_1x_2, x_2x_3, x_3x_4)$, and consider the following short exact sequence.

$$0 \longrightarrow S/J : (x_0x_4)(-2) \xrightarrow{x_0x_4} S/J \longrightarrow S/I \longrightarrow 0.$$

Then the mapping cone procedure using **Z**-graded minimal free resolutions of $S/J : (x_0x_4)$ and S/J provides the following free resolution of S/I_{C_5}.

$$\begin{tikzcd}
0 \arrow[r] & S/J : (x_0x_4)(-2) \arrow[r] & S/J \arrow[r] & S/I_{C_5} \arrow[r] & 0 \\
 & S(-2) \arrow[u] & S \arrow[u] & S \arrow[u, dashed] & \\
 & S^2(-3)] \arrow[u] & S^4(-2) \arrow[u] & S^5(-2) \arrow[u, dashed] & \\
 & S(-4) \arrow[u] & S^3(-3) \oplus S(-4) \arrow[u] & S^5(-3) \oplus S(-4) \arrow[u, dashed] & \\
 & 0 \arrow[u] & S(-5) \arrow[u] & S(-4) \oplus S(-5) \arrow[u, dashed] & \\
 & & 0 \arrow[u] & 0 \arrow[u, dashed] &
\end{tikzcd}$$

However, the above resolution of S/I_{C_5} is not minimal. The minimal free resolution for S/I_{C_5} is given by

$$0 \longrightarrow S(-5) \longrightarrow S^5(-3) \longrightarrow S^5(-2) \longrightarrow S \longrightarrow S/I_{C_5} \longrightarrow 0.$$

Fortunately, using this relationship between paths and cycles we can still obtain an explicit formula for the length of a minimal free resolution corresponding to C_n. We will mimic the procedure used for trees by removing an arbitrary edge, say $\{x_{n-1}, x_0\}$. Upon removing edge

$\{x_{n-1}, x_0\}$, we are left with P_{n-1} as shown above in Figure 5. We must be careful though, because Theorem 2.1.1 no longer applies.

5.0.6. Proposition. *Let C_n denote a cycle of length n. Then*

$$\operatorname{pd}(S/I_{C_n}) = \begin{cases} \lceil \frac{2n}{3} \rceil & \text{if } 3 \mid (n-1) \\ \left\lceil \frac{2(n-1)}{3} \right\rceil & \text{if } 3 \nmid (n-1). \end{cases}$$

Proof. Consider the exact sequence

$$0 \longrightarrow \big(S/I_{P_{n-1}} : (x_{n-1}x_0)\big)(-2) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{C_n} \longrightarrow 0. \tag{12}$$

Moreover,

$$I_{P_{n-1}} : (x_{n-1}x_0) = (x_1, x_{n-2}) + (x_2x_3, x_3x_4, \dots, x_{n-4}x_{n-3}).$$

Hence, (12) becomes

(13)

$$0 \longrightarrow (S/((x_1, x_{n-2}) + (x_2x_3, x_3x_4, \dots, x_{n-4}x_{n-3})))(-2) \longrightarrow S/I_{P_{n-1}} \\ \longrightarrow S/I_{C_n} \longrightarrow 0.$$

It should be noted that $(x_2x_3, x_3x_4, \dots, x_{n-4}x_{n-3}) \subset k[x_2, \dots, x_{n-3}]$ is isomorphic to $I_{P_{n-5}} \subset k[x_0, \dots, x_{n-5}]$, and so we shall set

$$I_{P'_{n-5}} := (x_2x_3, x_3x_4, \dots, x_{n-4}x_{n-3}).$$

Consequently, Theorem 3.1.1 provides that

$$\operatorname{pd}\left(S/((x_1, x_{n-2}) + I_{P'_{n-5}})\right) = 2 + \left\lceil \frac{2(n-5)}{3} \right\rceil = \left\lceil \frac{2(n-2)}{3} \right\rceil$$

and

$$\operatorname{pd}\left(S/I_{P_{n-1}}\right) = \left\lceil \frac{2(n-1)}{3} \right\rceil.$$

From the mapping cone construction, we obtain that

$$\operatorname{pd}(S/I_{C_n}) \le \max\left\{\operatorname{pd}\left(S/((x_1, x_{n-2}) + I_{P'_{n-5}})\right) + 1, \operatorname{pd}\left(S/I_{P_{n-1}}\right)\right\}.$$

We will proceed by showing that the last module of the free resolution for S/I_{C_n} obtained via the mapping cone construction cannot cancel, i.e., that

$$\text{pd}\ (S/I_{C_n}) = \max\left\{\text{pd}\ \left(S/((x_1, x_{n-2}) + I_{P'_{n-5}})\right) + 1, \text{pd}\ \left(S/I_{P_{n-1}}\right)\right\}.$$

Case (i). If $3 \mid (n-1)$, or $n \equiv 1 \bmod 3$; then P_{n-1} is not maximal. However, in this case, P_{n-5} is maximal. Additionally,

$$\text{pd}\ \left(S/((x_1, x_{n-2}) + I_{P'_{n-5}})\right) = \text{pd}\ \left(S/I_{P_{n-1}}\right),$$

and hence there can be no cancelation in the last module of the free resolution for S/I_{C_n} formed from the mapping cone construction. Therefore

$$\text{pd}\ (S/I_{C_n}) = \left\lceil \frac{2(n-1)}{3} \right\rceil + 1 = \left\lceil \frac{2n}{3} \right\rceil.$$

Case (ii). If $3 \nmid (n-1)$, or $n \equiv 0, 2 \bmod 3$; then P_{n-1} is maximal. Also

$$\text{pd}\ \left(S/((x_1, x_{n-2}) + I_{P'_{n-5}})\right) = \text{pd}\ \left(S/I_{P_{n-1}}\right) - 1.$$

Furthermore, the copy of S with the maximal shift in the last module of the free resolution for S/I_{C_n} obtained via the mapping cone construction cannot cancel, and consequently

$$\text{pd}\ (S/I_{C_n}) = \left\lceil \frac{2(n-1)}{3} \right\rceil. \qquad \square$$

The above proposition says that the length of a minimal free resolution corresponding to C_n agrees with the length of a minimal free resolution of P_{n-1} as long as $3 \nmid (n-1)$. However, in the alternate case, namely when $3 \mid (n-1)$, we see that the length of the minimal free resolution corresponding to C_n agrees with the length of a minimal free resolution for P_n.

In general, we notice that simple graphs are composed of trees and cycles. As seen in the case of cycles, even though Theorem 2.1.1 does not apply to a general simple graph G, we can still use the short exact sequence

$$0 \to \left(S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)\right)(-2) \xrightarrow{x_{n-1}x_n} S/I_{G\setminus\{x_{n-1},x_n\}} \to S/I_G \to 0,$$

where $\{x_{n-1}, x_n\}$ is an arbitrary edge of the graph G. Since a simple graph G can be constructed by the addition of edges to subgraphs of G that are paths and cycles, the above results can be used to generate estimates on the projective dimension of the more general module S/I_G.

REFERENCES

1. Shalom Eliahou and Michel Kervaire, *Minimal resolutions of some monomial ideals*, J. Algebra **129** (1990), 1–25.

2. Shalom Eliahou and Rafael H. Villarreal, *The second Betti number of an edge ideal*, XXXI National Congress of the Mexican Mathematical Society (Spanish) (Hermosillo, 1998), Aportaciones Mat. Comun., vol. 25, Soc. Mat. Mexicana, México, (1999), 115–119.

3. Christopher A. Francisco, Huy Tài Hà and Adam Van Tuyl, *Splittings of monomial ideals*, (to appear), arXiv:0807.2185.

4. Christopher A. Francisco and Adam Van Tuyl, *Sequentially Cohen-Macaulay edge ideals*, Proc. Amer. Math. Soc. **135** (2007), 2327–2337 (electronic).

5. Daniel R. Grayson and Michael E. Stillman, *Macaulay* 2, *A software system for research in algebraic geometry*, available at http://www.math.uiuc.edu/Macaulay2/.

6. Huy Tài Hà and Adam Van Tuyl, *Splittable ideals and the resolutions of monomial ideals*, J. Algebra **309** (2007), 405–425.

7. ———, *Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers*, J. Algebraic Combin. **27** (2008), 215–245.

8. Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Grad. Texts Math. **227**, Springer-Verlag, New York, 2005.

9. Uwe Nagel and Victor Reiner, *Betti numbers of monomial ideals and shifted skew shapes*, (to appear), arXiv:0712.2537v1.

10. Mike Roth and Adam Van Tuyl, *On the linear strand of an edge ideal*, Comm. Algebra **35** (2007), 821–832.

11. William Stein, *Sage*: *Open Source Mathematical Software* (Version 2.10.2), The Sage Group, 2008, http://www.sagemath.org.

12. Rafael H. Villarreal, *Rees algebras of edge ideals*, Comm. Algebra **23** (1995), 3513–3524.

13. Xinxian Zheng, *Resolutions of facet ideals*, Comm. Algebra **32** (2004), 2301–2324.

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