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On the equivalence of conditions on a branching process in continuous time and on its offspring distribution

By

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§1. Introduction

Let $[X(t,\omega);\ t\geq 0]$ be a strong Markov continuous time one dimensional branching process defined on a probability triplet (Ω, F, P). Assume the paths to be right continuous and to have left limits. Let the associated infinitestimal generating function be

$$u(z)=a[h(z)-z] \tag{1}$$

where

$$h(z)=\sum_{i=0}^{\infty} p_i z^i,\ p_h\geq 0,\ \sum_{i=0}^{\infty} p_i=1 \text{ and } 0<a<\infty .$$

We assume henceforth that for every $\epsilon>0$

$$\int_{1-\epsilon}^{1} \frac{1}{u(z)}dz=\infty . \tag{2}$$

Under (2) the process does not explode in finite time (see Chapter 5 in [2]) and thus for any t, $X(t,\omega)$ is a bonafide random variable. Further one can now interpret the infinitestimal generating function as follows. Consider a system where we start with $X(0)$ particles at $t=0$, each particle lives an exponential length of time with mean

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a^{-1} and on death creates (or splits into) a random number of new particles whose generating function is $h(z)$ and all particles behave independently of each other and identically. We can regard $X(t, \omega)$ as the number of particles in the system at time t.

Because of the Markov property we assume without any loss of generality that

(3) $$p_1 = 0.$$

If we set

(4) $$f(t, z) = \int_\Omega z^{X(t,\omega)} dP(\omega)$$

then $f(t, z)$ satisfies for any $t, u \geq 0$

(5) $$f(t+u, z) = f(t, f(u, z)),$$

thus making the family $\{f(t, z);\ t \geq 0\}$ a semigroup.

We call $u(z)$ the *infinitestimal generating function* of the semigroup in (5).

The question we wish to answer is how does one translate conditions on $h(z)$ into conditions on the semigroup $f(t, z)$. For example, Harris (see Chapter 5 in [2]) showed that if

$$\left.\frac{d^r h(z)}{dz^r}\right|_{z=1} < \infty$$

then

(7) $$EX^r(t) = \left.\frac{\partial^r f(t, z)}{\partial z^r}\right|_{z=1-} < \infty$$

for every positive integer r and $t > 0$. His method (differential equations) would fail if r is not an integer.

In this paper we establish an equivalence principle between conditions on the semigroup $f(t, z)$ and the infinitesimal generator $u(z)$ (see Theorem 1 of the next section). As a special case we establish the following fact

(8) $$EX(t) \log X(t) < \infty \Longleftrightarrow \sum_{j=0}^{\infty} j \log j p_j < \infty.$$

The importance of this problem arises from the fact† that one can generalize a number of results on Galton-Watson process (discrete time) to a continuous time Markov branching process $[X(t,\omega);\ t\geq 0]$ by exploiting the result that for every $\delta>0$ $[X(n\delta,\omega);\ n=0,1,2,\cdots]$ is a Galton-Watson process. One checks that for every $\delta>0$ the Galton-Watson process $[X(n\delta,\omega);\ n=0,1\ 2,\cdots]$ satisfies certain conditions which are usually in terms of $X(\delta,\omega)$. But our initial data is only $u(z)$. So the question of equivalence of conditions on $u(z)$ and $f(\delta,z)$ becomes very relevant here.

The present author after completing this work had a chance to talk with Howard Conner who outlined a proof of (8) in [1] by a slightly different technique. Here we prove completely a more general result. Professor S. Karlin showed me a purely analytic argument to establish (8).

§2. Main Result

Under the set up in §1, we have the following.

Theorem 1. *Let $\phi(x)$ be a function from $R^+=[0,\infty)$ to R^+ satisfying*

(i) *$\phi(x)$ is nondecreasing and ≥ 1,*

(*) (ii) *$\phi(x)$ is convex,*

(iii) *$\phi(xy)\leq K\phi(x)\phi(y)$ for some K and all x, y (K independent of x and y).*

Then

$$E\phi(X(t))<\infty \text{ for any } t>0$$

$$\Longleftrightarrow \quad \sum_{j=0}^{\infty}\phi(j)p_j<\infty.$$

Remark. In view of Lemma 0 below, Theorem 1 is still valid when $\phi(x)$ satisfies the following more general conditions:

† This fact has been a part of oral tradition in branching processes for a long time. It was put into print recently by H. Conner [1].

$$(**)\quad\begin{cases}\text{(i)}\ \ \phi(x) \text{ maps } R^{+}=[0,\infty) \text{ to } R^{+}\\ \text{(ii)}\ \ \text{There exists a } c\geq 0 \text{ and } K>0 \text{ such that } \phi(x) \text{ is}\\ \qquad \text{convex on } [c,\infty) \text{ and } \phi(xy)\leq K\phi(x)\phi(y) \text{ for } x,\ y\\ \qquad \text{in } [c,\infty) \text{ (of course, } K \text{ is independent of } x \text{ any } y)\\ \text{(iii)}\ \ \phi(x) \text{ is bounded and measurable in } [0,c].\end{cases}$$

Lemma 0. *Let $\phi(x)$ be a function on $R^{+}=[0, \infty)$ to R^{+} satisfying (**). Then there exists a $\tilde{\phi}$ satisfying (*) and further for any non-negative measure μ on the Borel sets of R^{+} with $\mu[0, c]<\infty$ we have*

$$\int_{0}^{\infty}\phi(x)\mu(dx)<\infty \Longleftrightarrow \int_{0}^{\infty}\tilde{\phi}(x)\mu(dx)<\infty.$$

Proof. We need to consider only the case of unbounded ϕ. Since ϕ is convex on $[c, \infty)$, bounded in $[0, c)$ and unbounded above on $[0, \infty)$ there exists a $c'\geq c$ such that

(a) $\phi(c')\geq \sup_{x\leq c'}\phi(x)\geq 1$

(b) ϕ is increasing on $[c', \infty)$.

Now set $\tilde{\phi}(x)=\phi(c')$ for $x\leq c'$

$\qquad\qquad =\phi(x)$ for $x>c'$.

Direct verification now shows that this $\tilde{\phi}$ is the desired one. Q.E.D.

Before proving Theorem 1 we establish a few corollaries.

Corollary 1. *For any $t>0$ and any $r\geq 1$ (r, not necessarily an integer)*

$$\begin{aligned} &EX^{r}(t)<\infty \\ (10)\qquad &\Longleftrightarrow \sum_{j=0}^{\infty} j^{r}p_{j}<\infty. \end{aligned}$$

Corollary 2. (*Conner* [1]) *For any $t>0$, $\alpha\geq 1$,*

$$
(11) \qquad \begin{aligned} & EX^{\alpha}(t) \log X(t) < \infty \\ \Longleftrightarrow & \sum_{j=0}^{\infty} j^{\alpha} \log j p_j < \infty . \end{aligned}
$$

Of course, the proofs of these corollaries are trivial since $\phi = x^r$ $(r \geq 1)$ and $\phi = x^{\alpha} \log x (\alpha \geq 1)$ satisfy (**).

Our approach exploits the concept of split times of branching processes. Under the condition (2) one can construct the process in the following manner. Let ξ_i, $i = 1, 2, \cdots$ be a sequence of independently and identically distributed random variables taking non-negative integer values with probability generating function $h(z)$. Set

$$
(12) \qquad \begin{aligned} S_n &= n_0 + \xi_1 + \cdots + \xi_n - n \\ N &= \inf \{n \colon S_n = 0\} \\ &= \infty \text{ if there is no such } n. \end{aligned}
$$

Let T_i, $i = 1, 2, \cdots, N$ be a sequence of random variables defined as follows:

T_1 is exponentially distributed with mean $(a n_0)^{-1}$ and further independent of all the ξ_i's. Next T_2 is exponentially distributed with mean (aS_1) and independent of T_1 and the ξ_i for $i \geq 2$ and given ξ_1, conditionally independent of ξ_1. In general T_i for $i \leq N$ is exponentially distributed with mean $(aS_{i-1})^{-1}$ and given S_{i-1} independent of all the ξ_i's and T_j's for $j \leq i-1$. We assume (Ω, F, P) is the "big" probability space on which all these random variable are defined.

We now set

$$
(13) \qquad \begin{aligned} \tau_0(\omega) &= 0 & & \\ \tau_i(\omega) &= T_1 + T_2 + \cdots + T_i, & & \text{for } i \leq N \\ &= \infty, & & \text{for } i > N \\ X(t, \omega) &= n_0 & & \text{for } 0 \leq t < \tau_1 \\ &= S_i(\omega) & & \text{for } \tau_i \leq t < \tau_{i+1} \\ &= 0 & & \text{for } t > \tau_N . \end{aligned}
$$

This is the process with $X(0)=n_0$. We can regard τ_i as the instant when the i-th death or split occurs and accordingly we define τ_i to be the i-th split time. Further ξ_i's can be interpreted as the number of progeny created at the i-th split. For $i\geq N(\omega)$, these do not make sense since $X(t,\omega)=0$ for $t>\tau_N$ and the population is extinct.

Let for any n

$$\begin{aligned} X_n(t,\omega)\equiv X_n(t)=X(t)=X(t,\omega) \qquad & \text{if } \tau_n>t \\ =0 \qquad & \text{otherwise} \end{aligned} \tag{14}$$

It is immediate that

$$X_n(t)\leq n_0+\sum_{j=1}^{n}\xi_j. \tag{15}$$

Here is the plan of our proof. In Lemma 1 we show that for our ϕ, $E\phi(\xi_1)<\infty$ implies $E\phi(X_n(t))<\infty$. We use Lemmas 2, 3, and 4 to show $\sup E\phi(X_n(t))<\infty$ and then appeal to monotone convergence theorem to finish the proof. The converse part uses some martingale arguments.

Lemma 1. *Let* $E\phi(\xi_1)\equiv\sum_{j=0}^{\infty}\phi(j)p_j<\infty$ *where* ϕ *satisfies* (*). *Then* $E\phi(X_n(t))<\infty$ *for every integer* n *and* $t>0$.

Proof: From (15) and (*)

$$\begin{aligned}\phi(X_n(t)) &\leq\frac{1}{2}\phi(2n_0)+\frac{1}{2}\phi(2\sum_{j=1}^{n}\xi_j)\\ &\leq\frac{1}{2}\phi(2n_0)+\frac{1}{2}\frac{1}{n}\sum_{j=1}^{n}\phi(2n\xi_j)\\ &\leq\frac{1}{2}\phi(2n_0)+\frac{K}{2n}\phi(2n)\sum_{j=1}^{n}\phi(\xi_j).\end{aligned}$$

Taking expectations

$$E\phi(X_n(t))\leq\frac{1}{2}\phi(2n_0)+\frac{K}{2}\phi(2n)E\phi(\xi_1)<\infty. \qquad \text{Q.E.D.}$$

Lemma 2. *Let* $m_n(t)=E\phi(X_n(t))$. *Assume* ϕ *satisfies* (*) *and* $E\phi(\xi_1)<\infty$. *Further let* $X(0)\equiv 1$. *Then* $\{m_n(t)\}$ *satisfies*

$$m_{n+1}(t)\leq c_1e^{-at}+c_2\int_0^t m_n(t-u)e^{-au}du \tag{16}$$

where

$$c_1=\phi(1);\quad c_2=KaE\phi(\xi_1).$$

Proof: For $n\geq 1$

$$\begin{aligned}E\phi(X_{n+1}(t))&=E\{\phi(X_{n+1}(t));\ \tau_1>t\}+E\{\phi(X_{n+1}(t));\ \tau_1\leq t\}\\&=I_1+I_2,\quad \text{say.}\end{aligned}$$

Clearly,

$$I_1=\phi(1)e^{-at}=c_1e^{-at}.$$

Now

$$I_2=E\{\phi(X_{n+1}(t));\ \tau_1\leq t\}.$$

Now on $\{\tau_1\leq t\}$ we have

$$X_{n+1}(t)\leq\sum_{j=1}^{\xi_1}\widetilde{X}_n^{(j)}(t-\tau_1) \tag{17}$$

(right side is zero if ξ_1 is zero)

where $\widetilde{X}_n^{(j)}(u)$ are independent copies of $X_n(u)$ for $j=1,2,\cdots$. The quickest way to see this is to note that the n-th split after τ_1 happens after t implies that in each of the k lines of descent engendered by the k particles present at τ_1+0, the n-th split occurs after t and further the total population $X(t)$ is the sum of the populations in all the lines. Thus

$$I_2\leq\int_0^t[\phi(0)p_0+\sum_{k\neq 0}p_kE[\phi(\sum_{j=1}^{k}\widetilde{X}_n^{(j)}(t-u))]]ae^{-au}du \tag{18}$$

But since ϕ satisfies (*) we have for $k\geq 1$

$$\phi(\sum_{j=1}^{k}\widetilde{X}_n^{(j)}(t-u))\leq K\frac{1}{k}\phi(k)\sum_{j=1}^{k}\phi(\widetilde{X}_n^{(j)}(t-u))$$

and hence for $k\geq 1$

$$E\phi(\sum_{j=1}^{k}\widetilde{X}_n^{(j)}(t-u))\leq K\phi(k)m_n(t-u).$$

Now $E\phi(\xi_1)=\sum_{k\neq 0} p_k\phi(k)+\phi(0)p_0$ is finite and $m_n(t-u)\geq\phi(0)\geq 1$ for all n, t, and u. Without loss of generality K can be assumed large enough to make $K\phi(0)>1$ so that

$$I_2\leq c_2\int_0^t m_n(t-u)e^{-au}du. \qquad \text{Q.E.D.}$$

Lemma 3. *For any set of constants c_1, c_2 and c_3 all satisfying $0<c_i<\infty$, $i=1, 2, 3$, there exists a unique non-negative and bounded (in finite intervals) solution to the integral equation*

$$m(t)=c_1e^{-c_3t}+c_2\int_0^t m(t-u)e^{-c_3u}du. \tag{19}$$

Proof: Check $m(t)=c_1e^{-(c_3-c_2)t}$ satisfies (19). Uniquencess is standard and omitted.

Lemma 4. *Assume ϕ satisfies (*) and $E\phi(\xi_1)<\infty$. Then*

$$\sup_n m_n(t)\leq m(t)=c_1e^{-(c_3-c_2)t} \tag{20}$$

where $c_1=\phi(1)$, $c_2=KaE\phi(\xi_1)$, and $c_3=a$.

Proof: Clearly $m_1(t)\leq m(t)$. Now use induction. Q.E.D.

We have all ingredients to prove the "if" part of Theorem 1.

Assume $E\phi(\xi_1)<\infty$ and ϕ satisfies (*). By monotone convergence theorem

$$\lim_{n\to\infty} E\phi(X_n(t))=E\phi(X(t)) \tag{21}$$

since $X_n(t)\uparrow X(t)$ and ϕ is increasing.

Thus from (20) and (21) we get

$$E\phi(X(t))\leq m(t)<\infty.$$

We now prove converse. That is assuming $E\phi(X(t_0))<\infty$ for some $t_0>0$ we wish to show that $E\phi(\xi_1)<\infty$. Since $X(\tau_1)=\xi_1$ it is equivalent to showing $E\phi(X(\tau_1))<\infty$. If $p_0=0$ then $X(s,\omega)$ is an increasing function of s. In this case,

$$
\begin{aligned}
& E\phi(X(t))<\infty && \text{for } t>0 \\
& \Longrightarrow E[\phi(X(t))<\infty;\ \tau_1\leq t]<\infty && \\
& \Longrightarrow E[\phi(X(\tau_1))<\infty;\ \tau_1\leq t]<\infty && \text{since } \phi \text{ is increasing} \\
& \Longrightarrow E[\phi(X(\tau_1))]P\{\tau_1\leq t\}>\infty && \text{since } X(\tau_1) \text{ and } \tau_1 \text{ are independent} \\
& \Longrightarrow E\phi(X(\tau_1)<\infty && \text{since } P\{\tau_1\leq t\}>0.
\end{aligned}
$$

So we need to consider only the case $p_0>0$.

Lemma 5. *Let ϕ satisfy (*) and let $t_0>0$ be such that $E\phi(X(t_0,\omega))<\infty$. Then*

(i) *$E\phi(X(t,\omega)<\infty$ for all t,*

(ii) *$EX(t,\omega)<\infty$ for all t and equal to $e^{\lambda t}$ for some real λ,*

(iii) *$\{\phi[X(t,\omega)e^{-\lambda t}],\ \mathcal{F}_t;\ t\geq 0\}$ is a non-negative submartingale where $\mathcal{F}_t\equiv\sigma\{X(s,\omega);\ s\leq t\}$ is the σ-algebra generated by $X(s,\omega)$ for $s\leq t$.*

Proof: From the convexity of ϕ and the fact that $X(t+u,\omega)$ can be regarded as $\sum_{j=1}^{X(t,\omega)}\widetilde{X}_j(u,\omega)$ where $\widetilde{X}_j(u,\omega)$ for $j=1,2,\cdots,X(t,\omega)$ are independent copies of $X(u,\omega)$ we obtain (i). Also since for any unbounded ϕ satisfying (*) there is a constant $c>0$ such that $\phi(x)\geq cx$ for large x. Thus $E\phi(X(t_0,\omega))<\infty$ implies $EX(t_0,\omega)<\infty$ and hence $EX(t,\omega)<\infty$ for all t and finally $EX(t,\omega)=e^{\lambda t}$ for some real λ.

Thus we obtain the fact that the family $\{X(t,\omega)e^{-\lambda t};\ \mathcal{F}_t;\ t>0\}$ is a non-negative martingale. Since ϕ is non-negative and convex (iii) would follow if we show

$$E\phi[X(t,\omega)e^{-\lambda t}]<\infty.$$

But this is immediate from

$$\phi(X(t,\omega)e^{-\lambda t})\leq K\phi(X(t,\omega))\phi(e^{-\lambda t}). \qquad \text{Q.E.D.}$$

Now we finish the proof of Theorem 1.

Since τ_1 and ξ_1 are independent, for any $t>0$,

$$E\{\phi(X(\tau_1);\ \tau_1\leq t\}=E\phi(X(\tau_1)P\{\tau_1\leq t\},$$

both sides being finite or infinite at the same time. Further

$$P\{\tau_1\leq t\}=1-e^{-n_0 t}>0 \quad \text{for } t>0.$$

So it suffices to show that $E\{\phi(X(\tau_1));\ \tau_1\leq t\}<\infty$. But

$$\phi(X(\tau_1))\leq K\phi(X(\tau_1)e^{-\lambda\tau_1})\phi(e^{\lambda\tau_1})$$

and on $\{\tau_1\leq t\}$ since ϕ is nondecreasing

$$\phi(e^{\lambda\tau_1})\leq\max\{\phi(e^{\lambda t}),\ \phi(1)\}.$$

Also by Doob's optional sampling theorem,

$$\begin{aligned}&E\{\phi[X(\tau_1)e^{-\lambda\tau_1}];\ \tau_1\leq t\}\\&\quad\leq E\{\phi[X(t)e^{-\lambda t}];\ \tau_1\leq t\}\\&\quad\leq E\{\phi[X(t)e^{-\lambda t}]\}\qquad\text{since } \phi \text{ is } \geq 0,\end{aligned}$$

and that is finite by Lemma 5. Q.E.D.

§3. Age Dependent Processes

Our main result §2 extends to age dependent processes. The nature of the proof is slightly different. Of course, the converse part of the theorem needs a different proof since no martingale argument will be available. In the direct part of the theorem the same proof works if instead of Lemmas 2 and 3 one uses Lemmas 2′ and 3′ below. Also the construction of split times are different. Although we gave the construction in the Markov case, we shall only remark that one can use Harris's theory (see Chapter 6 in [2]) to construct them in the age dependent case. We shall prove only the converse part in detail. The direct part briefly is as follows.

Retaining the same definitions of $X_n(t)$, $m_n(t)$, c_1, c_2 etc. one quickly establishes

Lemma 2′. *Under* (*), *if* $G(t)$ *is the lifetime distribution* *function*

$$m_{n+1}(t) \leq c_2(1-G(t)) + c_2\int_0^t m_n(t-u)dG(u).$$

Next one gets from renewal theory the following.

Lemma 3′. *For any set of constants c_1, c_2 satisfying $0<c_i<\infty$ for $i=1,2$ there exists a unique non-negative (and bounded in finite intervals) solution of the "renewal" equation*

$$m(t) = c_1(1-G(t)) + c_2\int_0^t m(t-u)dG(u)$$

provided $G(0+)=0$.

The rest of the proof is the Markov case.

The proof of the converse uses the following.

Lemma 6. *Let* (i) *N be a non-negative integer valued random variable,* (ii) *δ_i for $i=1,2,\cdots$ be a sequence of mutually independent and independent of N and identically distributed random variables with $P\{\delta_i=1\}=p=1-P\{\delta_i=0\}$, $0<p\leq 1$,* (iii) *$R_N=\sum_{i=1}^{N}\delta_i$, if $N\geq 1$ and 0 otherwise* (iv) *ϕ be any increasing function with*

$$\lim_{x\to\infty}\phi(x)=\infty \quad \text{and} \quad \varliminf_{l}\frac{E\phi(B_l)}{\phi(l)}=c>0$$

where B_l is a Binomial random variable with parameters l and p. Then

$$E\phi(R_N)<\infty \Longleftrightarrow E\phi(N)<\infty.$$

Proof: Since $R_N\leq N$ and ϕ is increasing

$$E\phi(N)<\infty \Longrightarrow E\phi(R_N)<\infty.$$

To prove the converse we observe that

$$E\phi(R_N)=\sum_{l=1}^{\infty}E\phi(B_l)P\{N=l\}+\phi(0)P(N=0)$$

$$=\sum_{l=1}^{\infty}\frac{E\phi(B_l)}{\phi(l)}\phi(l)P\{N=l\}+\phi(0)P(N=0)$$

But

$$\underline{\lim}_{l}\frac{E\phi(B_l)}{\phi(l)}=c>0.$$

Thus

$$E\phi(R_N)<\infty\Longrightarrow \text{ for some } 0<c'<c \text{ and } l_0$$
$$\sum_{l=l_0}^{\infty}\phi(l)P\{N=l\}\leq\frac{E\phi(R_N)}{c'}<\infty$$
$$\Longrightarrow E\phi(N)<\infty.$$

Q.E.D.

We need one more lemma.

Lemma 7. *Let ϕ satisfy* (*), $\phi(x)>0$ *for* $x>0$ *and* $\lim_{x\to\infty}\phi(x)=\infty$. *Then*

$$\underline{\lim}_{l}\frac{E\phi(B_l)}{\phi(l)}=c>0.$$

Proof: Let $\psi=1/\phi$. Then $\psi(x)$ is a bounded continuous and non-negative function on $[1,\infty)$. Now

$$\phi(l)\leq K\phi\left(\frac{l}{B_l}\right)\phi(B_l)$$
$$\Longrightarrow\frac{\phi(B_l)}{\phi(l)}\geq\frac{1}{K}\psi\left(\frac{l}{B_l}\right)$$
$$\Longrightarrow\frac{E\phi(B_l)}{\phi(l)}\geq\frac{1}{K}E\psi\left(\frac{l}{B_l}\right).$$

Since ψ is a bounded function and l/B_l converges w.p. 1 to $1/p$, we get

$$\underline{\lim}_{l}\frac{E\phi(B_l)}{\phi(l)}\geq\frac{1}{K}\psi\left(\frac{1}{p}\right)>0.$$

Q.E.D.

Now we prove the converse part of the main result, namely, that $E\phi(X(t))<\infty$ for some $t>0$ implies $E\phi(\xi_1)<\infty$ where ξ_1 is the number of particles created at the first split.

On the set $\tau_1\leq t$

$$X(t)=\sum_{j=1}^{\xi_1}\widetilde{X}_j(t-\tau_1)$$

where $\{\widetilde{X}_j(s);\ s\geq 0\}$ are independent copies of $\{X(t);\ t\geq 0\}$.

But

$$E\phi(X(t))=\phi(1)(1-G(t))+\int_0^t[E\{\phi(\sum_{j=1}^{\xi_1}\widetilde{X}_j(t-u));\ \xi_1\geq 1\}$$
$$+\phi(0)p_0]dG(u)$$

Thus there exists a u_0 in $[0,t]$ such that

$$E\{\phi(\sum_{j=1}^{\xi_1}\widetilde{X}_j(t-u_0);\ \xi_1\geq 1\}<\infty.$$

Let

$$\begin{aligned}\delta_j&=1 && \widetilde{X}_j(t-u_0)\geq 1\\ &=0 && \text{otherwise.}\end{aligned}$$

Then

$$X(t)\geq\sum_{j=1}^{\xi_1}\delta_j.$$

Since process is not degenerate, we have $0<P\{\delta_j=1\}\leq 1$. Using Lemmas 6 and 7 we see that $E\phi(X(t))<\infty\Longrightarrow E\phi(\xi_1)<\infty$. Q.E.D.

§4. Concluding Remarks

The arguments given here can be extended to the multitype case and the results have been deferred to a future publication.

References

[1] Conner, Howard, "A note on limit theorems for Markov branching processes", Proc. Amer. Math. Soc., **18**, (1967), 76-86.

[2] Harris, T. E., "The Theory of Branching Processes", Springer-Verlag, Berlin, 1963.

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