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Note on the generator of $\pi_7(SO(n))$

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The 7th homotopy group $\pi_7(SO(n))$ of the group $SO(n)$ of the rotations in the euclidean n-space is determined by Serre [5] without details. Let

$$\sigma: S^7 \to SO(8) \text{ and } \rho: S^7 \to SO(7) \subset SO(8)$$

be mappings defined by the formulas

$$\sigma(x)(y) = xy \text{ and } \rho(x)(y) = xy\bar{x} \text{ for } x, y \in S^7,$$

where the multiplication in S^7 is that of the Cayley numbers.

Denote by

$$\sigma_n \in \pi_7(SO(n)),\ n \geqq 8 \text{ and } \rho_n \in \pi_7(SO(n)),\ n \geqq 7$$

the classes represented by σ and ρ respectively, regarding $SO(8)$ as a subgroup of $SO(n)$, $n \geqq 8$ in the natural sense. About the element ρ_7, we have the knowledge of the result [8]:

$$p_* \rho_7 \neq 0$$

under the (projection) homomorphism $p_*: \pi_7(SO(7)) \to \pi_7(S^6) \approx Z_2$. From this we can prove that "*ρ_7 is not divisible by 2*". Furthermore, we shall prove

Theorem. i) *$\pi_7(SO(7))$ is a free cyclic group generated by ρ_7.* ii) *$\pi_7(SO(n))$, $n \geqq 9$, is a free cyclic group generated by σ_n.*

As a corollary we have $\pi_7(SO(8)) \approx Z + Z = \{\sigma_8\} + \{\rho_8\}$.

The proof of the theorem is mainly devoted to the following simple lemma and results on $\pi_6(S^3)$.

$SO(7)$ is the set of all $\alpha \in SO(8)$ such that α fixes the unit. Spin(7) is the set of all $\tilde{\alpha} \in SO(8)$ such that for some $\alpha \in SO(7)$ the relation

$$\alpha(x)\tilde{\alpha}(y) = \tilde{\alpha}(xy)$$

holds for all $x, y \in S^7$. In virtue of "*the principle of triality*" [3] we have just two of such $\tilde{\alpha}$ ($\tilde{\alpha}$ and $-\tilde{\alpha}$) for each α. By setting $f(\tilde{\alpha})=f(-\tilde{\alpha})=\alpha$, we have a double covering

$$f: \mathrm{Spin}(7)\to SO(7).$$

The projection $p: SO(8)\to S^7$ defines fiberings $p_1: \mathrm{Spin}(7)\to S^7$ (fibre: G_2) and $p_2: \mathrm{Spin}(5)\to S^7$ (fibre: S^3). Define a mapping

$$t: S^7\to S^7$$

by the formula $t(x)=x^3$. Obviously t is a mapping of degree 3.

Lemma. *There exists a mapping* $\tilde{\rho}: S^7\to \mathrm{Spin}(7)$ *such that* $f\circ\tilde{\rho}=\rho$ *and* $p_1\circ\tilde{\rho}=t$, i.e., *the diagram*

$$\begin{tikzcd} S^7 \arrow[r, "\rho"] \arrow[d, "t"'] \arrow[dr, "\tilde{\rho}"] & SO(7) \\ S^7 & \mathrm{Spin}(7) \arrow[l, "p_1"] \arrow[u, "f"'] \end{tikzcd}$$

is commutative.

Proof. In fact, we set $\tilde{\rho}(x)(y)=xyx^2$, and we shall prove the equality $(\rho(x)(y))(\tilde{\rho}(x)(z))=\tilde{\rho}(x)(yz)$. First we have the following formulas

$$x(yz)x=(xy)(zx) \quad \text{and} \quad (y\bar{x})(xzx)=(yz)x$$

for $x,y,z\in S^7$. The first formula is proved in [3], the second follows easily from the first and Lemma 2 in [4]. Now

$$\begin{aligned}(\rho(x)(y))(\tilde{\rho}(x)(z)) &= (xy\bar{x})(xzx^2)=(xy\bar{x})((xzx)x)\\ &=x((y\bar{x})(xzx))x=x((yz)x)x=x(xyz)x^2\\ &=\tilde{\rho}(x)(yz).\end{aligned}$$

Therefore $\rho=f\circ\tilde{\rho}$. Obviously $(p_1\circ\tilde{\rho})(x)=\tilde{\rho}(x)(1)=x^3=t(x)$. Then the lemma is proved.

We proceed to the proof of the theorem. It was proved in [2] that the characteristic class $\alpha\in\pi_6(S^3)$ of the fibering $\mathrm{Spin}(5)/S^3=Sp(2)/Sp(1)=S^7$ is a generator of $\pi_6(S^3)\approx Z_{12}$ which is represented by Blakers-Massey essential mapping [1]

$$g: S^6\to S^3.$$

Then in the diagram

$$\begin{tikzcd}[column sep=small] & \pi_7(S^6) \arrow[dl, "E"'] \arrow[dr, "g_*"] & & & & \pi_6(S^6) \arrow[dl, "E"'] \arrow[dr, "g_*"] & \\ \pi_8(S^7) \arrow[rr, "\Delta"] & & \pi_7(S^3) \arrow[r, "i_*"] & \pi_7(\mathrm{Spin}(5)) \arrow[r, "p_{2*}"] & \pi_7(S^7) \arrow[rr, "\Delta"] & & \pi_6(S^3) \end{tikzcd}$$

the commutativity holds. Since the suspension homomorphism E: $\pi_6(S^6)\to\pi_7(S^7)$ is an isomorphism and since $g_*:\pi_6(S^6)\to\pi_6(S^3)$ is onto, we have that $\Delta:\pi_7(S^7)\to\pi_6(S^3)$ is onto and that kernel $\Delta=$ image $p_{2*}=12(\pi_7(S^7))$. The group $\pi_7(S^3)$ has order 2 and is generated by the image $g_*(\eta)=\alpha\circ\eta$ of the generator η of $\pi_7(S^6)$ [7, Appendix]. Since $E:\pi_7(S^6)\to\pi_8(S^7)$ is an isomorphism, $\Delta:\pi_8(S^7)\to\pi_7(S^3)$ is onto. Then kernel $p_{2*}=$ image $i_*=$ image $(i_*\circ\Delta)=0$. Consequently we have an isomorphism

$$p_{2*}:\pi_7(\mathrm{Spin}(5))\approx 12(\pi_7(S^7)).$$

From the exactness of the sequences

$$\pi_7(\mathrm{Spin}(5))\to\pi_7(\mathrm{Spin}(6))\to\pi_7(S^5)\approx Z_2,$$

$$\pi_7(\mathrm{Spin}(6))\to\pi_7(\mathrm{Spin}(7))\to\pi_7(S^6)\approx Z_2,$$

we have that the cokernel of the injection homomorphism i_*: $\pi_7(\mathrm{Spin}(5))\to\pi_7(\mathrm{Spin}(7))$ has at most four elements. The mapping $\tilde{\rho}$ represents $f_*^{-1}(\rho_7)\in\pi_7(\mathrm{Spin}(7))$. By the above lemma, $p_{1*}(f_*^{-1}(\rho_7))$ generates $3(\pi_7(S^7))$. From the commutativity of the diagram

$$\begin{tikzcd}
\pi_7(\mathrm{Spin}(5)) \arrow[r, "p_{2*}"] \arrow[d, "i_*"] & \pi_7(S^7) \\
\pi_7(\mathrm{Spin}(7)), \arrow[ur, "p_{1*}"'] &
\end{tikzcd}$$

we see that the cokernel of i_* is mapped by p_{1*} into $\pi_7(S^7)/12(\pi_7(S^7))\approx Z_{12}$ and that the image contains $3(\pi_7(S^7))/12(\pi_7(S^7))\approx Z_4$. Therefore the cokernel of i_* has to be isomorphic to Z_4 and p_{1*} maps $\pi_7(\mathrm{Spin}(7))$ isomorphically onto $3(\pi_7(S^7))$. This shows that $\pi_7(\mathrm{Spin}(7))$ is an infinite cyclic group generated by $f_*^{-1}(\rho_7)$, and then i) of the theorem is proved, by operating the covering isomorphism $f_*:\pi_7(\mathrm{Spin}(7))\to\pi_7(SO(7))$.

As is well known [6], $\pi_7(SO(8))=\{\sigma_8\}+i_*\pi_7(SO(7))=\{\sigma_8\}+\{\rho_8\}$. It is also known [6] that the injection homomorphism

$$i_*:\pi_7(SO(8))\to\pi_7(SO(9))$$

is onto and its kernel is generated by $2\sigma_8-\rho_8$. Therefore $i_*(\sigma_8)=\sigma_9$ generates $\pi_7(SO(9))\approx Z$. Since i_*: $\pi_7(SO(9))\approx\pi_7(SO(n))$ and $i_*\sigma_9=\sigma_n$ for $n\geqq 9$, σ_n generates $\pi_7(SO(n))\approx Z$. This completes the proof of the theorem.

Corollary. $\pi_7(SO(5))\approx\pi_7(SO(6))\approx Z$. *The cokernels of the injection homomorphisms* $i_*:\pi_7(SO(i))\to\pi_7(SO(i+1))$, $i=5$, 6, *are* *isomorphic to* Z_2.

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