MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES, A
Vol. XXVIII, Mathematics No. 1, 1953.

On the independency of differential forms on algebraic varieties.

By

Yoshikazu NAKAI

(Received April 20, 1953)

In the classical algebraic geometry the following theorem has been hitherto admitted generally.

Let V^r be a projective model of an algebraic, variety, ω^i $(i=1,\cdots,s)$ linearly independent differential forms of the first kind on V, k a common field of definition for ω_i and W a generic hyperplane section of V with reference to k. Then ω_i's induce on W linearly independent differential froms $\bar{\omega}_i$ of the first kind.

Recently J. Igusa proved this rigorouly using the theory of harminic integrals.[1] It seems to be true that it holds also for the ground field of arbitrary characteristic, but the proof is not yet obtained. In this paper, modifying the above we shall prove the following:

Let V be an algebraic variety in a projective space, ω_i $(i=1,\cdots,s)$ linearly independent differential forms on V(they may be not of the first kind), k a common field of definition for ω_i and C_m a generic hypersuface section of V of order m with reference to k. Then the induced differential forms $\bar{\omega}_i$ on C_m by ω_i are also linearly independent provided m is sufficiently large.

I wish to express my sincere gratitude to Professors Y. Akizuki and J. Igusa for their valuable advices and kind encouragement.

§ 1. Some results on uniformizing parameters.[2]

Definition. Let X and Y bo two cycles on a Variety V and Q a Point on V. If any component of X containing Q intersect pro-

---

1) Cf. J. Igusa (1). The numbers in bracket refer to the bibliography at the end of the paper.

2) We seall use the notations and terminology adopted in Weil (5).

perly with every component of Y containing Q, then we shall say that the intersection product $X.\ Y$ is defined locally at Q.
We shall mean by $X \equiv Y$ (mod. Q) that the cycles X and Y contain the same components containing Q with the same multiplicities. For any number of cycles $X_1,\cdots,X_s$ the local intersection product at Q is defined in an analogous way.

Proposition 1. *Let V^r be a Variety, f_i ($i=1,\cdots, s \leq r$) functions on V and Q a Point on V where each function f_i is deffned and finite. Suppose that the intersection product $(f_1-f_1(Q))\cdots(f_s-f_s(Q))$ is defined locally at Q, then the functions $f_1,\cdots f_s$, are algebraically independent.*

Proof. Without loss of generalities we can suppose that $f_i(Q)=0$. We shall use the induction. For $s=1$ the assertion is trivial. Suppose that $f_1,\cdots,f_n$ are algebraically independent and f_{n+1} is algebraic with respect to $f_1,\cdots,f_n$. Let k be a common field of definition for f_i, P a generic Point of V over k, and put $f_i(P)=t_i$. Then $t_1,\cdots, t_n$ are independent variables over k and t_{n+1} is algebraic over $k(t_1,\cdots, t_n)$ by induction assumption. Let Z_n be a component of $(f_1)\cap\cdots\cap(f_n)$ containining Q and Z_{n+1} a component of $Z_n\cap(f_{n+1})$ containing Q. Then since f_i are defined over k, Z_n, Z_{n+1} are algebraic over k. Let M and N be generic Points of Z_n and Z_{n+1} respectively over k. Then $(P,\ t_1,\cdots,t_n)\rightarrow(M,\ 0,\cdots,\ 0)$ is a specialization over k, and this can be extended to a finite specialization $(M, 0,\cdots, 0, c)$ of $(P, t_1,\cdots, t_n, t_{n+1})$ over k. Since Z_{n+1} is a Subvariety of Z_n, N is a specialization of M over $\bar{k}$ and this can be extended to the specialization (N, c') of (M, c) over $\bar{k}$. Thus we see that $(N, 0,\cdots, 0,c')$ is a specialization of $(P, t_1,\cdots, t_n, t_{n+1})$ over k. But since $(N, 0,\cdots, 0, 0)$ is a specialization of $(P, t_1,\cdots, t_{n+1})$ and t_{n+1} has the uniquely determined specialization 0 over the specializa tion $P\rightarrow Q$, hence also it has the uniquely determined specialization over $P\rightarrow N$ with reference to k. Then we must have $c'=0$. By hypothesis t_{n+1} is algebraic over $k(t_1,\cdots, t_n)$ hence c can be so chosen that c is in $\bar{k}$, and c has the specialization 0 over $\bar{k}$. Then c must be 0, i.e. $(M, 0)$ is a specialization of (P, t_{n+1}) over k and Z_n must be contained in (f_{n+1}). Thus we have arrived at a contradiction and the assertion is proved. q.e.d.

Remark. The condition "*each f_i is defined and finite at Q*" is essential as is shown in the following example.

Example. In L^2, let $P=(1, x, y)$ be a generic Point of L over Π

(prime field) and define $f_i (i=1, 2)$ as follows.

$$f_1(\boldsymbol{P})=x, \quad f_2(\boldsymbol{P})=x+c, \quad c \in \Pi$$

They are not clearly independent, but they intersect properly at the point at infinity (0, 0, 1).

Theorem 1.[3] *Let $\tau_1,\cdots, \tau_r$ be functions on $\boldsymbol{V}^r$ and $\boldsymbol{P}'$ a simple Point of $\boldsymbol{V}$, then $\tau_1,\cdots,\tau_r$ are uniformizing parameters at $\boldsymbol{P}'$*[4] *on $\boldsymbol{V}$ if and only if the following conditions hold for (τ_i).*

(*i*) *each function τ_i is defined and finite at $\boldsymbol{P}'$.*

(*ii*) *Intersection product $(\tau_1-\tau_1(\boldsymbol{P}'))\cdots(\tau_r-\tau_r(\boldsymbol{P}'))$ is defined locally at $\boldsymbol{P}'$ and contain $\boldsymbol{P}'$ with multicity 1.*

Proof. Let k be a common field of definition for τ_i $(i=1,\cdots,r)$ $\boldsymbol{P}$ a generic Point of $\boldsymbol{V}$ over k, Γ_i the graph of τ_i in $\boldsymbol{V}\times S^1$, and put $\tau_i(\boldsymbol{P})=t_i$, τ_i $(\boldsymbol{P}')=t_i'$ and $\boldsymbol{Q}$, $\boldsymbol{Q}'$ points in S^r whose coordinates are $(t_1,\cdots,t_r)$, $(t_1',\cdots,t_r')$ respectively.

Suppose that (τ_i) are uniformizing parameters at $\boldsymbol{P}'$ on $\boldsymbol{V}$ and let $\boldsymbol{W}$ be the locus of $\boldsymbol{P}\times\boldsymbol{Q}$ over k in $\boldsymbol{V}\times S^r$. Then $\boldsymbol{W}$ has the properties described in W-F, VIII, prop. 10,[5] i.e. $\boldsymbol{W}$ is transversal to $\boldsymbol{V}\times\boldsymbol{Q}'$ at $\boldsymbol{P}'\times\boldsymbol{Q}'$; Moreover if $\boldsymbol{Z}$ is any Subariety of S^r which has $\boldsymbol{Q}'$ as a simple point, then $\boldsymbol{V}\times\boldsymbol{Z}$ and $\boldsymbol{W}$ are tranversal to each other at $\boldsymbol{P}'\times\boldsymbol{Q}'$. Let $\boldsymbol{X}_i$ be the components of $(\tau_i-\tau_i(\boldsymbol{P}'))$ containing P', then we shall show by induction that we have

$$(1) \qquad \boldsymbol{X}_1\cdots\boldsymbol{X}_n\equiv \mathrm{pr}_V[(\boldsymbol{V}\times t_1'\times\cdots\times t_n'\times S^{r-n}).\ \boldsymbol{W}] \qquad (\mathrm{mod}\ \boldsymbol{P}')$$

In fact we have

$$\mathrm{pr}_{oi}[(\boldsymbol{V}\times S^{i-1}\times t_i'\times S^{r-i}).\ \boldsymbol{W}]=\Gamma_i\cdot(\boldsymbol{V}\times t_i')$$

where pr_{oi} means the projection on the product of $\boldsymbol{V}$ and the i-th factor of S^1. Hence

$$\mathrm{pr}_V[(\boldsymbol{V}\times S^{i-1}\times t_i'\times S^{r-i}).\boldsymbol{W}]=\mathrm{pr}_V[\Gamma_i\cdot(\boldsymbol{V}\times t_i')]\equiv\boldsymbol{X}_i \quad (\mathrm{mod}\ \boldsymbol{P}')$$

From this we see at once that $\boldsymbol{X}_i$ is a Variety for every i and contained in (τ_i-t_i') with multiplicity 1, and the equality (1) is proved for $n=1$. Suppose that the equality (1) is already proved for a number $\leq n$, and $\boldsymbol{Z}_n$ be a component of $\boldsymbol{X}_1\cdots\boldsymbol{X}_n$ containing $\boldsymbol{P}'$, Then we have

---

3) This formulation is due to Prof. J. Igusa.

4) Cf. Definition 1 of Nakai (4).

5) This means "proposition 10 of chapter VIII of Weil (5)".

$$Z_n \equiv \mathrm{pr}_V[(V\times t_1'\times\cdots\times t_n'\times S^{r-n}).\ W] \qquad (\mathrm{mod}\ P'))$$

and we see by the property of W that Z_n is a Variety and contained in $X_1\cdots X_n$ with multiplicity 1. We shall next show that the intersection product

$$(2) \qquad (Z_n\times S^r)\cdot(V\times S^n\times t'_{n+1}\times S^{r-n-1}).W$$

is defined locally at $P'\times Q'$. Since the first and the last two members intersect properly, it is sufficient to show that

$$(2') \qquad (Z_n\times S^n\times t'_{n+1}\times S^{r-n-1}).W$$

is defined locally at $P'\times Q'$ by W–F, VII, Cor. of Th. 10. Now the projection from W to V is regular along Z_n, then we see easily that

$$(Z_n\times S^n\times t'_{n+1}\times S^{r-n-1})\cap W\subset(V\times t_1'\times\cdots\times t'_{n+1}\times S^{r-n-1})\cap W$$

and the right hand member containes only one component containing $P'\times Q'$ whose dimension is r-n-1, hence counting the dimension we see that the left hand side is defined locally at $P'\times Q'$. Then by W-F, VII, Th. 16 we see that Z_n and X_{n+1} intersect properly (locally at P') and we have

$$(3) \qquad Z_n\cdot X_{n+1}\equiv Z_n\cdot\mathrm{pr}_V[(V\times S_n\times t'_{n+1}\times S^{r-n-1}).W]$$
$$=\mathrm{pr}_V[(Z_n\times S^r).(V\times S^n\times t'_{n+1}\times S^{r-n-1}).W] \quad (\mathrm{mod.}\ P')$$

Moreover we have

$$(4) \qquad \mathrm{pr}_V[(Z_n\times S^r).W]\equiv Z_n\equiv pr_V[(V\times t_1'\times\cdots\times t'_n\times S^{r-n}).\ W] \quad (\mathrm{mod}\ P')$$

and since the projection from W to V is regular along Z_n, there is one and only one Subvariety of W which has the projection Z_n on V, and such a component must be contained in $(V\times t_1'\times\cdots\times t_n'\times S^{r-n})$. W. Then we can replace it in the position of $(Z_n\times S^r)$. W in (3), Thus we have

$$Z_n\cdot X_{n+1}\equiv\mathrm{pr}_V[(V\times t_1'\times\cdots\times t'_{n+1}\times S^{r-n-1}).\ W] \qquad (\mathrm{mod}\ P')$$

and the equality (1) is proved. In pariticular we have

$$X_1\cdots X_r\equiv\mathrm{pr}_V[(V\times Q').W]\equiv P' \qquad (\mathrm{mod}\ P')$$

and the condition (ii) is satisfied.

Conversely let (τ_i) satisfy the conditions (i) and (ii) and define W as in the above proof. Then the projection from W to V is regular at P'. By Prop. 1, $t_1,\cdots, t_r$ are independent variables over k, hence $\boldsymbol{W}$ has the projection S^r on S^r. Now we shall prove the equality (1) under these conditions. By induction suppose that (1) is proved for a number $\leq n$. The condition (ii) implies that the intersection product (2′) hence (2) are defined and the relation (3) also holds. By condition (i) we have the equality (4) and hence (1) too. Thus we have

$$(\boldsymbol{V}\times Q').\ \boldsymbol{W}\equiv\boldsymbol{P}'\times Q' \qquad (\text{mod } \boldsymbol{P}'\times Q')$$

and the assertion is proved. q. e. d.

Corollary 1. *Let $\boldsymbol{V}^r$ be a Variety, $\boldsymbol{U}^s$ its simple Subvariety, $\boldsymbol{P}'$ a simple Point of $\boldsymbol{V}$ which is also simple on $\boldsymbol{U}$ and $\tau_1,\cdots\tau_r$ uniformizing parameters at $\boldsymbol{P}'$ on $\boldsymbol{V}$. Then we can select among $(\bar{\tau}_i)$ uniformizing parameters at $\boldsymbol{P}'$ on $\boldsymbol{U}$, where $\bar{\tau}_i$ are functions on $\boldsymbol{U}$ induced by τ_i.*

Proof. Let P_0' be a representative of $\boldsymbol{P}'$, V_0, U_0 the representatives of $\boldsymbol{V},\boldsymbol{U}$ containing P_0' and T, N the tangential linear varieties to V_0, U_0 at P_0' Since $(\tau_i-\tau_i(P'))$ containes only one component $\boldsymbol{A}_i$ containing $\boldsymbol{P}'$ and $\boldsymbol{P}'$ is a simple Point of $\boldsymbol{A}_i$, there exist the tangential linear varieties M_i to A_{i0} at P_0', where A_{i0} are representatives of $\boldsymbol{A}_i$ $(i=1,\cdots,r)$. The assumption means that the linear varieties M_i are transversal to each other at P_0' in $\boldsymbol{T}^r$, i.e. when we denotes the indeterminates in T by X_i $(i=1,\cdots,r)$ and by $F_i(X)\equiv\sum a_{ij}X_j+a_i=0$ $(i=1,\cdots,r)$ the defining equations for M_i, the matrix $\|a_{ij}\|$ is regular. Let

$$H_a(X)\equiv\sum b_{aj}X_j+b_a=0 \quad (\alpha=1,\cdots r-s)$$

be the defining equations for N. Then to prove the assertion it is necessary and sufficient to show that there exist s-polynomials among $F_i(X)$ such that $H_a(X)$ $(\alpha=1,\cdots,r-s)$ and $F_{i_1}(X),\cdots,F_{i_s}(X)$ constitutes a set of linearly independent linear forms. Then an elementary considerations shows us that the assertion hold. q.e.d.

Corollary 2. *Let $\boldsymbol{V}^r$ be a Variety defined over k, $\boldsymbol{U}^s$ its simple Subvariety and $\boldsymbol{U}'$ a specialization of $\boldsymbol{U}$ over k. Let $\boldsymbol{Q}$ be a Point in $\boldsymbol{U}\cap\boldsymbol{U}'$ which has the following properties; (a) there is one and only one component $\boldsymbol{U}''$ of $\boldsymbol{U}'$ containing $\boldsymbol{Q}$, and $\boldsymbol{U}''$ is contained in $\boldsymbol{U}'$ with multiplicity 1; (b) $\boldsymbol{Q}$ is simple on $\boldsymbol{V}$, $\boldsymbol{U}$ and $\boldsymbol{U}''$; (c) $\boldsymbol{Q}$ is rational over k. Let $\tau_1,\cdots,\tau_r$ be uniformizing parameters at $\boldsymbol{Q}$ on $\boldsymbol{V}$ and suppose that $\bar{\tau}_1',\cdots,\bar{\tau}_s'$ are uniformizing parameters at $\boldsymbol{Q}$ on $\boldsymbol{U}''$, then $\bar{\tau}_1,\cdots,\bar{\tau}_s$ are uniformizing parameters at $\boldsymbol{Q}$ on $\boldsymbol{U}$, where $\bar{\tau}_i$ and $\bar{\tau}_i'$ are functions on $\boldsymbol{U}$ and $\boldsymbol{U}''$ respectively induced by the functions $\bar{\tau}_i$ on $\boldsymbol{V}$.*

Proof. Put $\tau_i(Q)=t_i$. Then we have by hypothesis

$$[(\bar{\tau}_1'-t_1)\cdots(\bar{\tau}_s'-t_s)]_{U''}\equiv Q \qquad (\text{mod } Q)$$

Hence we have

$$(\tau_1-t_1)\cdots(\tau_s-t_s)\cdot U''\equiv Q \qquad (\text{mod } Q)$$

by W-F, VII, Cor. of Th. 18. Since the left hand side is a component of a specializotion of

$$(\tau_1-t_1)\cdots(\tau_s-t_s)\cdot U$$

over k, with multiplicity 1, and (τ_i-t_i) and Q are invariant by the specialization $U\rightarrow U'$ over k we must have

$$(\tau_1-t_1)\cdots(\tau_s-t_s)\cdot U\equiv Q \qquad (\text{mod } Q)$$

i.e.
$$[(\bar{\tau}_1-t_1)\cdots(\bar{\tau}_s-t_s)]_U\equiv Q \qquad (\text{mod } Q)$$

Thus the condition (ii) of Th. 1 is satisfied by $\bar{\tau}_1,\cdots,\bar{\tau}_s$. The first condition is clearly satisfied by the assumption and the assertion is proved. q. e. d.

§ 2. The independency of differential forms.

Proposition 2. *Let $\varphi_i(1\leq i\leq s)$ be linearly independent functions on V and C_m an irreducible hypersurface sectiou of V of order m. Then if m is sufficiently large the functions $\bar{\varphi}_i$, which are the functions on C_m induced by φ_i, are linearly independent.*

Proof. Suppose that $\bar{\varphi}_i$ are linearly dependent on C_m, and let K be a common field of definition for V, φ_i and C_m, and P, Q the generic Points of V, C_m over K respectively. Then by the assumption the quantities $\varphi_i(Q)$ are linearly dependent over K and there exist quantities c_i in K such that we have $\sum c_i\ \varphi_i\ (Q)=0$ without being $\sum c_i\,\varphi_i=0$. Then $(\sum c_i\,\varphi_i)_0$ must contain C_m as its component. Since the linear system defined by the functions φ_i on V has a fixed degree, C_m cannot be a component of such a linear system if m is sufficiently large. Thus the assertion is proved.

Proposition 3. *Let $\omega_i(i=1,\cdots,s)$ be differential forms on V and k a common field of definition for ω_i.*[6] *Then if ω_i are linearly dependent over the constant field, they are already dependent over k.*

This can be proved in an analogous way as in the case of functions, and the proof is omitted.

6) cf. § 1 of Nakai (4).

Let V be a projective model of an algebraic Variety, ω_i $(1\leq i\leq s)$ linearly independent differential forms on V, k a common field of definition for ω_i and $P=(1, x_1,\cdots, x_N)$ a generic Point of V over k. Let H'_m be a hypersurface in the ambient projective space L^N defined by the equation

$$\sum_{i_0+\cdots+i_N=m} v_{i_0 i_1\cdots i_N} X_0^{i_0} X_1^{i_1}\cdots X_N^{i_N}=0$$

such that the intersection product $C'_m=V\cdot H'_m$ is irreducible and goes through P. Let $H_M (M\geq m)$ be a generic hypersurface defined by the equation

$$\sum_{i_0+\cdots+i_N=M} u_{i_0 i_1\cdots i_N} X_0^{i_0} X_1^{i_1}\cdots X_N^{i_N}=0$$

where $(u_{i_0 i_1\cdots i_N}, i_1+\cdots+i_N>0)$ are $\binom{N+M}{M}-1$ independent variables over $k(P)$ and $u_{M0\cdots0}$ is determined by the equation

$$u_{M0\cdots0}=-\sum_{M\geq i_1+\cdots+i_N>0} u_{i_0 i_1\cdots i_N} x_1^{i_1}\cdots x_N^{i_N}$$

Then as is well known $C_M=V\cdot H_M$ is irreducible. Under these conditions we have the

Proposition 4. *Let $\bar{\omega}_i$, $\bar{\omega}_i'$ be differential forms on C_M and C_m' respectively induced by differential forms ω_i $(i=1,\cdots s)$ on V. Then if $\bar{\omega}_i'$ are linearly independent on C_m', $\bar{\omega}_i$ are also linearly independent on C_M.*

Proof. Let τ_i be functions on V defined over k by $\tau_i(P)=x_i$. Then by Prop. 5 of Nakai (4) we can suppose that $\tau_1,\cdots,\tau_r$ are uniformizing parameters on V at P. Hence by Cor. 1 of Th. 1 we can assume that $\bar{\tau}_1',\cdots,\bar{\tau}'_{r-1}$ are uniformizing parameters at P on C'_m. Then $\bar{\tau}_1,\cdots,\bar{\tau}_{r-1}$ are also uniformizing parameters at P on C_M by Cor. 2 of Th. 1, where $\bar{\tau}_i$, $\bar{\tau}_i'$ are functions on C_M, C'_m respectively (In the following we shall denote by $-$ and $-'$ the functions on C_M and C'_m respectively induced by the function on V). Let

$$\omega_i=\sum_{i_1<\cdots<i_p}\varphi^{(i)}_{i_1\cdots i_p}\, d\tau_{i_1}\cdots d\tau_{i_p}$$

Then we have

$$\begin{aligned}\bar{\omega}_i&=\sum_{i_1<\cdots<i_p}\Big(\bar{\varphi}^{(i)}_{i_1\cdots i_p}-\frac{\bar{a}_{i_p}}{\bar{a}_r}\bar{\varphi}^{(i)}_{i_1\cdots i_{p-1}r}+\cdots+(-)^p\frac{\bar{a}_{i_1}}{\bar{a}_r}\bar{\varphi}^{(i)}_{i_2\cdots i_p r}\Big)d\bar{\tau}_{i_1}\cdots d\bar{\tau}_{i_p}\\&=\sum \phi^{(i)}_{i_1\cdots i_p}\, d\bar{\tau}_{i_1}\cdots d\bar{\tau}_{i_p}.\end{aligned}$$

where the sum is extended over all combinations of indices $i_1<\cdots<i_p$ taken from $1,\cdots, r-1$, and $\bar{a}_i$ are determined by the relation

$$d\Big(\sum_{M\geqq i_1+\cdots+i_N>0} u_{i_0\, i_1,\ldots,i_N}\bar{\tau}_1^{i_1}\cdots\bar{\tau}_N^{i_N}\Big)=0$$

and they are linear combinations of the functions of the form $\bar{\tau}_1^{i_1}\cdots\bar{\tau}_N^{i_N}\overline{\dfrac{\partial\tau_i}{\partial\tau_j}}$ with the coefficients in $k[u]$ (Cf. the proof of Th. 2 of Nakai (4)). In the same way we have

$$\bar{\omega}_i'=\sum_{i_1<\cdots<i_p}\Big(\bar{\varphi}'_{i_1,\ldots,i_p}-\frac{\tilde{a}_{i_p}}{\tilde{a}_r}\bar{\varphi}'_{i_1,\ldots,i_{p-1},r}+\cdots+(-)^p\frac{\tilde{a}_{i_1}}{\tilde{a}_r}\bar{\varphi}'_{i_2,\ldots,i_p,r}\Big)d\bar{\tau}'_{i_1}\cdots d\bar{\tau}'_{i_p}.$$

$$=\sum_{i_1<\cdots<i_p}\tilde{\Phi}_{i_1,\ldots,i_p}\, d\bar{\tau}'_{i_1}\cdots d\bar{\tau}'_{i_p}.$$

We shall remark here that $\varphi_{i_1,\ldots,i_p}(\boldsymbol{P})=\bar{\varphi}_{i_1,\ldots,i_p}(\boldsymbol{P})=\bar{\varphi}'_{i_1,\ldots,i_p}(\boldsymbol{P})$; moreover since $(M-m).\ \boldsymbol{X}_0+\boldsymbol{C}_m'$ is a specialization of $\boldsymbol{C}_M$ (where $\boldsymbol{X}_0$ is the intersection product of $\boldsymbol{V}$ with the hyperplane $X_0=0$) and $\boldsymbol{X}_0$ does not contain $\boldsymbol{P}$ we see that $\tilde{a}_i(\boldsymbol{P})$ are the uniquely determined specialization of $\bar{a}_i(\boldsymbol{P})$ over the specialization $(u)\to(v)$ with reference to $k(\boldsymbol{P})$.

Now suppose that $\bar{\omega}_i$ are linearly dependent, then there exist s-quantities in $k[u]$ such that we have

$\sum a_i(u)\tilde{\Phi}_{i_1,\ldots,i_p}(\boldsymbol{P})=0$, for all sets of indices $i_1<\cdots<i_p$.

By Th. 1 of Weil (6) there exist a valuation of $k(u,\boldsymbol{P})$ over $k(u)$ which has the value in the algebraic closure $\overline{k(v)}$ of $k(v)$. Let it be ν and suppose that $a_{i_0}(u)$ has the minimum value for ν, then the quantities $a_i(u)/a_{i_0}(u)$ has the finite specialization over the specialition $(u)\to(v)$ with reference to $k(\boldsymbol{P})$. Hence we have a non-identical relation of the form

$$\sum a_i'\tilde{\Phi}_{i_1,\ldots,i_p}(\boldsymbol{P})=0,\ \text{for all sets of indices } i_1<\cdots<i_p.$$

and a' are in $\overline{k(v)}$. But since $k(v,\boldsymbol{P})$ is a regular extension over $k(v)$, we see easily that there exists quantities a_i'' in $k(v)$ such that

$\sum a_i''\ \tilde{\Phi}_{i_1,\ldots,i_p}(\boldsymbol{P})=0$, for all sets of indices $i_1<\cdots<i_p$.

i.e. $\sum a_i''\,\bar{\omega}_i'=0$. It contradicts to our hypothsis and the assertion is proved.[7] q. e. d.

Lemma. *Let K be a field containing k such that* $\dim_k K=1$, *and $z_1,\cdots,z_n$ elements in K. Then for any valuation v of K over k, the value domain for the module of linear forms $o=\sum_{i=1}^{n}kz_i$ are bounded, i.e. there exist an integer N such that $|v(a)|<N$ for any element a of o.*

7) The device of the latter part of this proof is due to the remark by Prof. Y. Akizuki.

Proof.[8] As we easily see, it is sufficient to show the lemma under the additional condition that k is algebraically closed. Moreover we can suppose that $z_1,\cdots,z_n$ are linearly independent over k. We shall first show by induction that there exist the basis of o such that the value of the basis are all different form each other. Suppose that we have

$$v(z_1)<v(z_2)<\cdots<v(z_s)=v(z_{s+1})=\cdots=v(z_{s+j})<\cdots\leq v(z_n).$$

Then we can find elements in k such lhat $z_s/z_{s+i}\equiv a_i \pmod{\mathfrak{p}}$, $i=1,\cdots,j$, where $\mathfrak{p}$ is the valuation ideal. Put $z'_{s+t}=z_s-a_t z_{s+t}$. Then z'_{s+i} can be replaced by z_{s+i} and we have $v(z'_{s+t})>v(z_s)$. Take an element which has the minimum value among $\{z'_{s+i}, i=1,\cdots,j\}$ and $\{z_t, s+j<t\leq n\}$, and call it z''_{s+1}, then we get the basis of o whose first $s+1$ elements have dfferent values to each other and the remaining basis have the values not less than the values of the first $s+1$ basis elements. Continuing this process in finite number we will arrive at the required basis. Let them be $x_1,\cdots,x_n$, and suppose that $v(x_1)<\cdots<v(x_n)$. Then we see immediately that $v(x_1)\leq v(a)\leq v(x_n)$ for any a in o and the assertion is proved. q.e.d.

Proposition 5. *Let K be a field containing $k, z_1,\cdots,z_n$ elements in K and put $o=\sum kz_i$. Then for any independent variable x over k we can find infinitely many elements among $\{x^m\}$, $m=0, 1, 2,\cdots$ which are linearly independent over o.*

Proof. Put $K_0=k(z_1,\cdots,z_n,x)$ and L be any subfield of K_0 such that we have $\dim_L K_0=1$ and x is transcendental over L. Let v be a valuation of K_0 over L such that $v(x)>0$. Suppose that there are only a finite number of elements among $\{x^m\}$ which are linearly independent over $o'=\sum_{i=1}^{s} Lz_i\supset o$, and let them be $1, x^{m_1},\cdots,x^{m_s}$. Put $o''=\sum_{i,j} L\, z_i\, x^{m_j}$, then by the above lemma $v(a)$ is bounded for any element a in o''. By assumption, for any large N we have a relation of theform $a_N x^N=\sum_{i=1}^{s} a_i x^{m_i}$ where a's are in o'. But the right hand member is contained in o'' and has a bounded value for v and the left hand member may have any large value, and it is a contradiction. Hence there are infinitely many elements among $\{x^m\}$ which are linearly independent over o' hence also over o. q.e.d.

Now we are well prepared to prove our main theorem.

Theorem 2. *Let $V^r (r\geq 2)$ be a projective model of an algebraic*

8) I thank this proof to my friend M. Nagata.

Variety, $\omega_i (1 \leq i \leq s)$ liuearly independent differential forms on V, k a common field of definition for ω_i and C_m a generic hypersurface section of V of order m with reference to k. Then if m is sufficiently large, ω_i induce on C_m linearly independent differential forms $\bar{\omega}_i$.

Proof. Using the same notations as in Prop. 4, let

$$\omega_i = \sum_{i_1 < \cdots < i_p} \varphi^{(i)}_{i_1 \cdots i_p} \, d\tau_{i_1} \cdots d\tau_{i_p},$$

where the sum is extended over all sets of indices $i_1 < \cdots < i_p$ taken from $1, \cdots, r$ and the functions $\varphi^{(i)}_{i_1 \cdots i_p}$, τ_i are all defined over k. Let P be a generic Point of V over k and put $o = \sum_{\substack{i \\ (i_1 \cdots i_p)}} k.\ \varphi^{(i)}_{i_1 \cdots i_p}(P)$. Then by Prop. 5 we can suppose $1, x_1^{e_{12}}, \cdots, x_1^{e_{1r}}$ are linearly independent over o. Moreover as is seen from the proof of Prop. 5 we can choose e_{1j} in such a way that $e_{1j}+1$ is not divisible by the characteristic of the universal domain. Now put.

$$y_{11} = x_1$$
$$y_{12} = x_2 - x_1^{e_{12}+1}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$y_{1r} = x_r - x_1^{e_{1r}+1}$$

and let η_{1l} be functions on V defined over k by $\eta_{1l}(P) = y_{1l}$. Then we see easily that $\eta_{11}, \cdots, \eta_{1r}$, are also served as uniformizing parameters at P on V and we can express ω_i in the form

$$\omega_i = \sum_{j_1 < \cdots < j_p} \psi^{(i)}_{j_1 \cdots j_p} \, d\eta_{j_1} \cdots d\eta_{j_p}$$

where

$$\psi^{(i)}_{j_1 \cdots j_p} = \varphi^{(i)}_{j_1 \cdots j_p}. \qquad \text{if } j_1 > 1.$$

$$\psi^{(i)}_{1 j_2 \cdots j_p} = \varphi^{(i)}_{1 j_2 \cdots j_p} + \sum_{l \geq 2} (e_{1l}+1)\, \eta_{11}^{e_{1l}} \, \varphi^{(i)}_{l j_2 \cdots j_p}$$

It is to be noted that for any choice of indices $k_1 < \cdots < k_p$, $\varphi^{(i)}_{k_1 \cdots k_p}$ appears in the expression of $\psi^{(i)}_{1 k_2 \cdots k_p}$. Writing ω_i in the form

$$\omega_i = d\eta_{11} \cdot \omega_i^* + \omega_i^{(*)}.$$

we will see that ω_i^* are linearly independent differential forms on V. In fact suppose that ω_i^* are linearly dependent, then there exist the quanties c_i in k such that we have

$$\sum c_i \, \omega_i^*(P) = 0$$

i.e.

$$\sum c_i \, \psi^{(i)}_{1 j_2 \cdots j_p}(P) = 0 \text{ for all } j_2 < \cdots < j_p$$

$$\sum c_i [\varphi^{(i)}_{1 j_2 \cdots j_p}(P) + \sum_{l \geq 2} (e_{1l}+1) \cdot y_{11}^{e_{1l}} \, \varphi^{(i)}_{l j_2 \cdots j_p}(P)] = 0.$$

But since $1, y_{11}^{e_{12}}, \cdots, y_{1r}^{e_{1r}}$ are linearly independent over o we must have

$$\sum c_i \varphi^{(i)}_{i_1 \cdots i_p}(P)=0 \text{ for all } i_1<\cdots<i_p$$

It contradits to the independency of ω_i.

Next we shall transform the uniformizing parameters into

$$\begin{aligned} y_{21}&=y_{11}\\ y_{22}&=y_{12}\\ y_{23}&=y_{13}-y_{12}^{e_{23}+1}\\ &\cdots\cdots\cdots\\ y_{2r}&=y_{1r}-y_{12}^{e_{2r}+1} \end{aligned}$$

where $e_{2j}(j=3,\cdots,r)$ are so chosen that $1, y_{12}^{e_{23}}, \cdots, y_{12}^{e_{2r}}$ are linearly independent over $o'=\sum_{\substack{(i)\\(j_2\cdots j_p)}} k\ \varphi^{(i)}_{1j_2\cdots j_p}(P)$ and $e_{2j}+1$ are not divisible by the characteristic of the universal domain. Then by the same process as above we can express ω_i in the form

$$\omega_i=d\eta_{11}\ d\eta_{22}\ \omega_i^{**}+\omega_i^{(**)}$$

where η_{2j} are functions on V defined over k by $\eta_{2j}(P)=y_{2j}$ and ω_i^{**} are linearly independent. Continuing this process p-times we shall arrive at an expression of the form

$$\omega_i=d\eta_{p1}\ d\eta_{p2}\cdots d\eta_{pp}\ \varphi_i+\bar{\omega}_i$$

where φ_i are functions on V defined over k and linearly independent on V. From the above construction we see that y_{pj} $(j=\cdots 1,p)$ are contained in $k[x_1,\cdots,x_r]$; moreover if we put $y_{pj}=L_j(x)$, $L_j(X)$ has the form $X_j+G_j(X)$, where $G_j(X)$ are polynomials in $k[X_1,\cdots X_{j-1}]$. Henceforth we shall write η_j instead of η_{pj}.

Next we shall show that there exist an irreducible hypersurface section of V on which ω_i induce linearly independent differential forms $\bar{\omega}_i$. For this we shall divide into the three cases.

(I) The case $p\leq r-2$.

Let H_1 be a hypersurface in L^N defined by the equation of the form

$$H_1(X)\equiv v_0+v_{p+1}\cdot L_{p+1}(X)^{m_{p+1}}+\cdots+v_r\cdot L_r(X)^{m_r}=0.$$

where $v_{p+1},\cdots, v_r$ are independedt variables over $k(P)$ and v_0 is determined by

$$v_0=-\sum_{j=p+1}^{r} v_j\ L_j(x)^{m_j}$$

Then by Th. 2.4 of Matsusaka (3) we see easily that the intersection product $C_1 = H_1 \cdot V$ is irreducible for suitable choices of m_j $(j=p+1,\cdots, r)$ · (In the following we shall consider more two special hypersursurfaces. The irreducibility of the intersection product of V with these hypersurfaces can be seen by the same reasoning as this case and will not be mentioned explicitly.) We shall now show that $\bar{\eta}_1, \cdots \bar{\eta}_{r-1}$ are uniformizing parameters at P on C_1. For this purpose it is sufficient to show that the determinant

$$\begin{vmatrix} \partial L_j/\partial x_i, & j=1,\cdots, r-1 \\ \partial H_1/\partial x_i & \\ \partial F_s/\partial x_i, & s=1,\cdots, N-r \end{vmatrix} \qquad i=1,\cdots, N.$$

is not 0(cf. the Def. 1 of Koizumi (2)), where $F_s(X)$ are polynomials in the defining ideal $\mathfrak{p}$ of V such that $|\partial F_s/\partial x_i| \neq 0$ $(s=1,\cdots, N-r\,;\ i=r+1,\cdots, N)$. The existence of such polynomials is assured by the hypothesis that $\tau_1,\cdots,\tau_r$ are uniformizing parameters at P on V (cf. Prop. 5 of Nakai (4)). But it is clearly seen from the forms of $L_j(X)$ and $H_1(X)$. Then the induced differential forms $\bar{\omega}_i$ on C_1 can be written in the form

$$\bar{\omega}_i = \bar{\varphi}^{(i)}_{1\cdots p}\, d\bar{\eta}_1 \cdots d\bar{\eta}_p + *$$

But by Prop, 2, $\bar{\varphi}^{(i)}_{1\cdots p}$ are independent functions on C_1 if m_j are sufficiently large. Hence $\bar{\omega}_i$ are also linearly independent differential forms on C_1.

(II) The case $p=r-1 \geq 2$.

Take m_0 so large that on any irreducible hypersurface section of V of order $\geq m_0$, the induced functions $\bar{\varphi}^{(i)}_{1\cdots r-1}$ are linearly independent. In particular let H_2 be a hypersurface defined by the equation of the form

$$v_0 + v_r \cdot L_r(X) + \sum_{m \geq i_2 + \cdots + i_{r-1} > 0} v_{i_2 \cdots i_{r-1}} \cdot L_2(X)^{i_2} \cdots L_{r-1}(X)^{i_{r-1}} = 0$$

where $v_{i_2}, \cdots, v_{i_{r-1}}$, v_r are independent variables over $k(P)$ and v_0 is determined by the equation

$$v_0 = -v_r\, L_r(x) - \sum_{m \geq i_2 + \cdots + i_{r-1} > 0} v_{i_2 \cdots i_{r-1}} L_2(x)^{i_2} \cdots L_{r-1}(x)^{i_{r-1}}$$

and the sum is extended over all sets of indices such that $0 < i_2 + \cdots + i_{r-1} \leq m (m \geq m_0)$. Since $r \geq 3$ we see that $C_2 = V \cdot H_2$ is irreducible, and by the analogous reasoning as in the case (*I*) that $\bar{\eta}_1, \cdots, \bar{\eta}_{r-1}$ are uniformizing parameters at P on C_2. Then the induced differential forms have the form

$$\bar{\omega}_i = \bar{\varphi}^{(i)}_{1\ldots r-1}\ d\bar{\eta}_1 \cdots d\bar{\eta}_{r-1}$$

and $\bar{\varphi}^{(i)}_{1\ldots r-1}$ are linearly independent functions on $\boldsymbol{C}_2$, hence $\bar{\omega}_i$ are also independent on $\boldsymbol{C}_2$.

(III) The case $r=2$, $p=1$.

In this case we have

$$\omega_i = \varphi_1^{(i)}\ d\eta_1 + \varphi_2^{(i)}\ d\eta_2$$

and $\varphi_1^{(i)}\ (1 \leq i \leq s)$ are linearly independent function on $\boldsymbol{V}$. Let $\boldsymbol{H}_3$ be a hypersurface in $\boldsymbol{L}$ defined by the equation

$$v_0 + v_1\ X_1\ L_2(X)^{m-1} + v_2\ L_2(X)^m = 0.$$

where v_1 and v_2 are independent variables over $k(\boldsymbol{P})$ and v_0 is determined by

$$v_0 = -v_1\ x_1\ L_2(x)^{m-1} - v_2\ L_2(x)^m$$

and m is an integer not divisible by the characteristic of the universal domain. Then $\boldsymbol{C}_3 = \boldsymbol{V} \cdot \boldsymbol{H}_3$ is irreducible, and the induced differential forms have the form

$$\bar{\omega}_i = \left(\bar{\varphi}_1^{(i)} - \frac{v_1\ \bar{\eta}_2}{(m-1)v_1\bar{\eta}_1 + mv_2\bar{\eta}_2}\ \bar{\varphi}_2^{(i)}\right) d\bar{\eta}_1$$

Put

$$\psi_i = \{(m-1)v_1 \cdot \bar{\eta}_1 + mv_2 \cdot \bar{\eta}_2\}\bar{\varphi}_1^{(i)} - v_1\ \bar{\eta}_2\ \bar{\varphi}_2^{(i)}$$

Then they are functions on $\boldsymbol{V}$ defined over $k(v_1, v_2)$. We shall show that ψ_i are independent. Suppose that they are linearly dependent then there exist the quantities a_i in $k(v_1, v_2)$ such that

$$\sum_i a_i[\{(m-1) \cdot v_1 \cdot \bar{\eta}_1 + mv_2 \cdot \bar{\eta}_2\}\bar{\varphi}_1^{(i)} - v_1\ \bar{\eta}_2\ \bar{\varphi}_2^{(i)}] = 0$$

Since v_1 and v_2 are independent variables over $k(\boldsymbol{P})$ and $\boldsymbol{P}$ is a generic Point of $\boldsymbol{V}$ over $k(v_1, v_2)$ we can see immediately

$$\sum a_i \cdot mv_2\ \eta_2 \cdot \varphi_1^{(i)} = 0$$

i.e.

$$\sum a_i\ \varphi_1^{(i)} = 0$$

this is a contradiction. Moreover the degree of the linear system determined by the functions $\{\psi_i\}$ is bounded though the functions ψ_i varie as m varies. Hence if m is sufficiently large the induced functions $\bar{\psi}_i$ are linearly independent on $\boldsymbol{C}_3$ by Prop. 2. Hence the differential forms $\bar{\omega}_i$ are linearly independent on $\boldsymbol{C}_3$.

Now we see in any case if ω_i are linearly independent differential forms on $V^r (r \geq 2)$ of degree $p (1 \leq p \leq r-1)$, then there exist an irreducible hypersurface section C of V of order m where ω_i induce linearly independent differential forms $\bar{\omega}_i$. Then ω_i induce the linearly independent differential foams on a generic hypersurface section C_M of order M for all values of $M \geq m$, by prop. 4. Thus the theorem is completely proved. q.e.d.

We shall denote by $R_p(V)$ the number of linearly independent differential forms of the first kind of degree p on $\Delta^r (R_1(V)$ is the irregularity, and $R_r(V)$ is the geometric genus of V respectively). Then we have immediately the

Corolllary. The numbers $R_p(V)$ are bounded for any value of $p (1 \leq p \leq \dim. V)$.

Mathematical Institute
Kyoto University.

Bibliography.

(1) Igusa, J. On the Picard varieties attached to algebraic varieties, Amer. J. Math. Vol. LXXIV, 1952.

(2) Koizumi, S. On the differential forms of the first kind on algebraic varieties. J. Math. Soc. Japan, Vol. 2, 1951.

(3) Matsusaka, T. The theorem of Bertini on linear system in modular field. Mem. Coll. Sci. Univ. Kyoto, series A Vol XXVI, 1950.

(4) Nakai, Y. On the divisors of differential forms on algebraic varieties. forthcoming in J. Math. Soc. Japan.

(5) Weil, A. Foundations of algebraic geometry. Amer. Math. Soc. Colloq. publications.

(6) Weil, A. Arithmetic on algebraic varieties. Ann. of Math. Vol. 53, 1951.