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Free-fall and heteroclinic orbits to triple collisions in the isosceles three-body problem

By

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Abstract

We consider the spatial isosceles three-body problem where two masses are equal and the other may be different. We discuss free-fall orbits in the planar case and show that there exists a countable family of orbits converging to triple collisions forward and backward in time. In Devaney's coordinates, the orbits correspond to topologically transverse heteroclinic orbits between equilibria on the collision manifold in the blown-up equations (Theorem 1.1). By the "window theory" of Moeckel [*SIAM J. Math. Anal.*, **15** (1984), 857–876], we see that there exists an orbit shadowing a prescribed path on the graph consisting of these new heteroclinic orbits and already known, heteroclinic and periodic orbits. In particular, choosing appropriate infinite paths, we obtain a variety of new oscillatory orbits.

1. Introduction and Main Results

This paper is concerned with the Newtonian three-body problem given by the following set of ODEs:

$$m_\ell \frac{d^2\mathbf{x}_\ell}{dt^2} = \frac{\partial U}{\partial \mathbf{x}_\ell}, \quad \mathbf{x}_\ell = (x_\ell, y_\ell, z_\ell) \in \mathbb{R}^3, \quad \ell = 1,2,3, \quad m_\ell > 0 \tag{1.1}$$

where m_ℓ is the mass of the ℓ-th particle and $U(\mathbf{x})$ is the gravitational potential given by

$$U(\mathbf{x}) = \sum_{i<j} \frac{m_i m_j}{\|\mathbf{x}_i - \mathbf{x}_j\|}.$$

We consider the isosceles three-body problem: two masses are equal, $m_1 = m_2$, and m_3 remains on the z-axis in $\mathbb{R}^3$, while m_1 and m_2 remain symmetric with respect to this axis (Figure 1). We fix the center of mass at the origin without loss of generality. Then the isosceles three-body problem is given by the Hamiltonian system with Hamiltonian

$$H = \frac{1}{4}(p_x^2 + p_y^2) + \frac{\alpha+2}{4\alpha} p_z^2 - \frac{1}{2\alpha\sqrt{x^2+y^2}} - \frac{2}{\sqrt{x^2+y^2+z^2}}, \tag{1.2}$$

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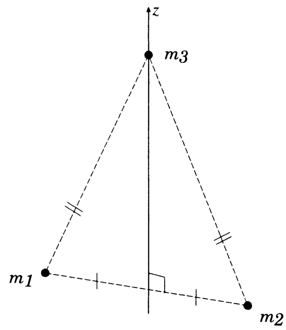

Figure 1. Isosceles three-body problem

where

$$\mathbf{x}_1 = \left(x, y, -\frac{\alpha}{\alpha+2}z\right), \mathbf{x}_2 = \left(-x, -y, -\frac{\alpha}{\alpha+2}z\right), \mathbf{x}_3 = \left(0, 0, \frac{2}{\alpha+2}z\right)$$

and (p_x, p_y, p_z) is the momentum corresponding to (x, y, z). Throughout we assume that all double collisions are regularized; by a change of variables their singularities are transformed to regular points (see Section 2), their behavior corresponds to that of elastic bounces. We discuss free-fall orbits, i.e. orbits with $p_x = p_y = p_z = 0$ at $t = 0$, and show that for each $k \in \{0, 1, 2, 3, \dots\}$ there exists an orbit which has triple collisions in the future and past and which experiences exactly k double collisions of m_1 and m_2 between the triple collisions.

In order to study behavior at and near triple collisions, it is convenient to use Devaney's coodinates [1], which we introduce in Section 3. In these coordinates the triple-collision singularity is blown-up to a two-dimensional invariant manifold called the *collision manifold* (Figure 3) on which there are six equilibrium points $E_\pm, E^*_\pm, C, C^*$. See Section 3 for the details. Orbits ending (resp. starting) at the triple collisions in the isosceles three-body problem correspond to those converging to C^* or $E^*_\pm$ as $t \to \infty$ (resp. C or $E_\pm$ as $t \to -\infty$) in Devaney's coordinate. Our main result is as follows.

Theorem 1.1. *For any $k \in \{0, 1, 2, 3 \dots\}$, there exists a topologically transverse heteroclinic orbit $\gamma^k_\pm$ on the energy surface* (*which we will denote $\mathscr{M}_+$ in Section* 3) *from $E_\pm$ to $E^*_\pm$ such that $\gamma^k_\pm$ has k double collisions between m_1 and m_2.*

Moeckel [3] analytically obtained heteroclinic orbits from C to C^* and from $E_\pm$ to $E^*_\pm$, which are denoted by κ_C and $\kappa_{E_\pm}$, respectively, such that the three masses move homothetically while maintaining a collinear or equilateral configuration. In particular, the heteroclinic orbits $\kappa_{E_\pm}$ are coincident with $\gamma^0_\pm$ and were proven to be transverse. McGehee [2] showed that there exist two-dimensional analytic manifolds called parabolic manifolds $\mathcal{P}_\pm$ and including parabolic orbits ($z \to \pm\infty$ and $p_z \to 0$ as $t \to +\infty$). The upper parts of $\mathcal{P}_\pm$ are also viewed as the stable manifolds of virtual hyperbolic periodic orbits $\beta_\pm$ for which m_1 and m_2 move according to the Kepler law while m_3 stays at infinity ($z = \pm\infty$). Moeckel [3] proved that there exist infinitely many heteroclinic orbits for $E_\pm$, $E_\pm$ and $\beta_\pm$ except $\kappa_{E_\pm}$. More precisely, there are

- a heteroclinic orbit η_+ from E_+ to β_+ (and η_- from E_- to β_-),
- a heteroclinic orbit θ_+ from β_+ to E_+^* (and θ_- from β_- to E_-^*),

on $\mathcal{C}$ for any mass ration;

- infinitely many heteroclinic orbits $\{\rho_+^k\}_{k\in\mathbb{N}}$ from E_+ to E_+^* (and $\{\rho_-^k\}_{k\in\mathbb{N}}$ from E_- to E_-^*),
- infinitely many heteroclinic orbits $\{\lambda_+^k\}_{k\in\mathbb{N}}$ from E_+ to E_-^* (and $\{\lambda_-^k\}_{k\in\mathbb{N}}$ from E_- to E_+^*),
- a heteroclinic orbit δ_+ from E_+^* to E_- (and δ_- from E_-^* to E_+)

on $\mathcal{C}$ for $m_3/m_1 > 0$ large; and

- a heteroclinic orbit α_+ from E_+^* to E_+ (and α_- from E_-^* to E_-)

returning $\mathcal{C}$ after leaving it for $m_3/m_1 > 0$ small. These heteroclinic orbits are different from $\gamma_\pm^k$ since $\rho_\pm^k$ and $\lambda_\pm^k$ pass near κ_C and have no double collision. On the other hand, Simó and Martínez [6] also proved the existence of infinitely many heteroclinic orbits from $E_\pm$ to $E_\pm^*$ which experience sufficiently many double collisions between triple collisions and which pass near the parabolic manifolds $\mathcal{P}_\pm$ sufficiently many times. These heteroclinic orbits may be coincident with $\gamma_\pm^k$ for k sufficiently large.

Figure 2 shows the heteroclinic orbits to $E_\pm$, $E_\pm^*$ and $\beta_\pm$; All above heteroclinic orbits except κ_C are topologically transverse on an invariant set $\mathscr{M}$

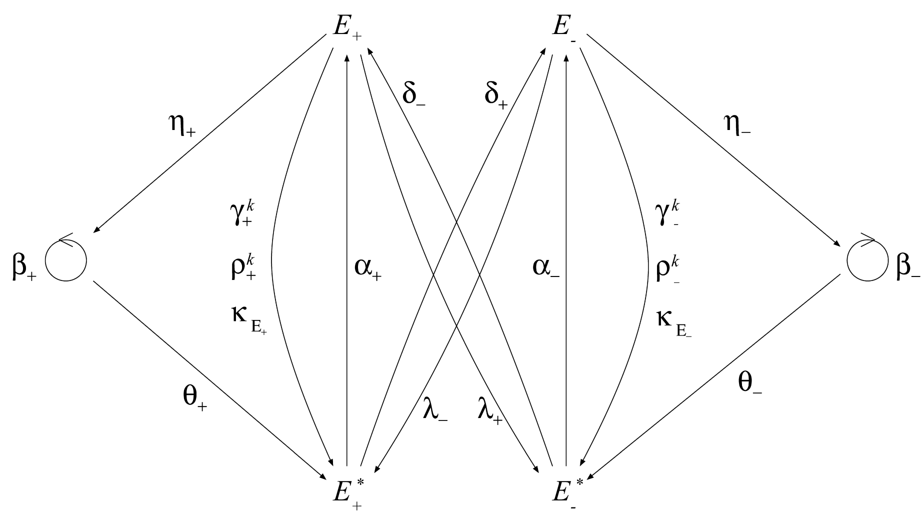

Figure 2. Topologically transverse heteroclinic orbits. The arrow $A \xrightarrow{\xi} B$ means that there exists a topologically transverse heteroclinic orbit ξ from A to B.

consisting of orbits with zero angular momentum (see Section 3 for the precise definition $\mathscr{M}$).

We now describe some immediate consequences of Theorem 1.1 based on the results of Moeckel [3]. Since orbits with a triple collision have zero angular momentum ($xp_y - yp_x = 0$), so do all above heteroclinic orbits. Moeckel [3] developed the window theory to show the existence of an orbit with small angular momentum shadowing the paths of heteroclinic orbits.

Let $\mathcal{H}$ be the set consisting of topologically transverse heteroclinic orbits on $\mathscr{M}$ and periodic orbits $\beta_\pm$. Using the window theory, we can obtain orbits shadowing the infinite sequences of the heteroclinic and periodic orbits.

Theorem 1.2 (Moeckel[3]). *Fix $h < 0$. For a given finite, path-connected subgraph $\mathcal{B}$ of $\mathcal{H}$, there is a positive number $\omega(\mathcal{B})$ such that for $0 < |\omega| < \omega(\mathcal{B})$, any infinite path in $\mathcal{B}$ is shadowed by at least one solution of the isosceles three-body problem with the angular momentum ω and with total energy $H = h$. In particular, the shadowing orbits converge to the infinite paths of heteroclinic orbits as $\omega \to 0$.*

Adding heteroclinic orbits of Theorem 1.1 to known ones, we know many new orbits experiencing near-triple collisions. In particular, we obtain new oscillatory orbits as below.

Definition 1.1. If an orbit $\mathbf{x}(t) = (\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t))$ of the three-body problem (1.1) satisfies

$$\liminf_{t\to\infty} \max_{i<j} \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| < \infty \text{ and } \limsup_{t\to\infty} \max_{i<j} \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| = \infty,$$

we call $\mathbf{x}(t)$ an *oscillatory orbit*.

Corollary 1.1 (Moeckel [3]). *In Theorem* 1.2 *if we choose a path which contains subsequences $\{\eta_\pm, \underbrace{\beta_\pm, \dots, \beta_\pm}_{j_i}, \theta_\pm\}_{i\in\mathbb{N}}$ satisfying $j_i \to \infty$ as $i \to \infty$, the orbit is an oscillatory orbit.*

Choosing a sequence which satisfies the condition of Corollary 1.1 and contains $\gamma^k_\pm$, we obtain a variety of new oscillatory orbits.

In Section 2 we consider free-fall orbits and show the existence of orbits having triple collisions forward and backward in time. In Section 3, in order to investigate the behavior at and near the triple collision, we introduce Devaney's coordinates. We give the proof of Theorem 1.1 in Section 4.

2. Free-fall orbits

In this section we only consider orbits on the subsystem $\{y = p_y = 0\}$ of the isosceles three-body problem. The problem is given by a Hamiltonian system with Hamiltonian

$$H(x, z, p_x, p_z) = \frac{1}{4}p_x^2 + \frac{\alpha+2}{4\alpha}p_z^2 - \frac{1}{2\alpha|x|} - \frac{2}{\sqrt{x^2+z^2}}.$$

Here we will blow-up double collision singularity $\{x = 0\}$ by so-called Levi-Civita coordinates. We fix the Hamiltonian energy at $H = h$, where $h < 0$ is a constant. We define canonical transformation by

$$x = \frac{1}{2}\xi^2, p_x = \frac{p_\xi}{\xi}, \tag{2.1}$$

and change the time by

$$dt = \xi^2 d\tau.$$

Then the equations become

$$\frac{d\xi}{d\tau} = \frac{1}{2}p_\xi \tag{2.2}$$

$$\frac{dz}{d\tau} = \frac{\alpha+2}{2\alpha}\xi^2 p_z \tag{2.3}$$

$$\frac{dp_\xi}{d\tau} = \frac{\alpha+2}{2\alpha}\xi p_z^2 - \frac{32\xi z^2}{(\xi^4+4z^2)^{3/2}} - 2h\xi \tag{2.4}$$

$$\frac{dp_z}{d\tau} = -\frac{16\xi^2 z}{(\xi^4+4z^2)^{3/2}} \tag{2.5}$$

which is a Hamiltonian system with Hamiltonian

$$\begin{aligned}\Gamma(p_\xi, p_z, \xi, z) &= H\left(\frac{p_\xi}{\xi}, p_z, \frac{1}{2}\xi^2, z\right)\xi^2 - h\xi^2 \\ &= \frac{1}{4}p_\xi^2 + \frac{\alpha+2}{4\alpha}\xi^2 p_z^2 - \frac{1}{\alpha} - \frac{4\xi^2}{\sqrt{\xi^4+4z^2}} - h\xi^2.\end{aligned}$$

We only consider orbits with $H = h$, i.e. with $\Gamma = 0$, and identify (ξ, z, p_ξ, p_z) with $(-\xi, z, -p_\xi, p_z)$ since the transformation (2.1) is 2-to-1 except $\xi = 0$. We consider orbits with $p_\xi = p_z = 0$ at $\tau = 0$, that is, free-fall orbits. It is easy to show that for any $\sqrt{-\frac{1}{h\alpha}} < \xi < \sqrt{-\frac{4\alpha+1}{h\alpha}}$ there is a unique $z > 0$ such that $\Gamma(0, 0, \xi, z) = 0$, so we use ξ as a parameter of the free-fall orbits.

Such free-fall orbits may have double collisions between m_1 and m_2 forward in time before m_3 reaches the origin. Let $N(\xi)$ be the number of such double collisions (including a double collision at $\tau = 0$ but excluding a triple collision). We set $N(\xi) = 0$ if no double collision occurs.

Proposition 2.1. (i) *The function* $N(\xi)$ *is well-defined for all* $\xi \in \left(\sqrt{-\frac{1}{h\alpha}}, \sqrt{-\frac{4\alpha+1}{h\alpha}}\right)$.

(ii) $N\left(\sqrt{-\frac{2\alpha+1}{h\alpha}}\right) = 0$

(iii) $N(\xi) \to \infty$ *as* $\xi \to \sqrt{-\frac{1}{h\alpha}} + 0$.

(iv) *For each* $k \in \mathbb{N}$, *there is a discontinuous point* $\xi_k \in \left(\sqrt{-\frac{1}{h\alpha}}, \sqrt{-\frac{4\alpha+1}{h\alpha}}\right)$ *of* N *such that* $N(\xi_k) = k$, $\lim_{\xi \to \xi_k - 0} N(\xi) = k$ *and* $\lim_{\xi \to \xi_k + 0} N(\xi) = k + 1$.

(v) *The free-fall orbit corresponding to ξ_k has a triple collision after k double collisions.*

Proof. (i) Assume a free-fall orbit $(\xi(\tau), z(\tau), p_\xi(\tau), p_z(\tau))$ is defined for $\tau < \tau_0$ where $\tau_0 \in (0, \infty]$ is maximal. Suppose $\tau_0 < \infty$. Then the orbit has a triple collision as $\tau \to \tau_0 - 0$, since singularities in the three body problem are only due to collisions (see [4]). Consider the case where $\tau_0 = \infty$ and suppose $z > \varepsilon > 0$ for all $\tau > 0$. From (2.5) we see that p_z monotonically decreases and $p_z < 0$ for $\tau > 0$ since $p_z(0) = 0$. Hence, by (2.3) we see that z also monotonically decreases, so that $dz/dt \to 0$ as $\tau \to \infty$. Thus, it follows from (2.2) and (2.3) that ξ and p_ξ tend to zero as $\tau \to \infty$ since $\lim_{\tau\to\infty} p_z(\tau) \neq 0$ by the monotonicity of p_z with $p_z(0) = 0$. This means that $\Gamma \to 1/\alpha$ as $\tau \to \infty$, which yields a contradiction to our assumption, $\Gamma = 0$. Consequently for any $\xi \in \left(\sqrt{-\frac{1}{h\alpha}}, \sqrt{-\frac{4\alpha+1}{h\alpha}}\right)$, m_3 reaches the origin in finite time or converges to the origin as $\tau \to \infty$, and hence $N(\xi)$ can be defined.
(ii) Let $\xi = \sqrt{-\frac{2\alpha+1}{h\alpha}}$ at $\tau = 0$. Then we have $z = (\sqrt{3}/2)\xi^2$ at $\tau = 0$, which means that the initial configuration is an equilateral triangle, and the initial velocity of each particle is zero. It is known that such a solution is homothetic and has a triple collision without double collision (see [3]). Hence, we have $N(\xi) = 0$.
(iii) For free-fall orbits, $z \to \infty$ as $\xi \to \sqrt{-1/h\alpha} + 0$ at $\tau = 0$. When $z > 0$ is sufficiently large, we see via (2.5) that p_z is sufficiently small, so that by (2.2) and (2.4) the (p_ξ, ξ)-dynamics is approximated by a harmonic oscillator, whose solutions are periodic and pass through $\xi = 0$ infinitely many times. Hence we have $N(\xi) \to \infty$ as $\xi \to \sqrt{-1/h\alpha} + 0$ at $\tau = 0$, since $\xi = 0$ represents double collision.
(iv) It immediately follows from (i)–(iii).
(v) It is sufficient to show that the free-fall orbit corresponding to ξ_k has a triple collision when m_3 reaches the origin for the first time. Assume that the free-fall orbit does not have a triple collision. By the continuity of orbits with respect to initial points, $N(\xi) = N(\xi_k)$ for ξ near ξ_k, since the orbit has no collision when m_3 arrives at the origin. This contradicts part (iv). Thus, the orbit has triple collision at the origin. □

Let γ_+^{2k} be the free-fall orbit corresponding to ξ_k. From the time reversibility, γ_+^{2k} has a triple collision before k double collisions backward in time. Therefore it has $2k$ double collisions between the triple collisions. It may be natural to think that there exists an orbit with any odd number of double collisions between triple collisions. We prove that this is the case in the following.

Consider an orbit with an odd number of double collisions between triple collisions. Consider an orbit with $(p_\xi, p_z, \xi, z) = (\frac{2}{\sqrt{\alpha}}, 0, 0, z)$ at $\tau = 0$ and let $M(z)$ be the number of double collisions (including one at $\tau = 0$) before m_3 reaches the origin. The function M has similar properties as N:

Proposition 2.2. (i) *The function M is well-defined for all $z > 0$.*
(ii) *There is a constant $c > 0$ such that $M(c) = 1$.*

(iii) *$M(z) \to \infty$ as $z \to \infty$.*
(iv) *For each $k \in \mathbb{N}$, there is a discontinuity point $z_k \in (c, \infty)$ of M such that $M(z_k) = k$, $\lim_{z \to z_k + 0} M(z) = k$ and that $\lim_{z \to z_k - 0} M(z) = k + 1$.*
(v) *The free-fall orbit corresponding to z_k has a triple collision after k double collisions including one at $\tau = 0$.*

Proof. We first note that there exists a periodic orbit experiencing double collisions only at $(p_\xi, p_z, \xi, z) = (\frac{2}{\sqrt{\alpha}}, 0, 0, \pm c)$ for $c > 0$ (see [5]). Hence we have $M(c) = 1$, which means part (ii). The remaining parts can be proven similarly as Proposition 2.1. $\square$

Let γ_+^{2k-1} be the free-fall orbit corresponding to z_k. Then γ_+^{2k-1} has a triple collision before k double collisions (including one at $\tau = 0$) backward in time. Hence, it has $2k - 1$ double collisions between the triple collisions. Thus, for any positive integer k, there exists an orbit γ_+^k which has k double collisions and satisfies $z > 0$ between the triple collisions. If $(p_\xi(\tau), p_z(\tau), \xi(\tau), z(\tau))$ is a solution, so is the reflection $(p_\xi(\tau), -p_z(\tau), \xi(\tau), -z(\tau))$. Let γ_-^k be the reflected solution of γ_+^k. Obviously, γ_-^k starts and ends at triple collisions and has k double collisions and satisfies $z < 0$ between the triple collisions.

3. Collision manifold

In order to investigate the behavior of orbits with a triple collision, we follow the approach of [3] and use the blow-up technique of [1] for the triple-collision singularity. Let

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{2\alpha}{\alpha+2} \end{pmatrix}$$

and define a canonical transformation

$$\zeta = M^{1/2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \eta = M^{-1/2} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}.$$

In the new coordinates the Hamiltonian becomes $H = \frac{1}{2}|\eta|^2 - U(\zeta)$, where

$$U(\zeta) = \frac{1}{\sqrt{2}} \left[\alpha^{-1}(\zeta_1^2 + \zeta_2^2)^{-1/2} + 4(\zeta_1^2 + \zeta_2^2 + (1 + 2\alpha^{-1})\zeta_3^2)^{-1/2} \right].$$

We introduce new variables

$$r = |\zeta|, \boldsymbol{s} = r^{-1}\zeta, \boldsymbol{z} = r^{1/2}\eta$$

and multiply the resulting vector field by $r^{3/2}$. Thus, we have

$$\begin{aligned} \dot{r} &= (\boldsymbol{s} \cdot \boldsymbol{z}) r, \\ \dot{\boldsymbol{s}} &= \boldsymbol{z} - (\boldsymbol{s} \cdot \boldsymbol{z}) \boldsymbol{s}, \\ \dot{\boldsymbol{z}} &= \frac{\partial U}{\partial \zeta}(\boldsymbol{s}) + \frac{1}{2}(\boldsymbol{s} \cdot \boldsymbol{z}) \boldsymbol{z}. \end{aligned} \tag{3.1}$$

We next introduce spherical coordinates $\boldsymbol{s} = (s_1, s_2, s_3) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$. Note that the vectors

$$\begin{aligned}
\boldsymbol{u}_1 &= \boldsymbol{s} \\
\boldsymbol{u}_2 &= \frac{\partial \boldsymbol{s}}{\partial \theta} = (-\sin\theta\cos\phi, \cos\theta\cos\phi, 0) \\
\boldsymbol{u}_3 &= \frac{\partial \boldsymbol{s}}{\partial \phi} = (-\cos\theta\sin\phi, -\sin\theta\sin\phi, \cos\phi)
\end{aligned}$$

form an orthogonal basis for $\mathbb{R}^3$. Writing $\boldsymbol{z} = v\boldsymbol{u}_1 + w_2\boldsymbol{u}_2 + w_3\boldsymbol{u}_3$ and using (3.1), we get

$$\begin{aligned}
\dot{r} &= vr \\
\dot{\theta} &= w_2 \\
\dot{\phi} &= w_3 \\
\dot{v} &= \frac{1}{2}v^2 + w_2^2\cos^2\phi + w_3^2 - U(\phi) \\
\dot{w}_2 &= -\frac{1}{2}vw_2 + 2w_2w_3\tan\phi \\
\dot{w}_3 &= U'(\phi) - \frac{1}{2}vw_3 - w_2^2\cos^2\phi\tan\phi.
\end{aligned}$$

where $U(\phi) = \frac{1}{\sqrt{2}}\alpha^{-1}\left[\sec\phi + 4\alpha^{3/2}(\alpha + 2\sin^2\phi)^{-1/2}\right]$. One can easily show that $\omega = r^{1/2}w_2\cos^2\phi$ is a constant of the motion which corresponds to the angular momentum.

To eliminate the singularities at $\phi = \pm\pi/2$, we replace w_3 by $w = w_3\cos\phi$ and define a new time variable T by $dt = r^{3/2}cos(\phi)dT$. Finally, we obtain

$$\begin{aligned}
\frac{dr}{dT} &= vr\cos\phi, \quad \frac{d\phi}{dT} = w, \\
\frac{dv}{dT} &= \left(U(\phi) - \frac{1}{2}v^2 + 2rh\right)\cos\phi, \\
\frac{dw}{dT} &= \frac{dU}{d\phi}(\phi)\cos^2\phi - \frac{1}{2}vw\cos\phi - \big(2U(\phi) - v^2 + 2rh\big)\sin\phi\cos\phi
\end{aligned} \tag{3.2}$$

with an energy relation

$$\frac{1}{2}\left(v^2\cos^2\phi + w^2 + \frac{\omega^2}{r}\right) - U(\phi)\cos^2\phi = rh\cos^2\phi. \tag{3.3}$$

In the coordinates the triple collision ($r = 0$) and the double collisions $\phi = \pm\frac{\pi}{2}$ singularities are blown-up. We call (r, ϕ, v, w) Devaney's coordinates. Moeckel [3] definedthe following invariant sets:

$$
\begin{aligned}
\mathscr{M}(\omega) &= \{(r,\phi,v,w) \mid (3.3) \text{ is satisfied for } \omega\} \\
\mathscr{M}_+ &= \{(r,\phi,v,w) \mid r \ge 0, v^2\cos^2\phi + w^2 - 2U(\phi)\cos^2\phi = 2rh\cos^2\phi\} \\
\mathscr{M}_0 &= \{(r,\phi,v,w) \mid r = 0, 2U(\phi)\cos^2\phi \ge v^2\cos^2\phi + w^2\} \\
\mathscr{C} &= \mathscr{M}_+ \cap \mathscr{M}_0 = \{(r,\phi,v,w) \mid r = 0, 2U(\phi)\cos^2\phi = v^2\cos^2\phi + w^2\} \\
\mathscr{M} &= \mathscr{M}_+ \cup \mathscr{M}_0.
\end{aligned}
$$

We will refer to $\mathscr{M}$ as the limiting variety of $\mathscr{M}(\omega)$ as $\omega \to 0$. Topologically $\mathscr{M}_0 \cong D^3\backslash\{4 \text{ points}\}$, $\mathscr{C} = \partial\mathscr{M}_0 \cong S^2\backslash\{4 \text{ points}\}$. We call $\mathscr{C}$ the collision manifold (Figure 3). Here the four points are $(\phi, v, w) = (\pm\frac{\pi}{2}, +\infty, 0)$ and $(\pm\frac{\pi}{2}, -\infty, 0)$. There are six equilibrium points

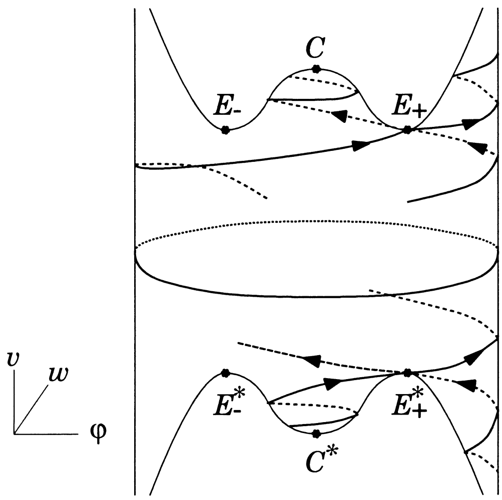

Figure 3. Collision manifold $\mathscr{C}$

$$
\begin{aligned}
C &= (0, \sqrt{2U(0)}, 0), & C^* &= (0, -\sqrt{2U(0)}, 0) \\
E_+ &= (\phi_+, \sqrt{2U(\phi_+)}, 0) & E_+^* &= (\phi_+, -\sqrt{2U(\phi_+)}, 0) \\
E_- &= (\phi_-, \sqrt{2U(\phi_-)}, 0) & E_-^* &= (\phi_-, -\sqrt{2U(\phi_-)}, 0)
\end{aligned}
$$

on $\mathscr{C}$ in the coordinates (ϕ, v, w), where $0, \phi_+(> 0)$ and $\phi_-(= -\phi_+)$ are the critical points of $U(\phi)$. It was shown in [1] that if an orbit converges to a triple collision forward (resp. backward) in time in the original system, the corresponding orbits converges to E_+^*, E_-^* or C^* (resp. E_+, E_- or C) as $T \to \infty$ (resp. $T \to -\infty$) in Devaney's coordinates. The state $\phi = 0$ represents a collinear configuration, while $\phi = \phi_+$ and $\phi = \phi_-$, respectively, represent upright and inverted equilateral ones. All the equilibrium points are saddle points.

Let $W^s(D)$ and $W^u(D)$ denote the stable and unstable manifolds of D, respectively, where D represents the equilibrium points. The dimensions of these manifolds are as follows.

$$\begin{aligned}\dim W^s(C) &= \dim W^u(C^*) = 3,\\ \dim W^u(C) &= \dim W^s(C^*) = 1,\\ \dim W^s(E_\pm) &= \dim W^u(E^*_\pm) = 2,\\ \dim W^u(E_\pm) &= \dim W^s(E^*_\pm) = 2.\end{aligned}$$

Note that $W^s(D)$ and $W^u(D^*)$ ($W^u(D)$ and $W^s(D)$) are symmetric with respect to the reflection $(v,w) \mapsto (-v,-w)$. For $D = C$ or $E_\pm$, $W^s(D)$ and $W^u(D^*)$ are included in M_0, and $W^u(D)$ and $W^s(D^*)$ are included in M_+. See [1, 3] for more detail.

4. Proof of Theorem 1.1

Theorem 1.1 follows from the following two lemmas.

Lemma 4.1. *The orbit $\gamma^k_\pm$ is a heteroclinic orbit from $E_\pm$ to $E^*_\pm$.*

Proof. As stated in the last paragraph of Section 3, The orbit $\gamma^k_\pm$ converges to either C^*, E^*_+ or E^*_- as $t \to \infty$. The stable manifold $W^s(C^*)$ of C^* is included in an invariant set $\{\phi = w = 0\} \subset \mathscr{M}_+$, which corresponds to the set of all collinear motions, $\{y = z = p_y = p_z = 0\}$, in the original problem (1.2) (see [3]). Since the corresponding free-fall orbit is not collinear, $\gamma^k_\pm$ converges to E^*_+ or E^*_- as $t \to \infty$. Moreover, the corresponding configurations are upright (resp. inverted) equilateral triangles when orbits converge to E^*_+ (resp. E^*_-). Since the z-component of corresponding free-fall orbit is positive (resp. negative) between the triple collisions, γ^k_+ (resp. γ^k_-) converges to E^*_+ (resp. E^*_-) as $t \to \infty$. Similarly $\gamma^k_\pm$ converges to $E_\pm$ as $t \to -\infty$. □

Lemma 4.2. *The heteroclinic orbit $\gamma^k_\pm$ is topologically transverse in $\mathscr{M}_+$.*

Proof. We mean "topological transversality" by "transversality" below, and set $T = 0$ when $t = 0$ without loss of generality.

We first claim that $\gamma^k_\pm$ transversely intersects the hyperplane $\{v = 0\}$ at $T = 0$. From the construction the orbit $\gamma^k_\pm$ has zero angular momentum ($\omega = 0$) and hence belongs to $\mathscr{M}_+$. Let us assume that k is odd. Since $p_z = 0$ and $x = y = 0$ at $t = 0$ in the original coordinates, $(p_z =)\sqrt{\frac{2\alpha}{\alpha+2}}r^{-1/2}(v\sin\phi + w) = 0$ and $\phi = \pm\frac{\pi}{2}$ at $T = 0$ in Devaney's coordinates. Hence it follows from (3.3) that $v = w = 0$ at $T = 0$. It is also easy to show that this is true when k is even. Using (3.2) and (3.3) and noting that $\lim_{\phi\to\pm\pi/2} U(\phi)\cos\phi \neq 0$, we see that $dv/dT = -U(\phi)\cos\phi < 0$ at $T = 0$. Thus, we have proved the claim.

Next we will show the transversality of $\gamma^k_\pm$ on $\mathscr{M}_+$. We use (r,v,w) as the coordinate system in a neighborhood of $\gamma^k_\pm(0)$ in $\mathscr{M}_+$. Let $\gamma^k_\pm(0)$ denote the position of $\gamma^k_\pm$ at $T = 0$. We can show that on the hyperplane $\{v = 0\}$, $W^s(E_\pm)$ transversely intersects the r-axis at $\gamma^k_\pm(0)$, as follows. Suppose that this is not

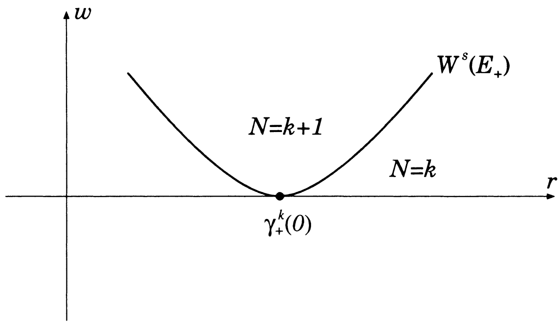

Figure 4.

true. Then all free orbits ($w = 0$) near but different from $\gamma_\pm^k(0)$ belong to the same region divided by $W^s(E_\pm)$ (see Figure 4). Hence, they have the same number of double collisions before m_3 arrives at the origin. This contradicts (iv) of Proposition 2.1 or 2.2.

On the other hand, since they are analytic manifolds, $W^s(E_\pm^*)$ and $W^u(E_\pm)$ of $E_\pm$ intersect $\{v = 0\}$ in an analytic curve. By the reversibility of (3.2), if $(r(T), \phi(T), v(T), w(T))$ is a solution, so is $(r(-T), \phi(-T), -v(-T), -w(-T))$. Hence $W^s(E_\pm^*)$ and $W^u(E_\pm)$ are symmetric with respect to the r-axis on $\{v = 0\}$.

We now prove the transversality of $\gamma_\pm^k$ in $\mathcal{M}_+ \cap \{v = 0\}$. Suppose that $W^s(E_\pm^*)$ and $W^u(E_\pm)$ do not intersect transversely. Then they coincide in a neighborhood of $\gamma_\pm^k$ on $\{v = 0\}$ since $W^s(E_\pm^*)$ intersects the r-axis transversely. Hence, the whole manifolds $W^s(E_\pm)^*$ and $W^u(E_\pm)$ also coincide since they are analytic. However, as stated in section 1, Mockel [3] proved that $\gamma_\pm^0 = \kappa_{E_\pm}$ are transverse. This yields a contradiction. Thus, we have completed the proof of this lemma. □

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