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Riemann surfaces obtained by conformal sewings

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Introduction. For Riemann surfaces of *infinite genus*, it is known that there occur many phenomena which are completely different from the function theory on plane regions or Riemann surfaces of finite genus. Actually we see them in various examples in classification theory of Riemann surfaces. Undoubtedly such phenomena depend on, in intuitive sense, the distributions of holes and handles representing genus. Now we take a countable number of disjoint cycles $\{\gamma_n\}$ on Riemann surface R so that $G=R-\cup\gamma_n$ becomes a planar region. G is conformally equivalent to a slit region. So, in this note, we consider plane region R which has an infinite number of disjoint slits $\{\gamma_n\}$ clustering nowhere in R. $G=R-\cup\gamma_n$ is a subregion of R whose boundary consists of $\cup\gamma_n$ and the ideal boundary of R. We construct Riemann surfaces by conformal sewings (cf. sec. 1) of G and investigate some relations by *extremal length methods* between the classes of such surfaces and the types (weakness, semiweakness and so on) of the slit regions. Furthermore we give some examples related these topics. The last one will give a relevant remark for the extension of classical Koebe's theorem to open Riemann surfaces.

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1. Let R be an open Riemann surface. By a *slit* in R, we shall mean an analytic arc which is homeomorphic to the closed interval $[-1,1]$ on the real axis in the complex plane. Consider a countable number of slits $\{\gamma_n\}_{n=0}^{\infty}$ in R which are mutually disjoint and cluster nowhere in R. Denote $G=R-\cup_n \gamma_n$. Now we define conformal sewings of G. (1) Take a parametric disk U_j around each γ_j so that γ_j is represented as $\{z_j; |\mathrm{Re}\, z_j|\leqq 1, \mathrm{Im}\, z_j=0\}$ by the local parameter z_j, then each γ_j has two sides, the upper edge $\gamma_j{}^+$ and the lower one $\gamma_j{}^-$. (2) Partition $\{\gamma_n\}_{n=0}^{\infty}$ into a countable number of finite sections $(\gamma_i)_{i\in I_k}$ $(k=0,1,\cdots)$. (3) For each arrangement of elements in every section $(\gamma_i)_{i\in I_k}$, say $(\gamma_{i_1},\gamma_{i_2},\cdots,\gamma_{i_{n(k)}})$, identify $\gamma_j{}^-$ with γ_{j+1}^+ $(j=i_1,i_2,\cdots,i_{n(k)-1})$ and $\gamma_{i_{n(k)}}^-$ with $\gamma_{i_1}^+$. Through these processes we can define a Riemann surface. Above all, the two points of the surface corresponding to the end points of slits have actually the following structure. For instance, the end point $\{z_{i_1}=1\}=\{z_{i_2}=1\}=\cdots=\{z_{i_{n(k)}}=1\}$ has a neighbourhood which is represented by n full circular disks in the complex plane. They are identified in the manner of an n-sheeted covering surface with a branch point, and we obtain a local parameter by taking an n-th root. We denote the resulting surface by $S(G)$ and call such a series of operations a *conformal sewing* of $G=R-\cup_n \gamma_n$. When we use the word 'any' (resp. 'some') conformal sewing, we mean conformal sewing for any (resp. some) (1) parametrization, (2) partition and (3) arrangement for $\{\gamma_n\}$. Especially, if each section consists of only one γ_k, then $S(G)=R$. More generally we map $U_j-\gamma_j$ conformally onto $U'_j-\gamma'_j$, where U'_j is a neighbourhood of $\gamma'_j=\{z'_j; |\mathrm{Re}\, z'_j|\leqq 1, \mathrm{Im}\, z'_j=0\}$. For such $\{\gamma'_j\}$ we can do the processes stated above to obtain a Riemann surface. We say such an operation a *general conformal sewing.*

Proposition 1. *Suppose that for each γ_n there exists a pair of parametric neighbourhoods U_n, V_n such that*

(a) $\gamma_n\subset U_n\subset \bar{U}_n\subset V_n,\ \bar{V}_m\cap\bar{V}_n=\phi\ (m\neq n)$

(b) $V_n-\bar{U}_n$ *is conformally equivalent to an annulus* A_n

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{\log \mathrm{mod}\, A_n}<\infty,$

where $\mathrm{mod}\, A_n$ *denotes the modulus of* $A_n(>1)$.

Then, $R\in O_G$ *if* $S(G)\in O_G$ *for some conformal sewing of* G. *Conversely, if* $R\in O_G$ *then* $S(G)\in O_G$ *for any conformal sewing of* G.

Moreover this proposition is also valid for O_{HB}, O_{HD}, U_{HB} and U_{HD} in place of O_G.

Proof. Let f_n be a bounded continuous function on R such that $f_n|_{R-V_n}=0$, $f_n|_{\bar{U}_n}=1$ and f_n is harmonic on $V_n-\bar{U}_n$. Set $f=\sum_{n=0}^{\infty}f_n$. Then by (c) we can easily find that f is a bounded continuous Dirichlet potential on R. While, we can regard f_n and consequently f as bounded continuous Dirichlet potentials on $S(G)$. If $R\notin O_G$ then there exists a non-constant bounded positive harmonic function on G vanishing continuously along the relative boundary ∂G with respect to $S(G)$. Hence $S(G)\notin O_G$.

We can prove similary the converse, and also the last statement.

2. Let R be an open Riemann surface with ideal boundary β and let R_0 be a parametric disk with compact relative boundary ∂R_0 in R. We consider the family of curves $F=F(R-\bar{R}_0, \partial R_0)$ in $R-\bar{R}_0$ such that each $c\in F$ consists of a finite number of disjoint Jordan closed curves and c separates the ideal boundary β from ∂R_0, i.e. c is homologous to ∂R_0. Let $\lambda(F)$ be the extremal length of F. Then we have the well known fact;

Lemma 1. *$R\in O_G$ if and only if $\lambda(F)=0$.*

Let $F'=F'(R-\bar{R}_0, \partial R_0)$ be the family of curves $c\in F$ such that each component of c is a dividing cycle in R. According to Kusunoki [6],

Definition. *We say that R belongs to class O' if $\lambda(F)=0$.*

That $\lambda(F')=0$ is an ideal boundary condition and is independent of the choice of R_0. (Such a remark will be omitted hereafter.)

Clearly $O'\subset O_G$. Moreover, if R belongs to O', then $R_S{}^*$ the Stoïlow compactification of R coincides with $R_M{}^*$ the Martin compactification of R (cf. [2], [6]).

3. From now on, we consider an open *planar* Riemann surface $R=\bar{C}-\beta$, where $\bar{C}, \beta$ stand for respectively the extended complex plane, a compact subset of $\bar{C}$ such that $\bar{C}-\beta$ is connected. Let $\{\gamma_n\}$ denote a sequence of slits as in sec. 1. Of course $\overline{\cup_n \gamma_n} - \cup_n \gamma_n \subset \beta$ (where closure is taken in $\bar{C}$). Let $G = R - \cup_n \gamma_n$, then G itself is a planar Riemann surface and β is a closed subset of the boundary of G. So, we can define the notion of weakness, semiweakness and parabolicity of β with respect to G. Here, we define them as follows. Let G_0 be a parametric disk in G. We introduce here two families F^0, F^1 of curves in $G-\overline{G}_0$. A curve $c \in F^1$ if and only if c consists of a finite number of Jordan closed curves in $G-\overline{G}_0$ such that c separates β from ∂G_0. While, $c \in F^0$ if and only if (i) c consists of a finite number of disjoint Jordan curves such that each component of c is closed or connects some γ_n with γ_m (may be $m=n$) and (ii) c separates β from ∂G_0 in $G-\overline{G}_0$. Let $\lambda(F^1)$ (resp. $\lambda(F^0)$) be the extremal length of F^1 (resp. F^0) with respect to $G-\overline{G}_0$.

Definition. *β is called weak (with respect to G) if $\lambda(F^1)=0$, and is said to be semiweak (with respect to G) if $\lambda(F^0)=0$.*

Definition. *β is called parabolic (with respect to G) if* $(G, \partial G) \in SO_{HB}$, i.e. there exist no HB-functions on G vanishing continuously along ∂G except constant zero.

We sometimes say that G is of *weak type* (resp. *semiweak type*, *parabolic type*), when β is weak (resp. semiweak, parabolic).

Definition. *We say that G belongs to class O_1 (resp. O_2) if $S(G) \in O_G$ for any (resp. some) conformal sewing.*

It is clear that $O_1 \subset O_2$. By Proposition 1, there exists non-trivial $G \in O_1$.

4. Let G be as in sec. 3, and let $\widehat{G}$ be the *double* of G along $\partial G = \cup_n \gamma_n$. Then there is a natural indirect conformal mapping φ of $\widehat{G}$ onto itself. For a subset E of G, we use the symbols $\widetilde{E}, \widehat{E}$ in place of $\varphi(E), E \cup \varphi(E)$ respectively.

Proposition 2. *G is of semiweak type if and only if $\widehat{G} \in O_G$.*

Proof. Let $F=F(\widehat{G}-\widehat{\overline{G}}_0, \partial\widehat{G}_0)$ be the family of curves in Lemma 1. For any $c\in F, c|_G$ the restriction of c onto G belongs to F^0. By $F|_G$, we mean the total of $c|_G$ such that $c\in F$. Then $\lambda(F^0)\leqq\lambda(F|_G)$ in $G-\overline{G}_0$. Clearly $\lambda(F|_G)\leqq\lambda(F)$ in $\widehat{G}-\widehat{\overline{G}}_0$. Since each $c\in F|_G$ is contained in $G-\overline{G}_0, \lambda(F^0)\leqq\lambda(F)$. So, if $\widehat{G}\in O_G$ then G is of semiweak type.

Conversely, let $\widehat{G}\notin O_G$. Then there exists the harmonic measure $\omega=\omega(\widehat{G}, \partial\widehat{G}_0)$ ($\not\equiv 0$) of the ideal boundary of $\widehat{G}$ with respect to $\widehat{G}-\widehat{\overline{G}}_0$. We take $ds=|\text{grad }\omega|\ |dz|$ as a linear density on $\widehat{G}-\widehat{\overline{G}}_0$, and set $ds_G=ds|_G$, which is a linear density on $G-\overline{G}_0$. For any $c\in F^0, \hat{c}=c\cup\tilde{c}\in F$. So, by means of the symmetricity of ω,

$$\int_c ds_G=\int_{\hat{c}} ds/2=\int_{\hat{c}}|\text{grad }\omega|\ |dz|/2\geqq\int_{\hat{c}}|{}^*d\omega|/2\geqq D(\omega)/2$$

where $D(\omega)$ means the Dirichlet integral of ω. Hence, $\lambda(F^0)\geqq D(\omega)/2>0$. Thus G is not of semiweak type.

If $\widehat{G}\in O_G$, then G belongs to SO_{HB}. Hence,

Corollary. *β is parabolic if β is semiweak.*

Proposition 3. *If β is weak, then every symmetric HB-function u defined near the ideal boundary of $\widehat{G}$ has a limit value at each Stoïlow ideal boundary point, where by the symmetricity of u we mean that $u=u\circ\varphi$.*

Proof. Take $\widehat{G}_0$ as in Proposition 2 so that u is defined on $\widehat{G}-\widehat{G}_0$. Since $\widehat{G}$ belongs to O_G, u is Dirichlet finite and $|\text{grad }u|\ |dz|$ becomes a linear density on $\widehat{G}-\widehat{\overline{G}}_0$. Let a, b be cluster values of u at a Stoïlow ideal boundary point p. For each $c\in F^1, c\cup\tilde{c}$ surrounds a symmetric ideal boundary neighbourhood. It is not difficult to see that there exists a determining sequence $\{V_n\}$ (symmetric with respect to φ) for p such that

$$\int_{c_n}|\text{grad }u|\ |dz|\to 0\quad(n\to\infty),$$

where we denote ∂V_n by c_n. Then by maximum principle,

$$|a-b|\leqq\max_{c_n} u-\min_{c_n} u\leqq\int_{c_n}|du|\leqq\int_{c_n}|\text{grad }u|\ |dz|.$$

Hence, $a=b$.

5. We consider conformal sewings of plane regions which are symmetric with respect to the real axis. More precisely, we assume that β is a compact set on the real axis and $\{\gamma_{2n}\}$ $(n=0, 1, \cdots)$ are contained in the upper half plane and $\gamma_{2n+1}=\{z\,;\bar{z}\in\gamma_{2n}\}$. Furthermore, in this section, $S(G)$ is the Riemann surface obtained by identifying γ_{2n} with γ_{2n+1} symmetrically with respect to the real axis. Then,

Proposition 4. *G is of weak type if and only if $S(G)$ belongs to class O'.*

Proof. The correspondence $z\rightsquigarrow\bar{z}$ induces a natural indirect conformal mapping φ (resp. ψ) of $S(G)$ (resp. $R=\boldsymbol{C}-\beta$) onto itself. $\psi(\gamma_{2n})=\gamma_{2n+1}$. Let G_0 be a symmetric (w.r.t. the real axis) neighbourhood around ∞ such that $G_0\cap(\cup_n\gamma_n)=\phi$, and let F^1 be the family of curves in $S(G)-\overline{G}_0$ defined in sec. 3 for weakness. F' denotes $F'(S(G)-\overline{G}_0, \partial G_0)$ in sec. 2 (i.e. $S(G)\in O'$ iff $\lambda(F')=0$) and $F''=\{c\in F'\,; c=\varphi(c)\}$. First we prove that $\lambda(F'')=0$ if and only if $\lambda(F')=0$. In fact, $\lambda(F'')=0$ implies $\lambda(F')=0$ since $F''\subset F'$. Conversely, let $\lambda(F')=0$. For any linear density ρ and $\varepsilon>0$, there exists a $c\in F'$ such that $\int_c(\rho+\rho\circ\varphi)|dz|=\int_{\varphi(c)}(\rho+\rho\circ\varphi)|dz|<\varepsilon$. Thus, $\int_{c\cup\varphi(c)}\rho|dz|\leqq\int_{c\cup\varphi(c)}(\rho+\rho\circ\varphi)|dz|<2\varepsilon$. Here, $c\cup\varphi(c)\in F''$. Hence, $\lambda(F'')=0$. Now we can prove semilarly that $\lambda(F^1)=0$ if and only if $\lambda(F^2)=0$, where $F^2=\{c\in F^1; c=\psi(c)\}$.

Clearly $F^2\subset F''$. So, if $\lambda(F^1)=0$, then $\lambda(F')=0$. Thus, $S(G)\in O'$ if G is of weak type.

Conversely, any $c\in F''$ does not meet $\cup_n\gamma_n$, since each component of c is a dividing cycle in $S(G)$. Thus $F''\subset F^2$. Hence, the converse statement follows.

In the same manner, for this $S(G)$ we have,

Proposition 5. *$S(G)\in O_G$ if and only if G is of semiweak type.*

It is easy to verify that if G is of weak type then G belongs to class O_2, and if G belongs to O_2 then G is of semiweak type.

Thus, we get the following schema of inclusion relations:

$$
\begin{tikzcd}
O_1 \arrow[dr, shift left] & & & & \\
& O_2 \arrow[ul, shift left, "/" marking] \arrow[dl, shift left, "/" marking] \arrow[r] & \text{semiweak} \arrow[r, shift left] & \widehat{G}\in O_G \arrow[l, shift left] \arrow[r, shift left] & \text{parabolic} \arrow[l, shift left, "/" marking] \\
\text{weak} \arrow[uu, "/" marking] \arrow[ur, shift left] & & & &
\end{tikzcd}
$$

Remark. When we define classes O_1 and O_2 for generel conformal sewings (cf. sec. 1) in place of conformal sewings, then we get besides in the schema, '$O_2 \rightleftarrows$ semiweak' and that if β consists of a finite number of components then '$O_1 \rightleftarrows$ weak'.

6. Examples.

In this section we give some examples. The examples (I) and (II) show the implications 'weak $\not\rightarrow$ O_1' and 'parabolic $\not\rightarrow$ semiweak' respectively. The example (III) shows '$O_2 \not\rightarrow O_1$' and also '$O_2 \not\rightarrow$ weak'. Besides them, these examples have other interesting properties.

(I) Let $R=\boldsymbol{C}$ (i.e. $\beta=\{\infty\}$) and $\{\gamma_n\}$ be given so that $\overline{\cup_n \gamma_n} - \cup_n \gamma_n \subset \beta$. Set $G=\boldsymbol{C}-\cup_n \gamma_n$, then clearly $\widehat{G}\in O_G$. Thus G is of semiweak type. (Generally this is true for any $\{\gamma_n\}$, if the logarithmic capacity of β with respect to $\bar{\boldsymbol{C}}$ is zero.) While, we insist that there exist some choice of $\{\gamma_n\}$ and some conformal sewing such that $S(G)$ belongs to a *hyperbolic* Riemann surface.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of strictly increasing positive numbers toward infinity. Set $\gamma_n=\{z=x+iy;\,|x|\leqq 1, y=a_n\}$ $(n=0,1,\cdots)$. Identify γ_{2n}^{+} with γ_{2n+1}^{-} and γ_{2n}^{-} with γ_{2n+1}^{+} symmetrically with respect to the line $y=(a_{2n}+a_{2n+1})/2$ so that we obtain Riemann surface $S(G)$.

Suppose $\sum_{n=1}^{\infty}(a_{2n}-a_{2n-1})=M<\infty$, then $S(G)\notin O_G$.

To see this, first we note that $G'=\sum_{n=1}^{\infty} G_n$ can be regarded as a subregion of $S(G)$ with $G_n=\{z=x+iy;\,|x|<1, a_{2n-1}<y\leqq a_{2n}\}$. It is clear that G' is conformally equivalent to the plane region Ω:

$$\Omega=\{z=x+iy;\,|x|<1, 0<y<M\}.$$

Under this conformal mapping, $\partial G'$ corresponds to $\{z=x+iy;\,x=\pm 1, 0<y<M\}\cup\{z;\,|x|\leqq 1, y=0\}$. Since $(G',\partial G')\notin SO_{HB}$, $S(G)$ is hyperbolic. Furthermore, we can easily show by definitions that G is of *weak* type provided that $a_{2n+1}/a_{2n}\geqq n\,(n=1,2,\cdots)$.

(II) Generally we can prove that slit region G is of parabolic type if there exists a conformal sewing such that the resulting surface $S(G)$ belongs to $O_{HB}-O_G$. Here we take radial slit disk $G=\boldsymbol{F}$ and Riemann surface $S(G)=\hat{\boldsymbol{F}}\in O_{HB}-O_G$ by some conformal sewing of G (cf. Tôki [10]). Then $G=\boldsymbol{F}$ is of parabolic type. While, G is not of semiweak type. For, $-\log|\pi(p)|$ is a non-constant positive superharmonic function on $\widehat{G}$, where π denotes the projection of $\widehat{G}$ onto the unit disk.

This example gives also a counter example showing that *subregions of a region of type NO_{HB}* (i.e. on which there exist no non-constant HB-functions whose normal derivatives vanish along the relative boundary) *are not always of type NO_{HB}*. (Note that every subregion of a region of type SO_{HB} is always of type SO_{HB}.) Now, for sufficiently small positive number r, $G_0=\{|z|<r\}$ becomes a parametric disk of $S(G)$ and G. Let $S(G)-\overline{G}_0=G'$, then the double $\widehat{G}'$ along $\partial G'=\partial G_0$ belongs to $O^2_{HB}-O_{HB}$. Thus $(G',\partial G')$ is of type NO_{HB}. While, $G-\overline{G}_0$ a subregion of G' is not of type NO_{HB}. For, the double of a subregion belongs to O_{HB} if and only if it is simultaniously of type NO_{HB} and SO_{HB} (cf. Kusunoki-Mori [7]). And clearly the double of $G-\overline{G}_0$ along $\partial(G-\overline{G}_0)$ does not belong to O_{HB}.

(III) Let $\{a_n\}_{n=0}^{\infty}$ be a strictly decreasing sequence of positive numbers converging to zero for $n\to\infty$. Let,

$$\beta=\{z=x+iy\,;\,|x|\leqq 1,\, y=0\},$$

$$\gamma_n=\{z\,;\,|x|\leqq 1,\, y=a_n\} \text{ and } \gamma_{-n}=\{z\,;\,|x|\leqq 1,\, y=-a_n\} \quad (n=0,1,\cdots).$$

Identify γ_n with γ_{-n} symmetrically with respect to the real axis to obtain $S(G)$. Clearly β is not weak. To prove the semiweakness of β, consider annuli A_n and A_n' in $S(G)$ such that

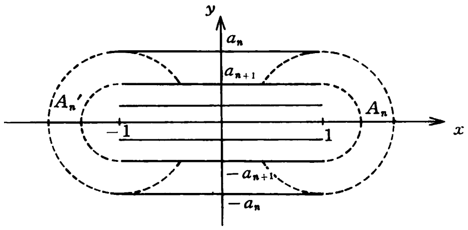
$$\begin{aligned} A_n=&\{z\,;a_{n+1}<|z-1|<a_n\}\\ &-\{z=x+iy\,;\,1-a_n\leqq x\leqq 1,\ |y|<a_{n+1}\},\\ A_n'=&\{z=x+iy\,;\,-x+iy\in A_n\}. \end{aligned}$$

F_n, F_n' and F_n'' are families of curves such that

$$F_n=\{c\,;\,c=A_n\cap\{|z-1|=r\},\, a_{n+1}<r<a_n\},$$

$F_n' = \{c'\,; c' = A_n' \cap \{|z+1| = r\}, a_{n+1} < r < a_n\}$,

$F_n'' = \{c \cup c'\,; c \in F_n, c' \in F_n'\}$.



Let G_0 bea neighbourhood of ∞ such that $G_0 \cap (\cup_{n=-\infty}^{\infty} \gamma_n) = \phi$, and let $F = F(S(G) - \bar{G}_0, \partial G_0)$ be as in Lemma 1 (i.e. $S(G) \in O_G$ iff $\lambda(F) = 0$). Since $\cup_n F_n'' \subset F$, we have

$$\frac{1}{\lambda(F)} \geqq \sum_{n=0}^{\infty} \frac{1}{\lambda(F_n'')}$$

and easily from the properties of extremal length,

$$\lambda(F_n) = \lambda(F_n') \leqq \frac{2\pi}{\log(a_n/a_{n+1})},$$

$$\lambda(F_n'') \leqq 2(\lambda(F_n) + \lambda(F_n')).$$

Thus, for any $m > 0$

$$\frac{1}{\lambda(F)} \geqq \sum_{n=0}^{m} \frac{1}{\lambda(F_n'')} \geqq \sum_{n=0}^{m} \frac{\log(a_n/a_{n+1})}{4\pi} = \frac{1}{4\pi} \log(a_0/a_m).$$

Hence, $\lambda(F) = 0$. G is of *semiweak type* by Proposition 5.

7. Let G and $S(G)$ be as in the example (III). We consider other properties that $S(G)$ has. And let U be a regular neighbourhood of the ideal boundary of $S(G)$ with $(\bar{U})^c \ni \{\infty\}$, then U is a (parabolic) end in the sense of Heins [3]. Denote by h the projection of U into the real axis (i.e. x-coordinate of the point in U), then h is a non-constant HB-function of U. And the full cluster set of h at the ideal boundary is clearly $[-1, 1]$. Hence, the harmonic dimension of U is at least two. While, we can find a bounded continuous function h' on $S(G) - K$ such that $h' = H_h^U / H_1^U$ on U and

$h'=0$ near K, where K is the closure of a regular neighbourhood of $\{\infty\}$ with $K\cap\bar{U}=\phi$. So, h' is a continuous function on S^* the Martin compactification of $S(G)-K$. The Martin ideal boundary part $\Delta(U)$ of S^* belonging to U is the fibre over $[-1,1]$, and it consists of the continuum including minimal points precisely two. Furthermore there exist no minimal points over $(-1,1)$, since minimal points are accessible. But it is an open problem how many minimal points $\Delta(U)$ does have in more general situations.

Let $w=f(z)$ be a univalent function which maps G onto a vertical slit region such that β reduces to the origin and the images of γ_n lie on the imaginary axis symmetrically with respect to the origin. Then $(f(z))^2$ induces a non-constant AB-function f^* on U. Clearly f^* has a limit value zero at $\Delta(U)$. Thus we can not conclude that AB-function reduces to zero, even if it converges to zero uniformly on curves $\{C_n\}$ converging to non-degenerate Martin ideal boundary $\Delta(U)$. This shows that *the extension of classical Koebe's theorem to open Riemann surfaces with Martin compactification does not hold under only assumption that* C_n *converges to a non-degenerate continuum.*

Finally, we give an example modified from (III). Let $\beta=\{|z|=1\}\cup\{z=re^{i\theta};0<\varepsilon\leqq r<1,\theta=0\}$ and $\gamma_n=\{z;\varepsilon\leqq r\leqq 1-a_n,\theta=b_n\}$, $\gamma_{-n}=\{\bar{z};z\in\gamma_n\}$, where $\{a_n\}$ and $\{b_n\}$ are sequences of strictly decreasing positive numbers converging to zero. Identify γ_n with γ_{-n} as in (III). Then we have *a hyperbolic Riemann surface whose Green function* G *does not have compact level curves with* $G(p)=r$ $(0<r\leqq-\log\epsilon)$.

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