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The elliptic hypergeometric functions associated to the configuration space of points on an elliptic curve I : Twisted cycles

By

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Abstract

We consider the Euler type integral associated to the configuration space of points on an elliptic curve, which is an analogue of the hypergeometric function associated to the configuration space of points on a projective line. We calculate the *twisted homology group* with coefficients in the local system associated to a power function g^α of an elliptic function g, and the related intersection form. Applying these calculations, we describe the *connection matrices* representing the linear isomorphisms induced from analytic continuations of the functions defined by integrations of g^α over twisted cycles.

1. Introduction

In many mathematical contexts, hypergeometric functions often have (or are defined by) Euler type integrals. In fact, the hypergeometric function associated to the configuration space of points on $\mathbb{P}^1$ has the following integral representation:

$$\int_{x_i}^{x_j} (t-x_1)^{\alpha_1}(t-x_2)^{\alpha_2}\cdots(t-x_n)^{\alpha_n}\,dt.$$

In this paper, instead of $\mathbb{P}^1$, we shall consider the configuration space of points on an elliptic curve.

Let C be an elliptic curve defined by $\mathbb{C}/\Gamma$, where $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is a lattice generated by ω_1 and ω_2. We assume $\Im\frac{\omega_2}{\omega_1} > 0$. We shall denote a point on C by an equivalence class $\overline{x} \in C$ for $x \in \mathbb{C}$. Our main purpose is to study the definite integrals of the multi-valued function g^α, where $\alpha \in \mathbb{C} \setminus \left(\frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z}\right)$ is a fixed complex number and

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$$g = \frac{\sigma(t - x_0)\sigma(t - x_1)\sigma(t - x_2)}{\sigma^3(t)\sigma(x_0)\sigma(x_1)\sigma(x_2)}.$$

Here σ is Weierstraß's σ function with the periods ω_1 and ω_2. Given x_0, x_1 and x_2, the function g depends only on the equivalence class $\overline{t} \in C$. So this function g is a meromorphic function on C, which is a counterpart of rational functions on $\mathbb{P}^1$, with a pole of the third order at $\overline{0}$ and zeros of the first order at $\overline{x_0}, \overline{x_1}, \overline{x_2}$. Thus the function g^α has branch points at $\overline{x_0}, \overline{x_1}, \overline{x_2}, \overline{0} \in C$, which is analogous to the case of the multi-valued function $t^{-3\alpha}(t - x_0)^\alpha(t - x_1)^\alpha(t - x_2)^\alpha$ on $\mathbb{P}^1$. However, g may depend on a representative x_0 (resp. x_1, x_2) of the equivalence class $\overline{x_0}$ (resp. $\overline{x_1}$, $\overline{x_2}$). We consider a subgroup W of Γ^3 defined by

$$W := \left\{ (\Pi_0, \Pi_1, \Pi_2) \in \Gamma^3 \;\middle|\; \Pi_0 + \Pi_1 + \Pi_2 = 0 \right\}$$

and a W-invariant subspace L of $\mathbb{C}^3$ defined by

$$L := \left\{ (x_0, x_1, x_2) \in \mathbb{C}^3 \;\middle|\; x_0 + x_1 + x_2 = 0 \right\}.$$

Put $\overline{S} := L/W$. Thus, g is a meromorphic function on $C \times \overline{S}$. For $q = \overline{(x_0, x_1, x_2)} \in \overline{S}$, we denote by X_q the punctured elliptic curve, that is, C minus four points $\overline{x_0}$, $\overline{x_1}$, $\overline{x_2}$ and $\overline{0}$. The integral of $g^\alpha dt$ over a path $\Xi_{\mu,q}$ (mentioned below) lying on X_q

$$F_\mu(q) := \int_{\Xi_{\mu,q}} g^\alpha dt \tag{1.1}$$

is a multi-valued function with respect to q defined over the space S of *configuration with constraint* (defined later), which is a subset of $\overline{S}$.

This function F_μ has *singularities* : Put $D^{ij} := \left\{\overline{(x_0, x_1, x_2)} \in \overline{S} \;\middle|\; x_i - x_j \in \Gamma\right\}$ and $D^i_\infty := \left\{\overline{(x_0, x_1, x_2)} \in \overline{S} \;\middle|\; x_i \in \Gamma\right\}$; when q belongs to D^{ij}, the topological type of X_q differs from that of C minus four points; when q belongs to D^i_∞, the integrand g^α is divergent. Hence we define the domain S of the function F_μ by removing the singular loci D^{ij} and D^i_∞ from $\overline{S}$:

$$S := \overline{S} \setminus \left(\bigcup_{i \neq j} D^{ij} \cup \bigcup_i D^i_\infty \right). \tag{1.2}$$

We introduce an integration path $\Xi_{\mu,q}$ of (1.1), the so-called *twisted cycle*. In case of projective space, Aomoto theory ([5]) tells us that a path of an Euler type integral can be regarded as a homology class with coefficients in the local system defined by the multi-valuedness of the integrand. In principle, this interpretation is valid for other cases. In our case, the local system $\mathcal{L}_{X_q}$ is defined by the multi-valuedness of g^α and we denote by $\{\Xi_{\mu,q}\}_\mu$ generators of $H_1(X_q, \mathcal{L}_{X_q})$. In Section 3, we give the concrete description of $\{\Xi_{\mu,q}\}_\mu$ (Theorem 3.1). In Section 4, we calculate the intersection form among the twisted cycles (Theorem 4.1).

The function F_μ on S can be seen as a flat section of a certain vector bundle over S or, equivalently, a solution of a certain linear differential equation ([8]): The sheaf $\mathcal{S}$ of germs of functions defined by $\mathbb{C}$-linear combinations of $\{F_\mu\}_\mu$ is locally constant and $\mathcal{S}\otimes\mathcal{O}_S$ is a flat (trivial) vector bundle over S. The analytic continuation along a path γ in S induces the linear isomorphism between the stalks of $\mathcal{S}$ over the initial point and the terminal point of γ ; the corresponding matrix is said to be the *connection matrix*. In Section 5, we consider the paths in S connecting two points $\overline{\left(\frac{\omega_i}{2}, \frac{\omega_j}{2}, \frac{\omega_k}{2}\right)}$ and $\overline{\left(\frac{\omega_{\tau(i)}}{2}, \frac{\omega_{\tau(j)}}{2}, \frac{\omega_{\tau(k)}}{2}\right)}$, where τ is a transposition of two of three letters i, j, k. For these paths, we calculate the connection matrices (Theorem 5.1).

Finally, we briefly mention the relation with other recent works. T. Mano and H. Watanabe study the *Riemann-Wirtinger* integrals ([10], [13]); the definite integrals of power products of an exponential function and theta functions, which are similar to our integrals (1.1). H. Watanabe considers in [11][12] a *fixed* configuration; the two-torsion points are removed from elliptic curves, which are deformed by the moduli parameter. He focuses the action of modular transformations on such elliptic curves and studies that action on (co)homology groups of the elliptic curves minus the two-torsion points with coefficients in certain local systems corresponding to the integrands of Riemann-Wirtinger integrals. Using the Riemann-Wirtinger integral representation of Gauß's hypergeometric function, he interprets the monodromy transformations of the hypergeometric function into the modular transformations. T. Mano considers in [9] the special solutions to the monodromy-preserving deformation of Fuchsian differential equations on an elliptic curve which are described Riemann-Wirtinger integrals. Their integrands have n branch points on an elliptic curve. He calculates the monodromy corresponding to interchange two of n branch points.

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2. The local system

Let C be an elliptic curve defined by $\mathbb{C}/\Gamma$, where $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is a lattice generated by ω_1 and ω_2. We assume $\Im\frac{\omega_2}{\omega_1} > 0$. We shall denote a point on C by an equivalence class $\overline{x} \in C$ represented by $x \in \mathbb{C}$. We consider a subgroup W of Γ^3 defined by

$$W := \left\{ \ (\Pi_0, \Pi_1, \Pi_2) \in \Gamma^3 \ \middle| \ \Pi_0 + \Pi_1 + \Pi_2 = 0 \ \right\}$$

and a W-invariant subspace L of $\mathbb{C}^3$ defined by

$$L := \left\{ \ (x_0, x_1, x_2) \in \mathbb{C}^3 \ \middle| \ x_0 + x_1 + x_2 = 0 \ \right\}.$$

Put $\overline{S} := L/W$. For $q = \overline{(x_0, x_1, x_2)} \in \overline{S}$, we denote by X_q the punctured elliptic curve, that is, C minus four points $\overline{x_0}$, $\overline{x_1}$, $\overline{x_2}$ and $\overline{0}$. We assume that these four points are distinct. (By the definition in (1.2), this condition is equivalent to $q \in S$.) We shall introduce a local system $\mathcal{L}_{X_q}$ which reflects the data of multi-valuedness of the function g^α defined as follows. Let g be the following holomorphic function on X_q:

$$g := \frac{\sigma(t - x_0)\sigma(t - x_1)\sigma(t - x_2)}{\sigma^3(t)\sigma(x_0)\sigma(x_1)\sigma(x_2)}.$$

Using the identity $\frac{\sigma'(t)}{\sigma(t)} = \zeta(t)$ (Weierstraß's ζ function), we have $\frac{dg}{g} = (\zeta(t - x_0) + \zeta(t - x_1) + \zeta(t - x_2) - 3\zeta(t))dt$. We shall consider the following sheaf $\mathcal{L}_{X_q}$:

$$\mathcal{L}_{X_q} := \mathrm{Ker}\left(d - \alpha\frac{dg}{g} : \mathcal{O}_{X_q} \longrightarrow \Omega^1_{X_q}\right),$$

where $\alpha \in \mathbb{C} \setminus \left(\frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z}\right)$ is a fixed complex number. This sheaf has no global section on X_q. We take the open part $\overset{\circ}{X}_q$ in X_q:

$$\overset{\circ}{X}_q := C \setminus \left(\overline{0x_0} \cup \overline{0x_1} \cup \overline{0x_2}\right),$$

where $\overline{0x_i}$ is the image of the segment $[0, x_i]$ in $\mathbb{C}$ through the natural projection $\mathbb{C} \longrightarrow C$. We should modify the segment $[0, x_i]$ in its homotopy class in such a way that the images of the segments do not have non-trivial intersection (nor self-intersection) except for the one at $\overline{0}$.

Lemma 2.1. *Regardless of the choice of a representative (x_0, x_1, x_2), $\overset{\circ}{X}_q$ is homotopic to the bouquet $S^1 \vee S^1$.*

Proof. The restriction to $[0, x_0] \cup [0, x_1] \cup [0, x_2]$ of the projection is a homeomorphism. We have a neighborhood $\overline{O}$ of $\overline{0x_0} \cup \overline{0x_1} \cup \overline{0x_2}$ and a neighborhood O of $[0, x_0] \cup [0, x_1] \cup [0, x_2]$ such that the restriction of the projection to $O : O \longrightarrow \overline{O}$ is also a homeomorphism because the projection is a covering map. Here $O \setminus ([0, x_0] \cup [0, x_1] \cup [0, x_2])$ is homeomorphic to $O \setminus 0$. So $\overline{O} \setminus \left(\overline{0x_0} \cup \overline{0x_1} \cup \overline{0x_2}\right)$ is homeomorphic to $\overline{O} \setminus \overline{0}$. Then, $C \setminus \left(\overline{0x_0} \cup \overline{0x_1} \cup \overline{0x_2}\right)$ is homeomorphic to $C \setminus \overline{0}$, which is homotopic to $S^1 \vee S^1$. □

Lemma 2.2.

1. *The sheaf $\mathcal{L}_{X_q}$ is a local system of rank 1 over $\mathbb{C}$ whose local sections are regarded as single-valued branches of some constant multiple of g^α.*
2. *The sheaf $\mathcal{L}_{X_q}$ has non-zero global sections on $\overset{\circ}{X}_q$.*

Proof.

1. Any local section of $\mathfrak{L}_{X_q}$ is a local solution of the first-order ordinary linear differential equation $\left(d - \alpha\frac{dg}{g}\right)\varsigma = 0$. Any local solution of this equation is a constant multiple of a single-valued branch of g^α. Hence, sections of $\mathfrak{L}_{X_q}$ on a small enough open set form a one dimensional vector space, and such a section is a single-valued branch of some constant multiple of g^α.

2. We shall prove that a single-valued solution of the equation $\left(d - \alpha\frac{dg}{g}\right)\varsigma = 0$ on $\overset{\circ}{X}_q$ does exist. The logarithm of any solution is given by an integral of $\alpha\frac{dg}{g}$ over a path. The single-valuedness is equivalent to the condition that the integral over any closed loop in $\overset{\circ}{X}_q$ vanishes. Due to Lemma 2.1, $\overset{\circ}{X}_q$ is homotopic to $S^1 \vee S^1$. We can pick two loops l_{ω_1}, l_{ω_2} on $\overset{\circ}{X}_q$ such that the inclusion map $l_{\omega_1} \cup l_{\omega_2} \hookrightarrow \overset{\circ}{X}_q$ is a deformation retract. To prove the single-valuedness, it suffices to verify the vanishing of integrals I_i given by

$$I_i(x_0, x_1, x_2) := \int_{l_{\omega_i}} \frac{dg}{g} = \int_{l_{\omega_i}} (\zeta(t - x_0) + \zeta(t - x_1) + \zeta(t - x_2) - 3\zeta(t))dt.$$

Using the identity $\zeta'(t) = -\wp(t)$, we have $\frac{\partial I_i}{\partial x_j} = \int_{l_{\omega_i}} \wp(t - x_j)dt = \int_{l_{\omega_i}} \wp(t)dt$. This is a constant, which we denote by π_i. Then the equality $I_i(x_0, x_1, x_2) = \pi_i x_0 + \pi_i x_1 + \pi_i x_2 + I_i(0, 0, 0)$ holds. Since $x_0 + x_1 + x_2 = 0$ (by the assumption) and $I_i(0, 0, 0) = 0$ (by its definition), the integrals $I_i(x_0, x_1, x_2)$ vanish, as desired.

□

Now we pick a non-zero global section ς over $\overset{\circ}{X}_q$, and fix it once for all in the sequel. For a path $l : I \longrightarrow X_q$ with the initial point in $\overset{\circ}{X}_q$, we denote by ς_l the analytic continuation of ς along l.

Remark 1. Note that X_q is homotopic to the bouquet $\underbrace{S^1 \vee \cdots \vee S^1}_{5\text{copies}}$. In fact, we can pick five loops l_0, l_1, l_2, l_{ω_1}, l_{ω_2} such that the inclusion map $l_0 \cup l_1 \cup l_2 \cup l_{\omega_1} \cup l_{\omega_2} \hookrightarrow X_q$ is a deformation retract. The loops l_{ω_1}, l_{ω_2} are the same ones as in the proof of Lemma 2.2, while the loop l_i ($i = 0, 1, 2$) is a closed loop in X_q around the point $\overline{x_i}$. The representation $\rho : \pi_1(X_q) \longrightarrow \mathbb{C}^*$ corresponding to the local system $\mathfrak{L}_{X_q}$ is given by the formulae $\rho(l_i) = c$ ($i = 0, 1, 2$), $\rho(l_{\omega_j}) = 1$ ($j = 1, 2$), where $c := e^{2\pi\sqrt{-1}\alpha}$.

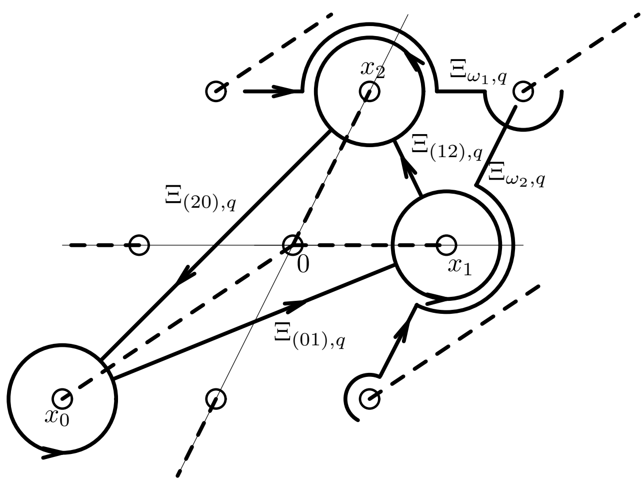
3. The twisted homology

We shall find generators of the *twisted homology group* $H_1(X_q, \mathfrak{L}_{X_q})$. The k-chain group $C_k(X_q, \mathfrak{L}_{X_q})$ with coefficients in $\mathfrak{L}_{X_q}$ is the complex vector space with the basis $\{\sigma \otimes s_\sigma\}$, where $\sigma : \Delta \longrightarrow X_q$ is a singular k-simplex and s_σ is a section of $\sigma^{-1}\mathfrak{L}_{X_q}$ over σ. The first boundary operator $\partial : C_1(X_q, \mathfrak{L}_{X_q}) \longrightarrow$

$C_0(X_q, \mathcal{L}_{X_q})$ is given by $\partial(l \otimes s_l) = l(1) \otimes s_{l,1} - l(0) \otimes s_{l,0}$. The other boundary operators are given in similar fashion. Then the complex $(C_\bullet(X_q, \mathcal{L}_{X_q}), \partial)$ defines the 1-st *twisted homology group* $H_1(X_q, \mathcal{L}_{X_q})$ in the usual manner. For two loops l_{ω_1}, l_{ω_2} in the proof of Lemma 2.2, we denote by $\Xi_{\omega_1,q}$ (resp. $\Xi_{\omega_2,q}$) the twisted cycle $l_{\omega_1} \otimes \varsigma_{l_{\omega_1}}$ (resp. $l_{\omega_2} \otimes \varsigma_{l_{\omega_2}}$), where ς is given in Section 2. We introduce other twisted cycles $\Xi_{(ij),q}$, which are the *regularizations* of the segments $\overline{x_i x_j}$. We denote by $S_\epsilon(x_i)$ a closed loop homotopic to the ϵ-circle centered at x_i. Let x_i^+ (resp. x_j^-) be the intersection point of the circle $S_\epsilon(x_i)$ (resp. $S_\epsilon(x_j)$) and the segment $[x_i, x_j]$. The *regularizations* of the segments $\overline{x_i x_j}$ is defined by

$$\Xi_{(ij),q} := \frac{1}{c-1}\overline{S_\epsilon(x_i)} \otimes \varsigma_{\overline{S_\epsilon(x_i)}} + \overline{[x_i^+, x_j^-]} \otimes \varsigma_{\overline{[x_i^+, x_j^-]}} - \frac{1}{c-1}\overline{S_\epsilon(x_j)} \otimes \varsigma_{\overline{S_\epsilon(x_j)}},$$

where ϵ is a sufficiently small positive number; we shall indicate by $\overline{}$ an image through the natural projection $\mathbb{C} \longrightarrow C$. Note that if ϵ is sufficiently small, this defines a homology class, independently of ϵ.



Theorem 3.1 (generators of H_1). *The twist cycles $\Xi_{(01),q}$, $\Xi_{(12),q}$, $\Xi_{(20),q}$, $\Xi_{\omega_1,q}$, $\Xi_{\omega_2,q}$ generate $H_1(X_q, \mathcal{L}_{X_q})$, and we have the relation $\Xi_{(01),q} + \Xi_{(12),q} + \Xi_{(20),q} = 0$ in $H_1(X_q, \mathcal{L}_{X_q})$ which defines the four dimensional vector space $H_1(X_q, \mathcal{L}_{X_q})$:*

$$H_1(X_q, \mathcal{L}_{X_q}) \cong \frac{\mathbb{C}\Xi_{(01),q} \oplus \mathbb{C}\Xi_{(12),q} \oplus \mathbb{C}\Xi_{(20),q} \oplus \mathbb{C}\Xi_{\omega_1,q} \oplus \mathbb{C}\Xi_{\omega_2,q}}{\mathbb{C}\left(\Xi_{(01),q} + \Xi_{(12),q} + \Xi_{(20),q}\right)}.$$

Proof. In view of Remark 1, $H_1(X_q, \mathcal{L}_{X_q})$ coincides with the first homology of the following complex:

$$0 \longleftarrow \mathbb{C} \xleftarrow{\partial} \bigoplus_{\mu \in \{0,1,2,\omega_1,\omega_2\}} \mathbb{C} l_\mu \otimes \varsigma_{l_\mu} \longleftarrow 0,$$

where $\partial(l_i \otimes \varsigma_{l_i}) = c-1$, and $\partial(l_{\omega_i} \otimes \varsigma_{l_{\omega_i}}) = 0$. The first homology of it is $\mathrm{Ker}\partial$, which is generated by

$$\left\{ \frac{1}{c-1} l_i \otimes \varsigma_{l_i} - \frac{1}{c-1} l_j \otimes \varsigma_{l_j},\ l_{\omega_k} \otimes \varsigma_{l_{\omega_k}} \right\}_{(ij)\in\{(01),(12),(20)\},k=1,2}.$$

The chain $\frac{1}{c-1} l_i \otimes \varsigma_{l_i} - \frac{1}{c-1} l_j \otimes \varsigma_{l_j}$ is a cycle homologous to $\Xi_{(ij),q}$, whence the assertion. □

4. The intersection form

The intersection form on the twisted cycles is a bilinear form between $H_1\left(X_q, \mathcal{L}_{X_q}\right)$ and $H_1\left(X_q, \mathcal{L}^\vee_{X_q}\right)$, where $\mathcal{L}^\vee_{X_q}$ is the local system dual to $\mathcal{L}_{X_q}$, i.e., the local system whose local sections are generated by $g^{-\alpha}$. To describe the intersection form explicitly, we shall take the generators $\Xi^\vee_{(01),q}$, $\Xi^\vee_{(12),q}$, $\Xi^\vee_{(20),q}$, $\Xi^\vee_{\omega_1,q}$ and $\Xi^\vee_{\omega_2,q}$ of $H_1\left(X_q, \mathcal{L}^\vee_{X_q}\right)$ which are given by replacing ς (resp. c) by $\frac{1}{\varsigma}$ (resp. $\frac{1}{c}$) in the definitions of $\Xi_{(01),q}$, $\Xi_{(12),q}$, $\Xi_{(20),q}$, $\Xi_{\omega_1,q}$ and $\Xi_{\omega_2,q}$, respectively:

$$\varsigma \longleftrightarrow \frac{1}{\varsigma}, \quad c \longleftrightarrow \frac{1}{c} \quad ; \quad \Xi = \sum_\gamma a_\gamma(c)\gamma \otimes \varsigma_\gamma \longleftrightarrow \Xi^\vee = \sum_\gamma a_\gamma\left(\frac{1}{c}\right)\gamma \otimes \frac{1}{\varsigma_\gamma},$$

where $\gamma \otimes \varsigma_\gamma$ is a twisted singular 1-simplex and $a_\gamma(X) \in \mathbb{C}(X)$. (Here, $\mathbb{C}(X)$ is the rational function field over $\mathbb{C}$ in one valuable X.)

We define the ordering on the index set $J := \{(01), (12), (20), \omega_1, \omega_2\}$ of the generators of H_1:

$$(01) \prec (12) \prec (20) \prec \omega_1 \prec \omega_2.$$

Now we have

Theorem 4.1 (Intersection form). *We assume that*

1. l_{ω_1} *is* $\overline{\frac{-\omega_1-\omega_2}{2}\,\frac{\omega_1-\omega_2}{2}}$, l_{ω_2} *is* $\overline{\frac{-\omega_1-\omega_2}{2}\,\frac{-\omega_1+\omega_2}{2}}$,
2. x_0, x_1 *and* x_2 *belong to the interior of the parallelogram spanned by the two segments* $[\frac{-\omega_1-\omega_2}{2}, \frac{\omega_1-\omega_2}{2}]$, $[\frac{-\omega_1-\omega_2}{2}, \frac{\omega_1-\omega_2}{2}]$,
3. $\arg x_0 < \arg x_1 < \arg x_2$.

Then the intersection form between $H_1\left(X_q, \mathcal{L}_{X_q}\right)$ *and* $H_1\left(X_q, \mathcal{L}^\vee_{X_q}\right)$ *is given by the matrix*

$$\left[\langle \Xi_{\mu,q}, \Xi^\vee_{\nu,q}\rangle\right]_{\mu,\nu\in J} = \begin{bmatrix} -\frac{c+1}{c-1} & \frac{1}{c-1} & \frac{c}{c-1} & 0 & 0 \\ \frac{c}{c-1} & -\frac{c+1}{c-1} & \frac{1}{c-1} & 0 & 0 \\ \frac{1}{c-1} & \frac{c}{c-1} & -\frac{c+1}{c-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

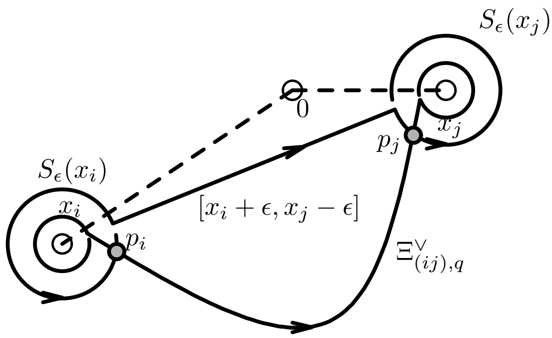
In particular, the (2,2)*-cofactor of this matrix is equal to* $\frac{c^2+c+1}{(c-1)^2}$.

Proof. The intersection number $\langle \Xi_{\mu,q}, \Xi^{\vee}_{\nu,q} \rangle$ can be calculated by summing up the local intersection numbers at each of the intersection points of the supports of $\Xi_{\mu,q}$ and of $\Xi^{\vee}_{\nu,q}$. The local intersection number $\langle \gamma_1 \otimes \varsigma_{\gamma_1}, \gamma_2 \otimes \frac{1}{\varsigma_{\gamma_2}} \rangle_p$ at the intersection point p of their supports is the product of the ordinary local intersection number $\langle \gamma_1, \gamma_2 \rangle_p$ and the ratio $\frac{\varsigma_{\gamma_1}(p)}{\varsigma_{\gamma_2}(p)}$.

<u>In the case of $\mu = \nu = (ij) \in \{(01), (12), (20)\}$:</u>

We change the representative of the homology class $\Xi^{\vee}_{\nu,q}$ in such a way that its support meets that of $\Xi_{\mu,q}$ at two points p_i, p_j, which belong to the supports of $\overline{S_\epsilon(x_i)} \otimes \varsigma_{\overline{S_\epsilon(x_i)}}$ and of $\overline{S_\epsilon(x_j)} \otimes \varsigma_{\overline{S_\epsilon(x_j)}}$, respectively. Then we have

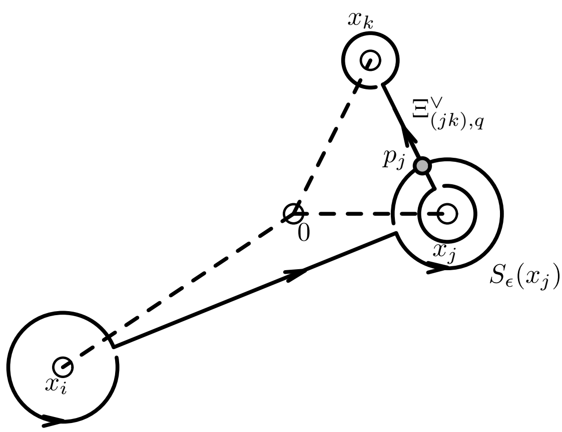
$$\begin{aligned}\langle \Xi_{\mu,q}, \Xi^{\vee}_{\nu,q} \rangle &= \frac{1}{c-1} \times (-1) \times \frac{\varsigma_{\overline{S_\epsilon(x_i)}}(p_i)}{\varsigma_{\overline{[x_i+\epsilon, x_j-\epsilon]}}(p_i)} - \frac{1}{c-1} \times 1 \times \frac{\varsigma_{\overline{S_\epsilon(x_j)}}(p_j)}{\varsigma_{\overline{[x_i+\epsilon, x_j-\epsilon]}}(p_j)} \\ &= \frac{1}{c-1} \times (-1) \times c - \frac{1}{c-1} \times 1 \times 1 = -\frac{c+1}{c-1}.\end{aligned}$$



<u>In the case of $\mu \neq \nu$, $\mu, \nu \in \{(01), (12), (20)\}$:</u>

We change the representative of the homology class $\Xi^{\vee}_{(jk),q}$ in such a way that its support meets that of $\Xi_{(ij),q}$ at a point p_j on the support of $\overline{S_\epsilon(x_j)} \otimes \varsigma_{\overline{S_\epsilon(x_j)}}$. Then we have

$$\begin{aligned}\langle \Xi_{(01),q}, \Xi^{\vee}_{(12),q} \rangle &= \langle \Xi_{(12),q}, \Xi^{\vee}_{(20),q} \rangle = \langle \Xi_{(20),q}, \Xi^{\vee}_{(01),q} \rangle \\ &= -\frac{1}{c-1} \times (-1) \times \frac{\varsigma_{\overline{S_\epsilon(x_j)}}(p_j)}{\varsigma_{\overline{[x_i+\epsilon, x_j-\epsilon]}}(p_j)} = \frac{1}{c-1}, \\ \langle \Xi_{(12),q}, \Xi^{\vee}_{(01),q} \rangle &= \langle \Xi_{(20),q}, \Xi^{\vee}_{(12),q} \rangle = \langle \Xi_{(01),q}, \Xi^{\vee}_{(20),q} \rangle \\ &= \frac{1}{c-1} \times 1 \times \frac{\varsigma_{\overline{S_\epsilon(x_j)}}(p_j)}{\varsigma_{\overline{[x_i+\epsilon, x_j-\epsilon]}}(p_j)} = \frac{c}{c-1}.\end{aligned}$$



<u>In the case of $\mu, \nu \in \{\omega_1, \omega_2\}$:</u>
The ordinary intersection form $\langle \bullet, \bullet \rangle : H_1(X_q, \mathbb{C}) \times H_1(X_q, \mathbb{C}) \longrightarrow \mathbb{C}$ is given by $\left[\langle l_{\omega_i}, l_{\omega_j} \rangle\right]_{i,j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $\frac{\varsigma_{l_{\omega_i}}(p)}{\varsigma_{l_{\omega_j}}(p)} = 1$ at the intersection point p of the supports of l_{ω_i} and of l_{ω_j}. Hence,

$$\left[\langle \Xi_{\omega_i,q}, \Xi^\vee_{\omega_j,q} \rangle\right]_{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

<u>In the other cases:</u>
We can change the representative of the homology class $\Xi^\vee_{\nu,q}$ in such a way that its support does not meet that of $\Xi_{\mu,q}$. □

5. The connection matrix

Put $D^{ij} := \left\{\overline{(x_0, x_1, x_2)} \in \overline{S} \;\middle|\; x_i - x_j \in \Gamma\right\}$ $(i, j = 0, 1, 2)$ and $D^i_\infty := \left\{\overline{(x_0, x_1, x_2)} \in \overline{S} \mid x_i \in \Gamma\right\}$ $(i = 0, 1, 2)$. We denote the space of *configuration with constraint* by $S := \overline{S} \setminus \left(\bigcup_{i \neq j} D^{ij} \cup \bigcup_i D^i_\infty\right)$ and the point $\overline{\left(\frac{\omega_i}{2}, \frac{\omega_j}{2}, \frac{\omega_k}{2}\right)}$ on S by $q_{(ijk)}$, where $\omega_0 := -(\omega_1 + \omega_2)$. The integral of $g^\alpha dt$ over a path $\Xi_{\mu,q}$ lying on X_q

$$F_\mu(q) := \int_{\Xi_{\mu,q}} g^\alpha dt$$

is a multi-valued function with respect to q defined over the space S. Let $\mathcal{S}$ be the sheaf of germs of the functions on S defined by $\mathbb{C}$-linear combinations of $\{F_\mu\}_{\mu \in \{(01),(20),\omega_1,\omega_2\}}$.

In this section, we shall describe the linear isomorphisms between stalks of $\mathcal{S}$ over $q_{(012)}$, $q_{(210)}$ and $q_{(102)}$, of which the corresponding matrices are called the *connection matrices*, induced by the analytic continuations of F_μ along four paths (given later) connecting two of $q_{(012)}, q_{(210)}, q_{(102)}$ on S. These linear isomorphisms are identified with the isomorphisms between the homology groups $\big\{H_1(X_q, \mathcal{L}_{X_q})\big\}_{q\in\{q_{(012)},q_{(210)},q_{(102)}\}}$, which are induced by translating homology classes along the paths. For a path $\gamma : I \longrightarrow S$, the pull back of the fibration $\bigcup X_q \longrightarrow S$ is a trivial one. Then we have the isomorphism between the homology groups of the fibers over $\gamma(0)$ and $\gamma(1)$. Put $\mathcal{H}(q) := H_1(X_q, \mathcal{L}_{X_q})$ for brevity.

We consider the paths on S connecting $q_{(012)}$ and $q_{(\tau_{(ij)}(0),\tau_{(ij)}(1),\tau_{(ij)}(2))}$, where $\tau_{(ij)}$ is the transposition (belonging to the symmetric group $\mathfrak{S}_3$) ; $\tau_{(ij)}(i) = j$, $\tau_{(ij)}(j) = i$ and $\tau_{(ij)}(k) = k$ for $k \neq i, j$. Put

$$q^{m_1,m_2}_{(ij)} := \overline{\left(\frac{\omega_0}{2}, \frac{\omega_1}{2}, \frac{\omega_2}{2}\right) + \frac{m_1\omega_1 + m_2\omega_2}{4}\mathsf{e}_i - \frac{m_1\omega_1 + m_2\omega_2}{4}\mathsf{e}_j},$$

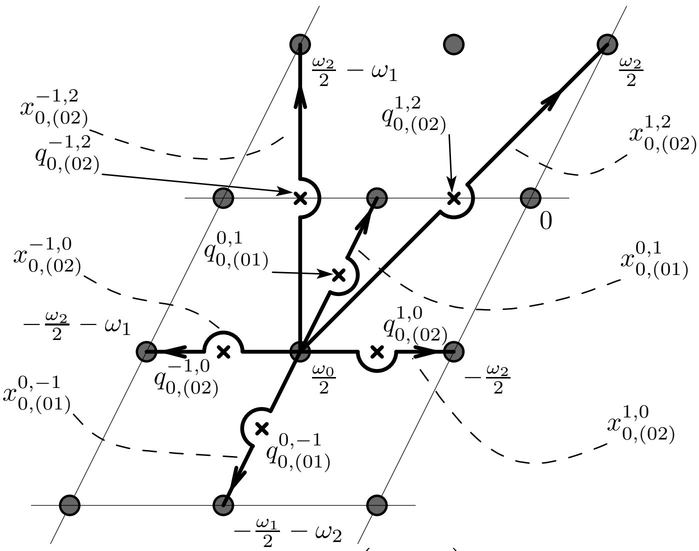
where $\{\mathsf{e}_0, \mathsf{e}_1, \mathsf{e}_2\}$ is the standard basis of $\mathbb{C}^3$. We denote by $\lambda^{m_1,m_2}_{(ij)}$ a path on the subset

$$\left\{\overline{(x_0, x_1, x_2)} \in \overline{S} \;\middle|\; x_k = \frac{\omega_k}{2}\ (k \neq i, j)\right\}$$

from $q_{(012)}$ to $q_{(\tau_{(ij)}(0),\tau_{(ij)}(1),\tau_{(ij)}(2))}$ via $q^{m_1,m_2}_{(i,j)}$:

$$\lambda^{m_1,m_2}_{(ij)}(s) := \overline{\left(\frac{\omega_0}{2}, \frac{\omega_1}{2}, \frac{\omega_2}{2}\right) + \frac{m_1\omega_1 + m_2\omega_2}{2}s\mathsf{e}_i - \frac{m_1\omega_1 + m_2\omega_2}{2}s\mathsf{e}_j}, \qquad 0 \le s \le 1.$$

The subset $\left\{\overline{(x_0, x_1, x_2)} \in \overline{S} \;\middle|\; x_k = \frac{\omega_k}{2}\ (k \neq i, j)\right\}$ meets the singular loci at six points ; for example, in case of $(i, j) = (0, 2)$, it meets D^{02} at four points $q^{1,0}_{(02)}$, $q^{-1,0}_{(02)}$, $q^{1,2}_{(02)}$, $q^{-1,2}_{(02)}$, meets both D^{01} and D^2_∞ at the same point $q^{0,2}_{(02)}$, and meets both D^{12} and D^0_∞ at the same point $q^{2,2}_{(02)}$. Thus, the path $\gamma^{m_1,m_2}_{(ij)}$ is defined by slightly deforming the path $\lambda^{m_1,m_2}_{(ij)}$ such that $\gamma^{m_1,m_2}_{(ij)}$ avoids $q^{m_1,m_2}_{(ij)}$. Note that the path $\gamma^{m_1,m_2}_{(ij)}$ can be realized as a composition of $\gamma^{1,0}_{(02)}$, $\gamma^{-1,0}_{(02)}$, $\gamma^{0,1}_{(01)}$ and $\gamma^{0,-1}_{(01)}$ up to homotopy. For example, $\gamma^{\pm1,2}_{(02)}$ is homotopic to $(\gamma^{0,1}_{(01)})^{\mp1} \circ (\gamma^{0,-1}_{(01)})^{\mp1} \circ (\gamma^{\pm1,0}_{(02)})^{\pm1} \circ (\gamma^{0,-1}_{(01)})^{\pm1} \circ (\gamma^{0,1}_{(01)})^{\pm1}$. (The following figure depicts the behavior of the x_0-component $x^{m_1,m_2}_{0,(ij)}(s)$ of a representative of $\gamma^{m_1,m_2}_{(ij)}(s)$, where $q^{m_1,m_2}_{0,(ij)}$ denotes the x_0-component of a representative of $q^{m_1,m_2}_{(ij)}$.)



We calculate the linear isomorphism $\left(\gamma^{m_1,m_2}_{(ij)}\right)_*$ between $\mathcal{H}\left(q_{(012)}\right)$ and $\mathcal{H}\left(q_{(\tau_{(ij)}(0),\tau_{(ij)}(1),\tau_{(ij)}(2))}\right)$ by the twisted Picard-Lefschetz formula ([7]):

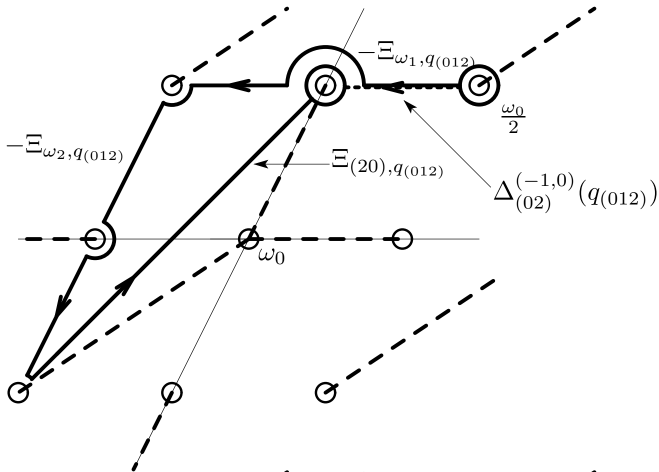
$$\begin{aligned}&\left(\gamma^{m_1,m_2}_{(ij)}\right)_*(\Xi)\\ &\quad = \Xi + \frac{(-c-1)\left\langle \Xi, \Delta^{m_1,m_2\vee}_{(ij)}\left(q_{(012)}\right)\right\rangle}{\left\langle \Delta^{m_1,m_2}_{(ij)}\left(q_{(012)}\right), \Delta^{m_1,m_2\vee}_{(ij)}\left(q_{(012)}\right)\right\rangle}\Delta^{m_1,m_2}_{(ij)}\left(q_{(\tau_{(ij)}(0),\tau_{(ij)}(1),\tau_{(ij)}(2))}\right),\end{aligned}$$

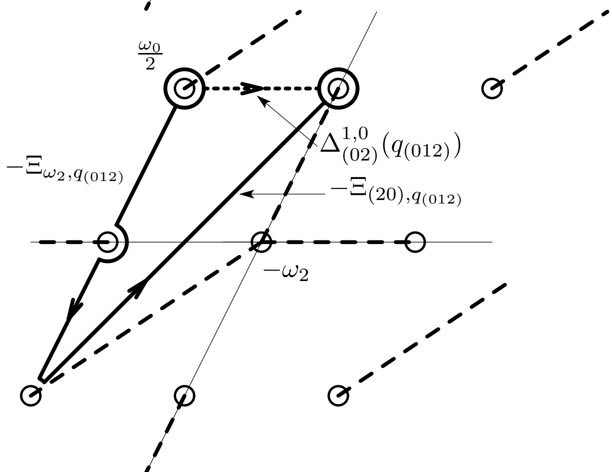
where Ξ is a twisted cycle over C minus the four points $\left\{\overline{0}, \overline{\frac{\omega_0}{2}}, \overline{\frac{\omega_1}{2}}, \overline{\frac{\omega_2}{2}}\right\}$ and $\Delta^{m_1,m_2}_{(ij)}(q)$ be the cycle *vanishing* at the singular point $q^{m_1,m_2}_{(ij)}$, as q tends to $q^{m_1,m_2}_{(ij)}$ along the path $\lambda^{m_1,m_2}_{(ij)}$. We have the following:

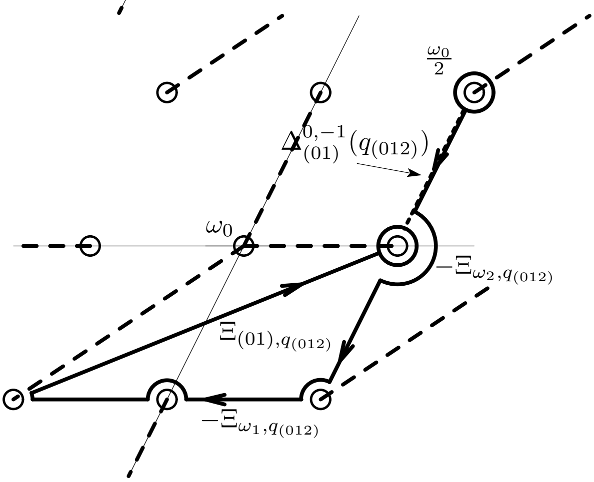
Proposition 5.1 (vanishing cycles)**.**

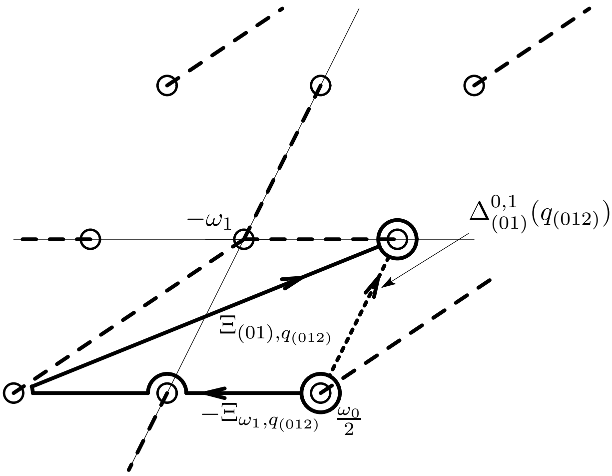
$$\begin{aligned}\Delta^{-1,0}_{(02)}(q_{(012)}) &= -\Xi_{(20),q_{(012)}} - \Xi_{\omega_1,q_{(012)}} - \Xi_{\omega_2,q_{(012)}},\\ \Delta^{1,0}_{(02)}(q_{(012)}) &= -\Xi_{(20),q_{(012)}} - \Xi_{\omega_2,q_{(012)}},\\ \Delta^{0,-1}_{(01)}(q_{(012)}) &= \Xi_{(01),q_{(012)}} - \Xi_{\omega_1,q_{(012)}} - \Xi_{\omega_2,q_{(012)}},\\ \Delta^{0,1}_{(01)}(q_{(012)}) &= \Xi_{(01),q_{(012)}} - \Xi_{\omega_1,q_{(012)}}.\end{aligned}$$

Proof. The vanishing cycle $\Delta^{m_1,m_2}_{(ij)}(q_{(012)})$ is the regularization of the segment $\overline{x^{m_1,m_2}_{i,(ij)}(0), x^{m_1,m_2}_{i,(ij)}(1)}$. Replacing it by the linear combination of $\Xi_{(01),q_{(012)}}$, $\Xi_{(20),q_{(012)}}$, $\Xi_{\omega_1,q_{(012)}}$ and $\Xi_{\omega_2,q_{(012)}}$, we obtain the assertion. (See the following figures, in which the dotted lines indicate the vanishing cycles.)


$-\Xi_{\omega_1,q_{(012)}}$
$\frac{\omega_0}{2}$
$-\Xi_{\omega_2,q_{(012)}}$
$\Xi_{(20),q_{(012)}}$
$\Delta^{(-1,0)}_{(02)}(q_{(012)})$
ω_0


$\frac{\omega_0}{2}$
$\Delta^{1,0}_{(02)}(q_{(012)})$
$-\Xi_{\omega_2,q_{(012)}}$
$-\Xi_{(20),q_{(012)}}$
$-\omega_2$


$\frac{\omega_0}{2}$
$\Delta^{0,-1}_{(01)}(q_{(012)})$
ω_0
$-\Xi_{\omega_2,q_{(012)}}$
$\Xi_{(01),q_{(012)}}$
$-\Xi_{\omega_1,q_{(012)}}$



□

We have the common basis for $\mathcal{H}(q_{(012)})$ and $\mathcal{H}\left(q_{(\tau_{ij}(0),\tau_{ij}(1),\tau_{ij}(2))}\right)$: Put

$$\begin{aligned}\Xi_{(kl)} :=& \frac{1}{c-1}\overline{S_\epsilon\left(\frac{\omega_k}{2}\right)} \otimes \varsigma_{\overline{S_\epsilon\left(\frac{\omega_k}{2}\right)}} \\ &+ \overline{\left[\frac{\omega_k}{2}+\epsilon, \frac{\omega_l}{2}-\epsilon\right]} \otimes \varsigma_{\overline{\left[\frac{\omega_k}{2}+\epsilon, \frac{\omega_l}{2}-\epsilon\right]}} - \frac{1}{c-1}\overline{S_\epsilon\left(\frac{\omega_l}{2}\right)} \otimes \varsigma_{\overline{S_\epsilon\left(\frac{\omega_l}{2}\right)}}, \\ \Xi_{\omega_k} :=& l_{\omega_k} \otimes \varsigma_{l_{\omega_k}},\end{aligned}$$

and we define the ordering on the index set $J' := \{(01), (20), \omega_1, \omega_2\}$ of this basis:

$$(01) \prec (20) \prec \omega_1 \prec \omega_2.$$

Now we have the following:

Theorem 5.1 (connection matrices)**.**
Let $M(\gamma^{m_1,m_2}_{(ij)}) = \left[m(\gamma^{m_1,m_2}_{(ij)})_{\mu,\nu}\right]_{\mu,\nu\in J'}$ *be the* 4×4*-matrix which represents the linear isomorphism* $\left(\gamma^{m_1,m_2}_{(ij)}\right)_*$*, that is,* $\left(\gamma^{m_1,m_2}_{(ij)}\right)_*(\Xi_\mu)$ $= \sum_\nu m(\gamma^{m_1,m_2}_{(ij)})_{\mu,\nu}\Xi_\mu$*. Then,*

$$M(\gamma^{-1,0}_{(02)}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ c & -c & c-1 & -c+1 \\ c & -c-1 & c & -c+1 \\ c & -c-1 & c-1 & -c+2 \end{bmatrix}, \quad M(\gamma^{1,0}_{(02)}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ c & -c & c-1 & 0 \\ 0 & 0 & 1 & 0 \\ c & -c-1 & c-1 & 1 \end{bmatrix},$$

$$M(\gamma^{0,-1}_{(01)}) = \begin{bmatrix} -c & 1 & -c+1 & c-1 \\ 0 & 1 & 0 & 0 \\ c+1 & -1 & c & -c+1 \\ c+1 & -1 & c-1 & -c+2 \end{bmatrix}, \quad M(\gamma^{0,1}_{(01)}) = \begin{bmatrix} -c & 1 & 0 & c-1 \\ 0 & 1 & 0 & 0 \\ c+1 & -1 & 1 & -c+1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proof. Substituting the result of Proposition 5.1 for the twisted Picard-Lefschetz formula, we obtain the assertion. □

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