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Notes on classification of Riemann surfaces

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1. Introduction.

To classify Riemann surfaces (simply, "surfaces") by Hardy classes seems to have long been an open question. Recently Heins solved this problem thoroughly in his Springer lecture note [2, pp. 34–51]. The objective of the present article is to show that surfaces R of class O_{H_p} $(0<p<\infty)$ or of class O_{AB} or of class O_{LA} [2, p. 35] are characterized by a certain topological property of analytic functions on R, where O_{H_p} denotes the totality of surfaces R on which Hardy class $H_p(R)$ contains only constant members. The reader should know what is meant by O_{AB}, O_{AD} and O_{HD} [1, pp. 200 and 198].

A complex-valued harmonic function f on a surface R is said to be open if $w=f(P)$, $P\in R$, carries open subsets of R to those of the w-plane. Given a surface R, we denote by $\mathscr{L}(R)$, $\mathscr{H}_p(R)\,(0<p<\infty)$, $\mathscr{B}(R)$ and $\mathscr{D}(R)$ the classes of open harmonic functions f on R such that $\log^+|f|$ has a harmonic majorant on R, $|f|^p$ has a harmonic majorant on R, f is bounded on R and f has finite Dirichlet integral on R, respectively. We denote by $O_X(X=\mathscr{L}, \mathscr{H}_p, \mathscr{B}, \mathscr{D})$ the class of surfaces R on which $X(R)$ is *empty*. Then we have

(1) $$O_{H_p}=O_{\mathscr{H}_p} \quad \text{for } 0<p<\infty.$$

(2) $$O_{AB}=O_{\mathscr{B}} \quad \text{and} \quad O_{LA}=O_{\mathscr{L}}.$$

(3) $$O_{AB} < O_{\mathscr{D}} \subset O_{AD},$$

where $<$ denotes the strict inclusion.

2. Proofs.

According to Yamaguchi [3, Theorem [I]], for any open harmonic function $f=u+iv$ on $R \in O_{AB}$ there exists a single-valued conjugate u^* of $u \equiv \mathrm{Re}\, f$ on R such that $\mathrm{Im}\, f \equiv v = \alpha u + \beta u^* + \gamma$, where α, β and γ are real constants and $\beta \neq 0$. Thus we have

(4) $$f = \beta g + bu + c \quad \text{on} \quad R \in O_{AB},$$

where $g = u + iu^*$, $b = 1 - \beta + i\alpha$ and $c = i\gamma$.

We first prove (1). The inclusion $O_{H_p} \supset O_{\mathscr{H}_p}$ is trivial since non-constant analytic function is an open map. Assume that there exists $f = u + iv \in \mathscr{H}_p(R)$ for some $R \in O_{H_p} \subset O_{AB}$. Then $|u|^p (\leqq |f|^p)$ admits a harmonic majorant on R. By (4) combined with $\beta \neq 0$ and by the well-known inequality [2, p. 10]: $(A+B)^p \leqq 2^p(A^p + B^p)$ for A, $B \geqq 0$, $0 < p < \infty$, we have $g \in H_p(R)$, so that g is a complex constant on $R \in O_{H_p}$. Therefore (4) shows that f is not open; this is a contradiction.

The proof of (2) is analogous to that of (1). For the proof of $O_{LA} = O_{\mathscr{L}}$ we use (4) and the inequality: $\log^+(A+B) \leqq \log^+ A + \log^+ B + \log 2$ for A, $B \geqq 0$.

To prove (3) we recall Tôki's theorem that there is no inclusion relation between O_{AB} and O_{HD} [1, p. 264]. Assume now that there exists $f = u + iv \in \mathscr{D}(R)$ for some $R \in O_{AB} \subset O_{AD}$. Then, in (4), $g \in AD(R)$ and hence g must be a constant; this contradicts openness of f. Thus we have $O_{AB} \subset O_{\mathscr{D}}$. Assume next that $O_{AB} = O_{\mathscr{D}}$. Then by $O_{HD} \subset O_{\mathscr{D}}$ we have $O_{HD} \subset O_{AB}$; a contradiction to Tôki's theorem.

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References

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[3] H. Yamaguchi, *Holomorphic functions and open harmonic mappings*, J. Math. Kyoto Univ. **9** (1969), 381-391.