J. Math. Kyoto Univ. (JMKYAZ)
21-2 (1981) 251-271

On the condition for the hyperbolicity of systems with double characteristic roots, II

By

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(Received September 18, 1979)

In this paper, we consider the Cauchy problem of systems with double characteristics;

$$\begin{cases} (1) \quad Pu \equiv D_t u + \sum_{i=1}^{n} A^i(t, x) D_{x_i} u + Bu = f, \qquad \text{in } \Omega \subset \boldsymbol{R}_t \times \boldsymbol{R}_x^n, \\ (2) \quad u(t_\circ, x) = u_\circ(x), \end{cases}$$

where $A^i(t, x)$ and $B(t, x)$ are of order N and $A^i(t, x)$ are real $(1 \leqq i \leqq n)$. We assume the following:

Assumption 1. Each characteristic root $\tau = \lambda_j(t, x; \xi)$ of $\det P_p(t, x; \tau, \xi) = 0$ is real, of constant multiplicity and at most double.

Let λ_j be double for $1 \leqq j \leqq r$ and be simple for $r+1 \leqq j \leqq s$. As well known, under the assumption 1, the following condition (L) is necessary for the $\mathcal{E}$ well-posedness.

$$\text{(L)} \qquad {}^{co}P_p P_s {}^{co}P_p + \frac{1}{2i} {}^{co}P_p \{P_p, {}^{co}P_p\} |_{\tau=\lambda_j} \equiv 0, \qquad (1 \leqq j \leqq r).$$

Here, P_p, P_s and ${}^{co}P_p$ are the principal symbol of P, the subprincipal symbol of P and the cofactor matrix of P_p. In the previous paper, we considered 1) the consequence of the condition (L), 2) the existence of stably non-hyperbolic operators with only real characteristic roots, 3) sufficients conditions for the $\mathcal{E}$ well-posedness, restricting ourselves to the case of $n=1$. In this article, we shall generalize the results in the sections 1 and 3 in [26]. However, there exists some difficulties proper to the case in higher dimension domains. Therefore the results in this article are a little rougher than those in [26].

We shall use the notation and the definitions given in [26] without mention of it. We shall name the sections in the previous paper [26] and this paper with the straight numbers. Then, we shall start from the section 4. (If we say "the section 1", it means "the section 1 in [26]".)

§ 4. Smoothness of the eigen-vectors—the case of higher dimension—.

In this and the next sections, we shall establish the similar theorems in the domain of dimension $n+1$ $(n\geqq 2)$ as those in the sections 1 and 3. If all of the coefficients of P_p depend only on t, we can obtain the results exactly corresponding to those in the previous sections. However, in the general case, some essential difficulties occur when $n\geqq 2$. In order to avoid one of such difficulties, we assume the following through this and the next sections. Let us set $R_j(t, x, \xi)=\operatorname{rank} P_p(t, x; \lambda_j(t, x; \xi), \xi)$, $(1\leqq j\leqq r)$ and $\Omega_w^j=\{(t, x, \xi)\in\Omega\times\boldsymbol{R}^n\backslash\{O\} \mid R_j(t, x, \xi)=N-1\}$.

Assumption 2. The projection of the boundary of each connected component of Ω_w^j to Ω lies on a family of some disjoint spacelike hypersurfaces $\{T_l^j\}_l$ $(1\leqq j\leqq r)$, in general, as a subset.

Remark The determination of such hypersurfaces is not unique.

4.1° Theorems of the smoothness of the eigen-vectors.

In this section, we show that the condition (L) implies the piece-wise smoothness of the eigen-vectors of $A_1(t, x; \xi)=\sum_{i=1}^{n} A^i(t, x)\xi_i$ under the assumptions 1 and 2.

$\{T_l^j\}_l$ devides Ω into a family of open connected subdomains $\{\Omega_k^j\}_k$ and a closed set Σ^j, where $R_j(t, x, \xi)$ is constant on each $\Omega_k^j\times\boldsymbol{R}^n\backslash\{0\}$, $\Sigma^j\supseteqq\bigcup_l T_l^j$ and $\mathring{\Sigma}^j=\varnothing$, $(1\leqq j\leqq r)$. If $R_j=N-2$ in $\bigcup_k \Omega_k^j\times\boldsymbol{R}^n\backslash\{O\}$, R_j becomes identically $N-2$ on $\Omega\times\boldsymbol{R}^n\backslash\{O\}$. Then P_p is strongly hyperbolic in Ω, if all R_j are equal to $N-2$ on $\bigcup_k \Omega_k^j\times\boldsymbol{R}^n\backslash\{O\}$, $(1\leqq j\leqq r)$. Therefore, we consider the case when R_{j_0} takes the value $N-1$ on $\Omega_{k_0}^{j_0}\times\boldsymbol{R}^n\backslash\{O\}$ for some j_0 and k_0, $(1\leqq j_0\leqq r)$.

Corresponding to Theorem 1.3, we can obtain the following theorem.

Theorem 4.1. (Smoothness of the eigen-vectors on $T^*(\overline{\Omega_k^j})\backslash\{O\}$.)

Under the assumption 1 and 2, we suppose that $P=P_p+B$ satisfies the condition (L).

(i) *If $R_j=N-1$ on $\Omega_k^j\times\boldsymbol{R}^n\backslash\{O\}$, we can take the real unit eigen-vector $\vec{e}_j(t, x, \xi)$ of $A_1(t, x; \xi)$ belonging to $\lambda_j(t, x, \xi)$ in $C^\infty(\overline{\Omega_k^j}\times\boldsymbol{R}^n\backslash\{O\})$, $(1\leqq j\leqq r)$.*

(ii) *When $R_j=N-1$ on $\Omega_k^j\cup\Omega_l^j$, by virtue of (i), we can take $\vec{e}_j$ in $C^\infty(\overline{\Omega_k^j}\times\boldsymbol{R}^n\backslash\{O\})$ and in $C^\infty(\overline{\Omega_l^j}\times\boldsymbol{R}^n\backslash\{O\})$, respectively. If $\vec{e}_j(t, x, \xi)$ is continuously connected on $(\partial\Omega_k^j\cap\partial\Omega_l^j)\times\boldsymbol{R}^n\backslash\{O\}$, it must belong to $C^\infty(\overline{\Omega_k^j\cup\Omega_l^j}\times\boldsymbol{R}^n\backslash\{O\})$.*

We shall prove this theorem at the last of this section.

Since no bicharacteristic curve is tangent to $\partial\Omega_k^j\times\boldsymbol{R}^n\backslash\{O\}$ under the assumption 2, the following corollary is immediately derived from Theorem 4.1.

Corollary 4.2. (Smoothness of the eigen-vectors along the bicharacteristic curves.)

Suppose that the same assumptions in Theorem 4.1 *hold and that* $R_j=N-1$ *in* $\Omega_k^j\times\boldsymbol{R}^n\setminus\{O\}$. *Let* $\pi(s)$ *be an arbitrary bicharacteristic curve belonging to* $\lambda_j(t, x, \xi)$. *Then,* $\vec{e}_j(t(s), x(s), \xi(s))$ *taken in Theorem* 4.1 *is smooth on* $\omega\cap\overline{\Omega_k^j}\times\boldsymbol{R}^n\setminus\{O\}$, *where* ω *is a neighbourhood of* $\pi(s)$ *in* $T^*(\Omega)$.

Remark. By virtue of Theorem 4.1 and Corollary 4.2, the condition stated in Y. Demay [3], [4] is not realized under the assumption 2, because ε_j in Corollary 4.9 must satisfy the equation of type $\frac{d}{ds}\varepsilon_j=a\varepsilon_j$ along the bicharacteristic curve in $T^*(\overline{\Omega_k^j})$, if there exists $S_j(t, x; \xi)$. (See the remark 2 of Theorem 1.3.)

Without some additional assumption like the assumption 2, the continuity of $\vec{e}_j(t, x, \xi)$ is not, in general, guaranteed in the case of $n\geqq 2$, even if the coefficients of P_p are real analytic. (See Theorem 1.5 and the example 4 in the section 1.) However, we can relax the assumption 2 in this case.

Theorem 4.3. (Real analyticity of the eigen-vectors.)

Suppose the real analyticity of the coefficients of P_p *and the assumption* 1. *Then, only one of the following two cases arises for each* j;

(I) $R_j=N-2$ *in* $T^*(\Omega)\setminus\{O\}$.

(II) $R_j=N-1$ *in* $T^*(\Omega)\setminus\{O\}$ *except an analytic set.*

Here, the exceptional set is the support of the discontinuity of $R_j(t, x, \xi)$. *If its projection to* Ω *is a family of spacelike hypersurfaces in* Ω *with respect to* $D_t-\lambda_j(t, x; D_x)$ *(except its singular points), we can take* $\vec{e}_j(t, x, \xi)$ *in the real analytic class in* $T^*(\Omega)\setminus\{O\}$.

This theorem is brought from theorem 4.1 and de l'Hopital's theorem.

4.2° Proof of Theorem 4.1 —Reduction—.

From now on, we prove Theorem 4.1. The proof depends on the same idea as that in Theorem 1.3. However, since there exists an essential difference in the case of $n\geqq 2$, we give the detailed proof again.

Suppose that $P=P_p+B$ satisfies the condition (L). We consider the behaviour of $\vec{e}_j(t, x, \xi)$ near $\partial\Omega_k^j\times\boldsymbol{R}^n\setminus\{O\}$ through reducing the condition (L) to the first order partial differential equations with respect to the elements of $A_1(t, x; \xi)$. We consider mainly the property (i). The property (ii) will be obtained immediately through the proof of (i).

At first, let us reduce the equation (1) to a family of systems of type 2×2 and scalar equations.

Lemma 4.4. 1) *The condition* (L) *is invariant under an arbitrary spacelike transformation.*

2) *Let* $\mathcal{N}_0(t, x; \xi)$ *be regular and homogeneous of degree* 0 *in* ξ, *and moreover, let* $\mathcal{N}_{-1}(t, x; \xi)$ *and* $\mathcal{N}'_{-1}(t, x; \xi)$ *be of degree* -1. *Let us set* $\mathcal{N}(t, x; D_x)=\mathcal{N}_0(t, x; D_x)+\mathcal{N}_{-1}(t, x; D_x)$ *and* $\mathcal{N}'(t, x; D_x)=(\mathcal{N}_0)^{-1}(t, x; D_x)+\mathcal{N}'_{-1}(t, x; D_x)$. *If* P *satisfies the condition* (L), *the principal and subbrincipal parts of* $\mathcal{N}(t, x; D_x)\cdot P(t, x; D_t, D_x)\cdot\mathcal{N}'(t, x; D_x)$ *also satisfy the condition* (L).

Proof. 1) was proved in H. Yamahara [25]. 2) is easily proved by the following:

$$\tag{4.1} \begin{cases} \sigma_1(\mathcal{N}P\mathcal{N}')=\mathcal{N}_0P_p\mathcal{N}_0^{-1}\ (\equiv\tilde{P}_p), \quad {}^{co}\tilde{P}_p=\mathcal{N}_0{}^{co}P_p\mathcal{N}_0^{-1}, \\ \sigma_0(\mathcal{N}P\mathcal{N}')=\mathcal{N}_0B\mathcal{N}_0^{-1}+\sum_{i=0}^{n}\mathcal{N}_0P_p^{(i)}\mathcal{N}_0^{-1}{}_{(i)}+\sum_{i=0}^{n}\mathcal{N}_0{}^{(i)}P_{p(i)}\mathcal{N}_0^{-1} \\ \qquad +\sum_{i=1}^{n}\mathcal{N}_0{}^{(i)}P_p\mathcal{N}_0^{-1}{}_{(i)}+\mathcal{N}_{-1}P_p\mathcal{N}_0^{-1}+\mathcal{N}_0P_p\mathcal{N}'_{-1}. \end{cases}$$

Q. E. D.

Lemma 4.5. *For arbitrary* (t, x, ξ) *in* $\Omega\times\boldsymbol{R}^n\backslash\{O\}$, *there exists a neighbourhood* ω *of* (t, x) *in* Ω *and a conical neighbourhood* Γ *of* ξ *in* $\boldsymbol{R}^n\backslash\{O\}$ *such that* $A_1(t, x; \xi)$ *has a real* C^∞-"*emblocking*" *matrix* $\mathcal{B}_0$ *which is homogeneous of degree* 0 *in* ξ. *Here,* $\mathcal{B}_0$ *satisfies*

$$\tag{4.2} A_1\mathcal{B}_0=\mathcal{B}_0C'_1, \quad \text{where} \quad C'_1=\begin{pmatrix} C_1^{1'} & & & & & \\ & C_1^{2'} & & & 0 & \\ & & \ddots & & & \\ & & & C_1^{r'} & & \\ & 0 & & & \lambda_{r+1} & \\ & & & & & \ddots \\ & & & & & & \lambda_s \end{pmatrix}$$

and $C_1^{j'}$ *is of type* 2×2, *real and homogeneous of degree* 1 *in* ξ, $(1\leqq j\leqq r)$.

Proof. The projection p_j to the root space belonging to λ_j is given by

$$\tag{4.3} p_j=(1/2\pi i)\int_{\gamma_j}(\eta I-A_1/|\xi|)^{-1}d\eta,$$

where γ_j is a circle including only $\lambda_j/|\xi|$. Obviously, p_j is smooth in a neighbourhood of (t, x, ξ) and homogeneous of degree 0 in ξ. This implies Lemma 4.5.

Q. E. D.

We use $\sim$ in order to express the asymptotic equivalence.

Lemma 4.6. *For arbitrary* (t, x, ξ) *in* $\Omega\times\boldsymbol{R}^n\backslash\{O\}$, *there exists a neighbourhood* ω *of* (t, x) *and a conical neighbourhood* Γ *of* ξ *such that* $P(t, x; D_t, D_x)$ *has an* "*emblocking operator*" $\mathcal{B}(t, x; D_x)$ *which satisfies the followings in* $\omega\times\Gamma$.

(i) $\mathcal{B}(t, x; \xi)\sim\sum_{k=0}^{\infty}\mathcal{B}_{-k}(t, x; \xi)$, *where* $\mathcal{B}_{-k}(t, x; \xi)$ *is homogeneous of degree* $-k$ *and the right-hand side is a formal series.*

(ii)

$$(4.4)\qquad P(t, x; D_t, D_x)\cdot \mathscr{B}(t, x; D_x)\equiv \mathscr{B}(t, x; D_x)\cdot \mathcal{C}(t, x; D_t, D_x) \quad \mathrm{mod.}\, S^{-\infty}.$$

$$(4.5)\qquad \mathcal{C}(t, x; D_t, D_x)=\begin{pmatrix} \mathcal{C}^1 & & & & & \\ & \mathcal{C}^2 & & & 0 & \\ & & \ddots & & & \\ & & & \mathcal{C}^r & & \\ & 0 & & & \mathcal{C}^{r+1} & \\ & & & & & \ddots \\ & & & & & & \mathcal{C}^s \end{pmatrix},$$

where $\mathcal{C}^j(t, x; \tau, \xi)\sim \mathcal{C}_1^j(t, x; \tau, \xi)+\sum_{k=0}^{\infty} \mathcal{C}_{-k}^j(t, x; \xi)$, $\mathcal{C}_{-k}^j$ *is of type* 2×2 *when* $1\leqq j\leqq r$ *and is scalar when* $r+1\leqq j\leqq s$, $(-1\leqq k)$. *Here* $\mathcal{C}_1^j(t, x; \tau, \xi)=\tau I-\mathcal{C}_1^{j'}(t, x; \xi)$ *when* $1\leqq j\leqq r$ *and* $\mathcal{C}_1^j(t, x; \tau, \xi)=\tau-\lambda_j(t, x, \xi)$ *when* $r+1\leqq j\leqq s$. $\mathscr{B}_0$ *and* $\mathcal{C}_1^{j'}$ *are those in Lemma* 4.5, $(1\leqq j\leqq r)$.

(iii) $\mathcal{C}_1^j+\mathcal{C}_0^j$ *satisfies the condition* (L) *with respect to* $\lambda_j(t, x, \xi)$, *if and only if* P *satisfies the condition* $L_j(t, x, \xi)\equiv 0$ *in* $\Omega\times \boldsymbol{R}^n\backslash\{O\}$, $(1\leqq j\leqq r)$.

Proof. Let us set formally $\mathscr{B}(t, x; D_x)\sim \sum_{k=0}^{\infty} \mathscr{B}_{-k}(t, x; D_x)$. Then,

$$(4.6)\qquad P\cdot \mathscr{B}\sim \sum \frac{1}{\alpha !} P_h{}^{(\alpha)} \mathscr{B}_{-k(\alpha)}=P_P\mathscr{B}_0+\Big(\sum_{i=0}^{n} P_p{}^{(i)} \mathscr{B}_{0(i)}+B\mathscr{B}_0+P_p\mathscr{B}_{-1}\Big)+\cdots,$$

where $P_1=P_p$, $P_0=B$, $P_{-k}=0$ $(1\leqq k)$ and i, k and α in the summation run over $0\leqq h\leqq 1$, $k\geqq 0$ and $|\alpha|\geqq 0$, respectively. On the other hand,

$$\mathscr{B}\cdot \mathcal{C}\sim \sum \frac{1}{\alpha !} \mathscr{B}_{-k}{}^{(\alpha)}\mathcal{C}_{-l(\alpha)}=\mathscr{B}_0\mathcal{C}_1+\Big(\sum_{i=0}^{n} \mathscr{B}_0{}^{(i)}\mathcal{C}_{1(i)}+\mathscr{B}_0\mathcal{C}_0+\mathscr{B}_{-1}\mathcal{C}_1\Big)+\cdots,$$

where l, k and α in the summation run over $l\geqq -1$, $k\geqq 0$ and $|\alpha|\geqq 0$, respectively. Multiplying $\mathscr{B}_0^{-1}$ to the above formulas from left, we get

$$(4.7)\qquad \mathcal{C}_1(\mathscr{B}_0^{-1}\mathscr{B}_{-m})-(\mathscr{B}_0^{-1}\mathscr{B}_{-m})\mathcal{C}_1=\sum \frac{1}{\alpha !} \mathscr{B}_0^{-1}\mathscr{B}_{-k}{}^{(\alpha)}\mathcal{C}_{-l(\alpha)} -\sum \frac{1}{\alpha !} \mathscr{B}_0^{-1}P_h{}^{(\alpha)} \mathscr{B}_{-k(\alpha)}+\mathcal{C}_{-(m-1)},$$

where h, l, k and α run over $0\leqq h\leqq 1$, $-1\leqq l\leqq m-2$, $0\leqq k\leqq m-1$ and $|\alpha|\geqq 0$ satisfying $-h+k+|\alpha|=m-1$ or $l+k+|\alpha|=m-1$. Let us set

$$(4.8)\qquad \tilde{\mathscr{B}}_{-m}\equiv \mathscr{B}_0^{-1}\mathscr{B}_{-m}=\begin{pmatrix} \mathscr{B}_{-m}^{11}, & \mathscr{B}_{-m}^{12}, & \cdots\cdots, & \mathscr{B}_{-m}^{1s} \\ \mathscr{B}_{-m}^{21}, & \mathscr{B}_{-m}^{22}, & \cdots\cdots, & \mathscr{B}_{-m}^{2s} \\ \vdots & \vdots & & \vdots \\ \mathscr{B}_{-m}^{s1}, & \mathscr{B}_{-m}^{s2}, & \cdots\cdots, & \mathscr{B}_{-m}^{ss} \end{pmatrix}$$

where $\mathscr{B}_{-m}^{ij}$ is of type 2×2 when $1\leqq i$, $j\leqq r$, 2×1 when $1\leqq i\leqq r$ and $r+1\leqq j\leqq s$, 1×2 when $r+1\leqq i\leqq s$ and $1\leqq j\leqq r$, and a scalar when $r+1\leqq i$, $j\leqq s$. We also set

$$C_{-m}=\begin{pmatrix} C_{-m}^{1} & & & \\ & C_{-m}^{2} & & \\ & & \ddots & \\ & & & C_{-m}^{s} \end{pmatrix}, \tag{4.9}$$

where C_{-m}^{j} is of type 2×2 when $1\leqq j\leqq r$ and a scalar when $r+1\leqq j\leqq s$. First, we set $\tilde{\mathscr{B}}_0=I$ and $C_1^j=\tau I-C_1^{j'}(t, x; \xi)$. Then equation (4.7) is equivalent to the following (4.7′).

$$C_1^{i'}\mathscr{B}_{-m}^{ij}-\mathscr{B}_{-m}^{ij}C_1^{j'}=\text{the known terms}-\delta_{ij}C_{-(m-1)}^{j}, \tag{4.7′}$$

($m\geqq 1$ and δ_{ij} is Kronecker's δ.)

If $i\neq j$, we have the unique solution $\mathscr{B}_{-m}^{ij}$ of (4.7′) by virtue of $\lambda_i\neq\lambda_j$, and if $i=j$, we put $C_{-(m-1)}^{j}=(-1)\times$(" the known terms " in the right-hand side of (4.7′)) and $\mathscr{B}_{-m}^{ij}=0$. Here, we notice that $C_{-(m-1)}^{j}$ and $\mathscr{B}_{-m}^{ij}$ are smooth in $\omega\times\Gamma$ even if the rank of $\lambda_j I-C_1^{j'}$ changes. Since $\mathscr{B}_0$ is regular and smooth in $\omega\times\Gamma$, $\mathscr{B}_{-m}=\mathscr{B}_0\tilde{\mathscr{B}}_{-m}$ and C_{-m} belong to $C^\infty(\omega\times\Gamma)$ and moreover they are homogeneous in ξ.

By virtue of the determination of C_{-l}^{j} ($l\geqq 0$), we have

$$\begin{aligned} C_0^j=\text{the corresponding part of } \Big[&\mathscr{B}_0^{-1}B\mathscr{B}_0+\sum_{i=0}^{n}\mathscr{B}_0^{-1}P_p^{(i)}\mathscr{B}_{0(i)} \\ &+\sum_{i=0}^{n}\mathscr{B}_0^{-1(i)}P_{p(i)}\mathscr{B}_0+\sum_{i=1}^{n}\mathscr{B}_0^{-1(i)}P_p\mathscr{B}_{0(i)}-\sum_{i=1}^{n}\mathscr{B}_0^{-1(i)}\mathscr{B}_{0(i)}C_1 \\ &+C_1\mathscr{B}_0^{-1}\mathscr{B}_{-1}+\mathscr{B}_0^{-1}\mathscr{B}_{-1}C_1\Big]. \end{aligned} \tag{4.8}$$

Therefore, we can prove the invariance of the condition (L) by the same way as the proof of Lemma 4.4. Moreover, since C is emblocked, the condition (L) of C is equivalent to those of C^j ($1\leqq j\leqq r$). (See also K. Kajitani [11].) Q. E. D.

4.3° Proof of Theorem 4.1 —Continued—.

Let us take $P=P_p+B$ which satisfies the condition (L). By virtue of Lemma 4.6, $P(t, x; D_t, D_x)$ is reduced to a family of systems of type 2×2 and scalar operators modulo $S^{-\infty}$. $C_1^j+C_0^j$ satisfies the condition (L) ($1\leqq j\leqq r$) since P satisfies the condition (L). We set $C_1^j=(\tau-\lambda_j(t, x; \xi))I-\tilde{A}_1^j$ and

$$\tilde{A}_1^j=\begin{bmatrix} a^j(t, x; \xi) & b^j(t, x; \xi) \\ c^j(t, x; \xi) & d^j(t, x; \xi) \end{bmatrix}.$$

$\tilde{A}_1^j$ has a double eigen-value 0. Then,

$$\begin{cases} d^j(t, x; \xi)=-a^j(t, x; \xi), \quad (a^j(t, x; \xi))^2+b^j(t, x; \xi)c^j(t, x; \xi)=0, \\ (\tilde{A}_1^j(t, x; \xi))^2=0 \text{ and } {}^{co}C_1^j(t, x; \xi)=(\tau-\lambda_j(t, x; \xi))I+\tilde{A}_1^j(t, x; \xi) \quad \text{in } \omega\times\Gamma. \end{cases} \tag{4.9}$$

Let us omit the suffix j. We assume that

$$R(t, x, \xi)\equiv\operatorname{rank} P_p(t, x; \lambda(t, x; \xi), \xi)=N-1 \quad \text{in } \Omega_k\times \boldsymbol{R}^n\backslash\{O\},$$

that is, $|b(t, x; \xi)|+|c(t, x; \xi)|\neq 0$ in $\Omega_k\times \boldsymbol{R}^n\backslash\{O\}$, and that a part of $\partial\Omega_k$ consists of $T: t=\psi(x)$ and Ω_k is in the side of $t<\psi(x)$. Let us write A instead of $\tilde{A}_1$. By virtue of $A_{(i)}^{(i)}A+A^{(i)}A_{(i)}+A_{(i)}A^{(i)}+AA_{(i)}^{(i)}=0$ $(0\leqq i\leqq n)$, the left-hand side in the condition (L) is equal to

$$(4.10)\qquad \{(\xi_0-\lambda)I+A\}\Big[C_0-\frac{1}{2}\sum_{i=0}^{n}\{(\xi_0-\lambda)_{(i)}^{(i)}I-A_{(i)}^{(i)}\}\Big]\{(\xi_0-\lambda)I+A\}$$

$$+\frac{1}{2}\{(\xi_0-\lambda)I+A\}\Big[\sum_{i=0}^{n}\{(\xi_0-\lambda)^{(i)}I-A^{(i)}\}\{(\xi_0-\lambda)_{(i)}I+A_{(i)}\}$$

$$-\sum_{i=0}^{n}\{(\xi_0-\lambda)_{(i)}I-A_{(i)}\}\{(\xi_0-\lambda)^{(i)}I+A^{(i)}\}\Big]\Big|_{\xi_0=\lambda}$$

$$=AA_{(0)}+\sum_{i=1}^{n}A\{\lambda_{(i)}A^{(i)}-\lambda^{(i)}A_{(i)}\}+\sum_{i=1}^{n}AA_{(i)}A^{(i)}+AC_0A\,.$$

Let us set $a_{x_i}=\dfrac{\partial}{\partial x_i}a(t, x; \xi)$, $a_{\xi_i}=\dfrac{\partial}{\partial \xi_i}a(t, x; \xi)$ and

$$C_0=-\sqrt{-1}\begin{bmatrix}\alpha_0(t, x; \xi) & \beta_0(t, x; \xi)\\ \gamma_0(t, x; \xi) & \delta_0(t, x; \xi)\end{bmatrix}.$$

By virtue of (4.9), (4.10)=0 becomes the following;

$$(4.11)\qquad (-a/b)_t-\Big\{\sum_{i=1}^{n}\lambda_{\xi_i}(-a/b)_{x_i}-\sum_{i=1}^{n}\lambda_{x_i}(-a/b)_{\xi_i}\Big\}+b\sum_{i=1}^{n}(-a/b)_{x_i}(-a/b)_{\xi_i}$$

$$-\beta_\circ(-a/b)^2-(\alpha_\circ-\delta_\circ)(-a/b)+\gamma_\circ=0\,,\qquad \text{when } b\neq 0\,,$$

$$(4.12)\qquad (a/c)_t-\Big\{\sum_{i=1}^{n}\lambda_{\xi_i}(a/c)_{x_i}-\sum_{i=1}^{n}\lambda_{x_i}(a/c)_{\xi_i}\Big\}+c\sum_{i=1}^{n}(a/c)_{x_i}(a/c)_{\xi_i}$$

$$-\gamma_0(a/c)^2-(\delta_\circ-\alpha_\circ)(a/c)+\beta_\circ=0\,,\qquad \text{when } c\neq 0\,.$$

Now, we set $g=-a/b$, $h=a/c$, $\alpha=\mathrm{Re}\,\alpha_\circ$, $\beta=\mathrm{Re}\,\beta_\circ$, $\gamma=\mathrm{Re}\,\gamma_\circ$, and $\delta=\mathrm{Re}\,\delta_\circ$. Then, the real parts of (4.11) and (4.12) become

$$(4.11')\qquad g_t-\sum_{i=1}^{n}\lambda_{\xi_i}g_{x_i}+\sum_{i=1}^{n}(\lambda_{x_i}-a_{x_i}-b_{x_i}g)g_{\xi_i}-\beta g^2-(\alpha-\delta)g+\gamma=0\,,$$

$$(4.12')\qquad h_t-\sum_{i=1}^{n}\lambda_{\xi_i}h_{x_i}+\sum_{i=1}^{n}(\lambda_{x_i}+a_{x_i}-c_{x_i}h)h_{\xi_i}-\gamma h^2-(\delta-\alpha)h+\beta=0\,.$$

Here, $gh=1$ holds if $g\neq 0$ and $h\neq 0$. (4.11′) and (4.12′) have the solution $g=-a/b$ and $h=a/c$ when $b\neq 0$ and when $c\neq 0$, respectively.

If a, b and c depend only on t, (4.11′) and (4.12′) are expressed as the following:

$$(4.11')_t\qquad g_t-\beta g^2-(\alpha-\delta)g+\gamma$$

$$(\equiv g_t-\sum_{i=1}^{n}\lambda_{\xi_i}g_{x_i}+\sum_{i=1}^{n}\lambda_{x_i}g_{\xi_i}-\beta g^2-(\alpha-\delta)g+\gamma=0)\,,$$

$$(4.12')_t \qquad h_t-\gamma h^2-(\delta-\alpha)h+\beta$$

$$(\equiv h_t-\sum_{i=1}^{n}\lambda_{\xi_i}h_{x_i}+\sum_{i=1}^{n}\lambda_{x_i}h_{\xi_i}-\gamma h^2-(\delta-\alpha)h+\beta=0).$$

Here, both of the derivations operate on g and h along the bicharacteristic curves. Then, we can obtain the result as detailed as Theorem 1.3.

Let us seek for the solutions of (4.11′) and (4.12′) which coincide with $-a/b$ when $b\neq 0$, and with a/c when $c\neq 0$ in $(\omega\cap\Omega_k)\times\Gamma$, respectively. Let $\dot{a}$ be $\dfrac{d}{ds}a$, that is, the derivative of a along the characteristic curve of (4.11′) or (4.12′).

$$(4.11'')\qquad \begin{cases} \dot{t}=1, \quad \dot{x}=-\nabla_\xi\lambda, \quad \dot{\xi}=\nabla_x\lambda-\nabla_x a-g\nabla_x b, \\ \dot{g}=\beta g^2+(\alpha-\delta)g-\gamma, \end{cases}$$

$$(4.12'')\qquad \begin{cases} \dot{t}=1, \quad \dot{x}=-\nabla_\xi\lambda, \quad \dot{\xi}=\nabla_x\lambda+\nabla_x a-h\nabla_x c, \\ \dot{h}=\gamma h^2+(\delta-\alpha)h-\beta. \end{cases}$$

We consider the case when ω intersects $T: t=\psi(x)$. We may assume that $\omega\cap\overline{\Omega_k}$ does not intersect $\{T_l\}$ without T and that $\bar{\omega}$ is compact. (See Figure 6.)

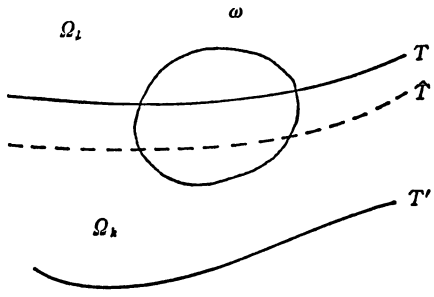

Figure 6.

At first, we consider (4.11″) and (4.12″) in $(\omega\cap\Omega_k)\times\Gamma$. Let us set

$$(4.13)\qquad x(0)=x^\circ, \quad \xi(0)=\xi^\circ, \quad t(0)=\psi(x^\circ)-\varepsilon_\circ \ (\equiv t^\circ),$$

where $(t^\circ, x^\circ, \xi^\circ)\in\omega\times\Gamma$ and $\varepsilon_\circ>0$. Moreover, we set

$$(4.14)\qquad g(0)=-a(t^\circ, x^\circ; \xi^\circ)/b(t^\circ, x^\circ; \xi^\circ) \quad \text{if} \quad b(t^\circ, x^\circ; \xi^\circ)\neq 0,$$

or

$$(4.15)\qquad h(0)=a(t^\circ, x^\circ; \xi^\circ)/c(t^\circ, x^\circ; \xi^\circ) \quad \text{if} \quad c(t^\circ, x^\circ; \xi^\circ)\neq 0.$$

At least, one of (4.14) and (4.15) can be always defined. If $g(0)$ makes sense, there exists the solution $(t(s), x(s), \xi(s), g(s))$ near $(t^\circ, x^\circ, \xi^\circ, (-a/b)(t^\circ, x^\circ; \xi^\circ))$ and if $h(0)$ makes sense, there exists the solution $(t(s), x(s), \xi(s), h(s))$ near $(t^\circ, x^\circ, \xi^\circ, (a/c)(t^\circ, x^\circ; \xi^\circ))$.

Lemma 4.7. *Suppose that* (4.14) *makes sense and let* $(t(s), x(s), \xi(s), g(s))$ *be* *the solution* (4.11″) *with the data* (4.13) *and* (4.14). $t=t(s)$, $x=x(s)$, $\xi=\xi(s)$ *and* $h=1/g(s)$ *are the solutions of* (4.12″), (4.13) *and* (4.15) *as long as* $g(s)\neq 0$. *Conversely, the assertion exchanging the places of* g *and* h *is also true.*

Proof. By the uniqueness of the solutions, $t(s)$, $x(s)$, $\xi(s)$ and $g(s)$ satisfy

$$g(s)=-a(t(s),\ x(s)\,;\xi(s))/b(t(s),\ x(s)\,;\xi(s))\,, \tag{4.16}$$

as far as $b(t(s), x(s); \xi(s))$ does not vanish. $g(s)\neq 0$ implies $c(t(s), x(s); \xi(s))\neq 0$ by virtue of (4.9). Then, because of the equality $h=1/g(s)=a(t(s), x(s); \xi(s))/c(t(s), x(s); \xi(s))$, we can see the followings;

$$\begin{cases} \dot{\xi}(s)=\nabla_x\lambda-\nabla_x a-(-a/b)\nabla_x b\,|_{(t,x,\xi)=(t(s),x(s),\xi(s))} \\ \qquad =\nabla_x\lambda+\nabla_x a-(a/c)\nabla_x c\,|_{(t,x,\xi)=(t(s),x(s),\xi(s))} \\ \dot{h}(s)=(1/g(s))^{\cdot}=-\dot{g}(s)/(g(s))^2 \\ \qquad =-\{\beta g^2+(\alpha-\delta)g-\gamma\}/g^2\,|_{(t,x,\xi,g)=(t(s),x(s),\xi(s),g(s))} \\ \qquad =\gamma h^2+(\delta-\alpha)h-\beta\,|_{(t,x,\xi,h)=(t(s),x(s),\xi(s),1/g(s))}\,. \end{cases} \tag{4.17}$$

This shows that $t=t(s)$, $x=x(s)$, $\xi=\xi(s)$ and $h=1/g(s)$ are the solutions of (4.12″), (4.13) and (4.15). The converse can be shown by the same way. Q.E.D.

By virtue of Lemma 4.7, the characteristic curves of (4.11′) and (4.12′) coincide each other in $\overline{\Omega_k}$.

Let us put $M=\sup\{|\nabla_\xi\lambda|,\ |\nabla_x\lambda|,\ |\nabla_x a|,\ |\nabla_x b|,\ |\nabla_x c|,\ |\alpha|,\ |\beta|,\ |\gamma|,\ |\delta|\}$, where (t, x, ξ) runs over $\omega\times S_\xi^{n-1}$, and let θ_1 and θ_2 be Arctan 2 and Arctan 3, respectively.

Lemma 4.8. *For* $\varepsilon_0<(\theta_2-\theta_1)/2M$, *let the initial surface be* $\hat{T}\times \boldsymbol{R}^n\backslash\{O\}$, *where* $\hat{T}$ *is defined by* $t=\phi(x)-\varepsilon_0$.

1) *If* $|g(0)|=|a(t^\circ, x^\circ; \xi^\circ)/b(t^\circ, x^\circ; \xi^\circ)|\leqq 1$, (*where* $t^\circ=\phi(x^\circ)-\varepsilon_0$), (4.14) *makes sense in a neighbourhood* U *of* $(t^\circ, x^\circ, \xi^\circ)$ *in* $\hat{T}\times\boldsymbol{R}^n\backslash\{O\}$ *and there exists the solution of* (4.11″) *with the data* (4.13) *and* (4.14) *starting from* U *across* $T\times\boldsymbol{R}^n\backslash\{O\}$ *in* C^∞*-class.*

2) *If* $|h(0)|=|a(t^\circ, x^\circ; \xi^\circ)/c(t^\circ, x^\circ; \xi^\circ)|\leqq 1$, (4.15) *makes sense in a neighbourhood* $\tilde{U}$ *of* $(t^\circ, x^\circ, \xi^\circ)$ *in* $\hat{T}\times\boldsymbol{R}^n\backslash\{O\}$ *and there exists the solution of* (4.12″) *with the data* (4.13) *and* (4.15) *starting from* $\tilde{U}$ *across* $T\times\boldsymbol{R}^n\backslash\{O\}$ *in* C^∞*-class.*

3) *If* $\frac{1}{2}<|a(t^\circ, x^\circ; \xi^\circ)/b(t^\circ, x^\circ; \xi^\circ)|<2$, *there exist both the solutions of* (4.11″) *with the data* (4.13)—(4.14) *and of* (4.12″) *with the data* (4.13)—(4.15), *which start from a neighbourhood of* $(t^\circ, x^\circ, \xi^\circ)$ *in* $\hat{T}$ *and which cross over* T. *Here, neither* $g(s)$ *nor* $h(s)$ *vanishes on the interval from* $\hat{T}$ *to* T *and* $g(s)$ *and* $h(s)$ *satisfy the equality* $g(s)h(s)\equiv 1$.

Proof. (4.11″) and (4.12″) have the same major equations:

$$\begin{cases} \dot{X}=M\,,\quad \dot{\Xi}=\sqrt{n}M(2+G)\Xi\,,\quad \dot{\Theta}=-\sqrt{n}M(2+G)\Theta\,, \\ \dot{G}=2M(G^2+1)\,, \end{cases} \tag{4.18}$$

where $|x_i(s)|\leqq X(s)$ $(1\leqq i\leqq n)$, $|\xi(s)|\leqq \Xi(s)$, $\Theta(s)\leqq\max_{1\leqq i\leqq n}|\xi_i(s)|$ and $|g(s)|\leqq G(s)$. Then, we prove only 1), since 2) can be proved by the same way as 1) and since 3) is immediately derived from 1) and 2). (4.18) have the following solutions and the estimates:

$$(4.19)\qquad \begin{cases} X=X_\circ+Ms\,, \quad \Xi\leqq\Xi_\circ\exp(5\sqrt{n}Ms)\,, \quad \Theta\geqq\Theta_\circ\exp(-5\sqrt{n}Ms) \\ G=\tan(2Ms+\operatorname{Arctan}G_\circ), \quad \text{when } G\leqq 3. \end{cases}$$

The projection of the solution of (4.11″) to (t, x)-space is tangent to the family of the projections of the bicharacteristic curves of P_p. $g(s)$ exists across T because of $\varepsilon_\circ<(\theta_2-\theta_1)/2M$ and $|g(0)|<2$. Moreover, we see that $|x(s)-x^\circ|\leqq M\varepsilon_\circ$ and $(\max_i|\xi_i^\circ|)\exp(-5\sqrt{n}M\varepsilon_\circ)\leqq\max_{s,i}|\xi_i(s)|\leqq\max_s|\xi(s)|\leqq|\xi^\circ|\exp(5\sqrt{n}M\varepsilon_\circ)$, where i and s run over $1\leqq i\leqq n$ and $0\leqq s\leqq\varepsilon_\circ$. Then, $|\xi(s)|$ never vanishes if $|\xi^\circ|\neq 0$. It is obvious that the solutions of (4.11′) belongs to C^∞-class as far as they makes sense. Q. E. D.

By virtue of Lemma 4.8, there is a neighbourhood $\omega'\times\Gamma'$ $(\subset\subset\omega\times\Gamma)$ of $(\bar t, \bar x, \bar\xi)\in(T\cap\omega)\times\Gamma$ such that there exists the solution $g(t, x, \xi)$ of (4.11′) or $h(t, x, \xi)$ of (4.12′) which satisfies the equality $g(t, x, \xi)=-a(t, x; \xi)/b(t, x; \xi)$ or $h(t, x, \xi)=a(t, x; \xi)/c(t, x; \xi)$, respectively, in $(\omega'\cap\Omega_k)\times\Gamma'$. Moreover, if g makes sense and does not vanish, h also makes sense and satisfies the relation $gh\equiv 1$. The converse is also valid.

From now on, we write ω and Γ instead of ω' and Γ'. Now, let us define the unit eigen-vector $\vec e(t, x, \xi)$ on $(\omega\cap\overline{\Omega_k})\times\Gamma$. The unit eigen-vector of C_1^j belonging to $\lambda(t, x, \xi)$ is given by $\pm(1/\sqrt{a^2+b^2})^t(b, -a)=\pm(1/\sqrt{1+g^2})^t(1, g)$ if $b\neq 0$, and by $\pm(1/\sqrt{a^2+c^2})^t(a, c)=\pm(1/\sqrt{1+h^2})^t(h, 1)$ if $c\neq 0$, when $t<\phi(x)$. Let us define

$$(4.20)\qquad \vec e'(t, x, \xi)=\begin{cases} \pm(1/\sqrt{1+g^2})^t(0, \cdots, 0, \overset{2j-1}{1}, \overset{2j}{g}, 0, \cdots, 0)\,, \\ \pm(1/\sqrt{1+h^2})^t(0, \cdots, 0, \overset{2j-1}{h}, \overset{2j}{1}, 0, \cdots, 0)\,, \end{cases}$$

using the solutions g of (4.11′) and h of (4.12′), where the signature is chosen suitably as $\vec e'(t, x, \xi)$ becomes continuous. Thus, we can get $\vec e'(t, x, \xi)$ in $C^\infty((\omega\cap\overline{\Omega_k})\times\Gamma)$. Then, $\vec e=|\mathscr{B}_0\vec e'|^{-1}\mathscr{B}_0\vec e'$ is the unit eigen-vector of $A_1(t, x; \xi)$ belonging to $\lambda(t, x, \xi)$ in $(\omega\cap\overline{\Omega_k})\times\Gamma$ which is real and belongs to $C^\infty((\omega\cap\overline{\Omega_k})\times\Gamma)$.

$\vec e(t, x, \xi)$ can be connected on $\overline{\Omega_k}\times\boldsymbol{R}^n\backslash\{O\}$ in C^∞-class since the eigen-space of A_1 belonging to λ is of dimension one and since each $\vec e$ in $(\omega\cap\overline{\Omega_k})\times\Gamma$ is taken real. Then, the proof of the property (i) is completed.

Now, we prove the property (ii). Though $g(t, x, \xi)$ or $h(t, x, \xi)$ is defined across $T\times\Gamma$ starting from $\hat T\times\Gamma$, (4.20) does not always express the eigen-vector of C_1^j in $t>\phi(x)$. However, if $R_j=N-1$ in $(\omega\backslash T)\times\boldsymbol{R}^n\backslash\{O\}$, we have the solution $g^\pm$ of (4.11′) or $h^\pm$ of (4.12′) in ω which starts from $\hat T_\pm\times\Gamma$ and coincides $-a(t, x; \xi)/b(t, x; \xi)$ or $a(t, x; \xi)/c(t, x; \xi)$ in $(\omega\cap\Omega_\pm)\times\Gamma$, where $\hat T_\pm=\{t=\phi(x)\pm\varepsilon_\circ\}$, $\Omega_+=\{t>\phi(x)\}$, $\Omega_-=\{t<\phi(x)\}$ and we take the same signature in all terms. If $|(-a/b)(t^\circ, x^\circ; \xi^\circ)|$ is less than or equal to 1 on $(\hat{T}_+\cap\omega)\times\Gamma$, g^+ exists in $\omega\times\Gamma$ and it is bounded on $T\times\Gamma$. If g^- or $1/h^-$ coincides with g^+ on $T\times\Gamma$, they coincide each other on $\overline{\omega\cap\Omega_-}\times\Gamma$ because of the uniqueness of the solution of the Cauchy problem. Therefore, g^+ coincides with $-a(t, x; \xi)/b(t, x; \xi)$ in $(\omega\backslash T)\times\Gamma$ and it belongs to $C^\infty(w\times\Gamma)$. Then, we can take $\vec{e}(t, x, \xi)$ in C^∞-class in ω. In the case when $|(a/c)(t^\circ, x^\circ; \xi^\circ)|$ is less than or equal to 1, we also obtain $\vec{e}(t, x, \xi)$ in $C^\infty(\omega\times\Gamma)$ under the assumption in (ii). Continuing them, we obtain $\vec{e}(t, x, \xi)$ in $C^\infty(\overline{\Omega_+\cup\Omega_-}\times \boldsymbol{R}^n\backslash\{O\})$. Thus, the proof of Theorem 4.1 is completed. Q. E. D.

4.4° Jordan's normalizer.

Theorem 4.1 implies the following corollary, which will be used in the next section.

Corollary 4.9. (Jordan's normalizer.)

Under the assumptions 1 *and* 2, *suppose that the condition* (L) *is satisfied. If* ω *in Lemma* 4.6 *is covered by* $\overline{\Omega_k\cup\Omega_l}$, *there exist "Jordan's normalizers"* $J^{j,k}$ *and* $J^{j,l}$ *of* C^j *in* $(\omega\cap\overline{\Omega_k^j})\times\Gamma$ *and* $(\omega\cap\overline{\Omega_l^j})\times\Gamma$, *respectively*, $(1\leqq j\leqq r)$, *that is*,

$$C^j(t, x; D_t, D_x)\cdot \mathcal{J}^{j,\nu}(t, x; D_x)\equiv \mathcal{J}^{j,\nu}(t, x; D_x)\cdot \mathcal{D}^j(t, x; D_t, D_x), \quad \bmod S^{-\infty}, \ (\nu=k, l),$$

$$\mathcal{D}^j(t, x; \tau, \xi)\sim \mathcal{D}_1^j(t, x; \tau, \xi)+\sum_{s=0}^{\infty}\mathcal{D}_{-s}^j(t, x; \xi),$$

$$\mathcal{D}_1^j(t, x; \tau, \xi)=\tau I-\begin{bmatrix}\lambda_j(t, x; \xi) & \varepsilon_j(t, x; \xi)\\ 0 & \lambda_j(t, x; \xi)\end{bmatrix},$$

$\varepsilon_j(t, x; \xi)=b^j(t, x; \xi)-c^j(t, x; \xi)$ *in* $\omega\times\Gamma$, *where* $C_1^j=(\tau-\lambda_j)I-\begin{bmatrix}a^j & b^j\\ c^j & d^j\end{bmatrix}$,

$$\mathcal{D}_0^j(t, x; \xi)=\begin{bmatrix}* & *\\ 0 & *\end{bmatrix} \quad \text{if } \varepsilon_j(t, x; \xi)\neq 0,$$

$$\mathcal{J}^{j,\nu}(t, x; \xi)\sim \sum_{s=0}^{\infty}\mathcal{J}_{-s}^{j,\nu}(t, x; \xi), \qquad (\nu=k, l),$$

where $\mathcal{D}_{-s}^j$ *and* $\mathcal{J}_{-s}^{j,\nu}$ *are homogeneous of degree* $-s$ *in* ξ.

Proof. If $\operatorname{rank} C_1^j(t, x; \lambda_j(t, x; \xi), \xi)=0$ in Ω_ν^j, $\mathcal{J}^{j,\nu}$ may be I, and if $\operatorname{rank} C_1^j(t, x; \lambda_j(t, x; \xi), \xi)=1$ in Ω_ν^j, the existence of $\mathcal{J}^{j,\nu}$ is guaranteed by Theorem 4.1. When $\varepsilon_j\neq 0$, the condition (L) implies that the type of $\mathcal{D}_0^j$ is $\begin{bmatrix}* & *\\ 0 & *\end{bmatrix}$. (See, for example, H. Yamahara [25].) Q. E. D.

§ 5. Theorems for the hyperbolicity in the domain of higher dimension.

In this section, we establish two theorems on the hyperbolicity of P in Ω under the condition (L), supposing the assumptions 1, 2 and an additional one, where Ω is a domain in $\boldsymbol{R}^{n+1}$ $(n\geqq 2)$.

5.1° Case with real analytic coefficients.

Theorem 5.1. (Hyperbolicity in the case with real analytic coefficients.)

Suppose the assumption 1, *and the real analyticity and the boundedness of the coefficients of* P_p *and suppose that the projection of the support of the discontinuity of* $R_j(t, x, \xi)$ *to* Ω *is a family of space-like hypersurfaces in* Ω *with respect to* $D_t-\lambda_j(t, x; D_x)$, $(1\leqq j\leqq r)$, (*except their singular points*). *Then* P *is hyperbolic in* Ω *under the condition* (L).

Here, the loss of regularity in Ω *is* 1 *if the case II arises for some* j *and it is* 0 *if the case II does not arise for any* j, $(1\leqq j\leqq r)$.

Remark. For the local hyperbolicity in Ω, we need not the boundedness of the coefficients of P_p.

By virtue of Theorem 4.3, we can obtain the above result by the same way as the proof of V. M. Petkov [19], [20] or H. Yamahara [25]. (H. Yamahara set the stronger assumptions than those of V. M. Petkov, but they can be relaxed to Petkov's through a little more precise consideration.)

5.2° Case with coefficients in C^∞-class.

If the coefficients of P_p belong only to C^∞-class, we need some additional conditions besides (L) for the hyperbolicity of P in Ω. (See the section 3.) Here, we only propose the theorem corresponding to Theorem 3.2.

First, we state a proposition on the finite propagation under the assumptions 1, 2, and 3.

Assumption 3. $\{T^j_l\}_l$ in the assumption 2 does not accumulate on arbitrary compact set K in Ω for each j, that is,

(4) There exists a positive constant δ_K such that

$$\mathrm{dist}(T^j_k\cap K,\ T^j_l\cap K)\geqq\delta_K, \qquad \text{if } k\neq l,\quad (1\leqq j\leqq r).$$

Remark. Under the assumption 3, $\Sigma^j=\bigcup_l T^j_l$.

Proposition 5.2. (Finite propagation.)

Under the assumption 1, 2 *and* 3, *if the condition* (L) *is satisfied, the solution of the Cauchy problem* (1)—(2) *has a finite propagation speed* $\lambda_{\max}(t, x)=\max_{\xi,j}|\lambda_j(t, x;\xi)|$, *where* ξ *and* j *run over the unit sphere and* $1\leqq j\leqq s$, *respectively.*

Proof. Let $\{\Omega^j_k\}_k$ be the family of the connected subdomains of Ω devided by $\bigcup_j\{T^j_l\}_l$. In order to obtain the proposition, we need establish only the local uniqueness at each point in Ω. On the other hand, obviously, the propagation speed is $\lambda_{\max}(t, x)$ in each $\overline{\Omega_k^j}$. (See H. Yamahara [25].)

Now, let $(t_\circ, x_\circ)$ be a point on $\bigcup_j \{T_l^j\}_l$ and let $T_1, \cdots, T_\mu$ be the elements of $\bigcup_j \{T_l^j\}_l$ which pass on $(t_\circ, x_\circ)$. Here, μ is at most r by the assumption 2 and there exists a lense-shaped neighbourhood ω of $(t_\circ, x_\circ)$ which contains only $T_1, \cdots, T_\mu$ by the assumption 3. Let $\hat{T}_i$ be the piecewise smooth hypersurface which is composed by the i-th piece of $\{T_i \cap \Omega_{t_\circ}^+\}_i \cup \{t=t_\circ\}$. Obviously, $\hat{T}_1 = \{t=t_\circ\}$. If there is not the i-th piece, we adopt the $(i-1)$-th piece. (See the figure 7.)

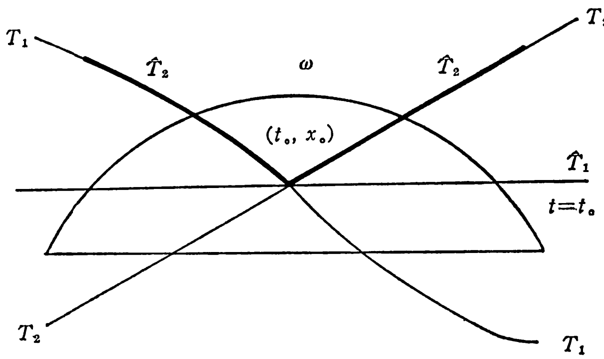

Figure 7.

If $u_\circ(x)=0$ on Ω_{t_0} and $f(t, x)=0$ in Ω, each of the solutions $u(t, x)$ vanishes on the closures of the domains surrounded by $\hat{T}_1$, $\hat{T}_2$ and $\partial\omega$, because the propagation speed in each $\overline{\Omega_k^j}$ is $\lambda_{\max}(t, x)$, and $\hat{T}_i$ and $\partial\omega$ are piecewisely space-like. Therefore, especially, $u(t, x)$ vanishes on $\hat{T}_2 \cap \bar{\omega}$. This implies that $u(t, x)$ vanishes on the closures of the domains surrounded by $\hat{T}_2$, $\hat{T}_3$ and $\partial\omega$. Thus, step by step, we can see $u(t, x)=0$ on $\overline{\omega_{t_0}^+}$. Q.E.D.

Now, we are in a position to state the theorem.

Theorem 5.3. (Hyperbolicity in the case with C^∞-coefficients.)
Suppose that the coefficients of P_p belong to $C^\infty(\Omega) \cap B^0(\Omega)$ and that the assumptions 1, 2 *and* 3 *are satisfied. Then, P is hyperbolic in Ω under the condition* (L).

Here, the loss of regularity on K from $t_\circ$ is at most $\max_j \sharp\{\Omega_k^j;\ \Omega_k^j \cap K_{t_0}^+ \neq \varnothing$ *and* $R_j(t, x, \xi)=N-1$ *on* $T^*(\Omega_k^j)\setminus\{O\}\}$, *where K is lense-shaped.*

Remark. For the local hyperbolicity, we need not the boundedness of the coefficients of P_p in Ω.

5.3° Proof of Theorem 5.3 —Reduction—.

For convenience sake, we set $\Omega = I_\circ \times \boldsymbol{R}^n$ $(\equiv [t_1, t_2] \times \boldsymbol{R}^n)$. Since the principle of superposition holds good by virtue of Proposition 5.2 and the boundedness of

$\bar{\lambda}_{\max}=\sup_{(t,x)\in\Omega}\lambda_{\max}(t, x)$, we only need show the solvability of the Cauchy problem (1)—(2) in $[t_\circ, t_\circ+\varepsilon]\times \boldsymbol{R}^n$ for arbitrary $u_\circ(x)\in\mathcal{D}(K)$, $f(t, x)\in\mathcal{E}_t(I_\circ; \mathcal{D}(K))$ and $t_\circ\in I_\circ$, where K is an arbitrary compact set and ε is independent of the choice of $t_\circ$. We take a compact set K' which satisfies the relations $K'\supset K$ and $\mathrm{dist}(K, K')>\bar{\lambda}_{\max}\cdot(t_2-t_1)$.

By virtue of the assumption 3, we can take a positive constant $\delta_\circ$ such that the distance between arbitrary $T_k^j\cap(I_\circ\times K')$ and $T_l^j\cap(I_\circ\times K')$ is at least $4\delta_\circ$. Then, for arbitrary point $(t_\circ, x_\circ)\in I_\circ\times K'$, $\omega_\circ=I'_\circ\times\theta_\circ$ $(\equiv[t_\circ-\delta_\circ, t_\circ+\delta_\circ]\times\{x\,|\,|x-x_\circ|<\delta_\circ\})$ intersects at most one of $\{T_l^j\}_l$, $(1\leqq j\leqq r)$. On the other hand, we can take a positive $\delta'_\circ$ such that Lemma 4.6 and Corollary 4.9 hold good for a finite $\{\Gamma_i\}$ in $\omega'_\circ=[t_\circ-\delta'_\circ, t_\circ+\delta'_\circ]\times\{x\,|\,|x-x_0|<\delta'_\circ\}$, where $\delta'_\circ$ depends only on K'. We set $\delta=\min\{\delta_\circ, \delta'_\circ\}$, $\delta_1=\delta/10$, $\varepsilon_1=\delta_1(\bar{\lambda}_{\max})^{-1}$, $I_1=[t_\circ-\varepsilon_1, t_\circ+\varepsilon_1]$ and $\omega\equiv I_1\times\theta=I_1\times\{x\,|\,|x-x_\circ|<\delta\}$. Moreover, we take K_i $(0\leqq i\leqq 5)$ in $\boldsymbol{R}^n$ such that $K_i\subset K_{i+1}$ and $\mathrm{dist}(K_i, K_{i+1})=\delta_1$ $(0\leqq i\leqq 4)$, where $K_5=\theta$. By virtue of the principle of superposition and the compactness of K, we only need consider the Cauchy problem in $I_1\times\boldsymbol{R}^n$ for the data $u_\circ(x)$ in $\mathcal{D}(\theta)$ and the right-hand side $f(t, x)$ in $\mathcal{E}_t(I_\circ; \mathcal{D}(\theta))$.

We seek for a Fourier integral operator $U(t, s; x, y, D_y)$ of size $N\times N$ which satisfies

$$(5.1)\qquad \begin{cases} P(t, x; D_t, D_x)\cdot U(t, s; x, y, D_y)\equiv 0, \quad \text{mod. } S^{-\infty}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (t_\circ\leqq s\leqq t\leqq t_\circ+\varepsilon_1), \\ U(s, s; x, y, D_y)=\zeta(x)I, \end{cases}$$

and

$$(5.2)\qquad \mathrm{supp}_{(t,x)}U(t, s; x, y, D_y)\subseteqq\{(t, x)\,|\,\mathrm{dist}(x, K_4)\leqq\bar{\lambda}_{\max}\cdot(t-s),\ t\geqq s\},$$

where $\zeta(x)\in C_0^\infty(K_4)$ and $\zeta(x)\equiv 1$ on K_3. By the property (5.2), we can modify the coefficients of P out of $I_\circ\times K'$. We can find a modification such that the assumption 1, 2 and 3 are kept correct in Ω and the coefficients are independent of x outside a compact set. Of course, the condition (L) may be violated out of $I_\circ\times K'$. We follow the process in H. Kumano-go [13]. We write P in order to express the modified P. Now, for each $(t_\circ, x_\circ)\in I_1\times\boldsymbol{R}^n$, we can take $\mathcal{B}(t, x; \xi)$ in $\omega\times\Gamma_i$ and $\mathcal{G}^{j,\nu}(t, x; \xi)$ in $(\omega\cap\overline{\Omega_\nu^j})\times\Gamma_i$.

Let $\{\eta_i(\xi)\}_i$ be a partition of the unity belonging to $\{\Gamma_i\}$, which is homogeneous of degree 0. We construct the solution U through the principle of superposition with respect to the following U_i.

$$(5.1')\qquad \begin{cases} P(t, x; D_t, D_x)\cdot U_i(t, s; x, y, D_y)\equiv 0, \quad \text{mod. } S^{-\infty}, \\ U_i(s, s; x, y, D_y)=I\zeta(x)\eta_i(D_y). \end{cases}$$

From now on, we omit the suffix i. Let us extend $\mathcal{B}(t, x; \xi)$ out of Γ as it is regular and homogeneous of degree 0 in ξ on $I_1\times\boldsymbol{R}^n\times\boldsymbol{R}^n\setminus\{O\}$. Set

$$(5.3)\qquad U(t, s; x, y, D_y)\equiv\mathcal{B}(t, x; D_x)\cdot V(t, s; x, y, D_y), \quad \text{mod. } S^{-\infty}.$$

We have the following:

$$\mathcal{C}(t,\ x\ ;\ D_t,\ D_x)\cdot V(t,\ s\ ;\ x,\ y,\ D_y)\equiv 0\ ,\quad \mathrm{mod.}\ S^{-\infty}. \tag{5.4}$$

(5.4) is equivalent to the following (5.4′).

$$\mathcal{C}^j(t,\ x\ ;\ D_t,\ D_x)\cdot V^j(t,\ s\ ;\ x,\ y,\ D_y)\equiv 0\ ,\quad \mathrm{mod.}\ S^{-\infty},\quad (1\leqq j\leqq s), \tag{5.4$'$}$$

where $V^j={}^t((2j-1)$-th row vector of V, $2j$-th row vector of $V)$, $(1\leqq j\leqq r)$, and $V^j=(j+r)$-th row vector of V, $(r+1\leqq j\leqq s)$. Here, we consider only the case where $1\leqq j\leqq r$ since the case where $r+1\leqq j\leqq s$ is easy.

5.4° Determination of W^j.

Let us extend $\mathcal{G}^{j,\nu}$ suitably out of $(\omega\cap\overline{\Omega_\nu^j})\times\Gamma$ as it becomes regular and homogeneous of degree 0 in ξ on $I_1\times \boldsymbol{R}^n\times\boldsymbol{R}^n\backslash\{O\}$. Setting

$$V^j(t,\ s\ ;\ x,\ y,\ D_y)\equiv\mathcal{G}^{j,\nu}(t,\ x\ ;\ D_x)\cdot W^j(t,\ s\ ;\ x,\ y,\ D_y)\ ,\quad \mathrm{mod.}\ S^{-\infty}, \tag{5.5}$$

we obtain the following:

$$\mathcal{D}^j(t,\ x\ ;\ D_t,\ D_x)\cdot W^j(t,\ s\ ;\ x,\ y,\ D_y)\equiv 0\ ,\quad \mathrm{mod.}\ S^{-\infty}. \tag{5.6}$$

Let the symbol of $W^j(t,\ s\ ;\ x,\ y,\ D_y)$ be

$$\begin{cases} W^j(t,\ s\ ;\ x,\ y,\ \xi)=w^j(t,\ s\ ;\ x,\ \xi)\exp[i\phi_j(t,\ s\ ;\ x,\ \xi)-iy\cdot\xi]\ , \\ w^j(t,\ s\ ;\ x,\ \xi)\sim\sum_{m=-1}^{\infty} w_{-m}^j(t,\ s\ ;\ x,\ \xi)\ , \end{cases} \tag{5.7}$$

where $w_{-m}^j(t,\ s\ ;\ x,\ \xi)$ is homogeneous of degree $-m$ in ξ, and where $\phi_j(t,\ s\ ;\ x,\ \xi)$ is real and homogeneous of degree 1 in ξ. $\phi_j(t,\ s\ ;\ x,\ \xi)$ is defined by

$$\begin{cases} \dfrac{\partial}{\partial t}\phi_j-\lambda_j(t,\ s\ ;\ x,\ \nabla_x\phi_j)=0\ , \\ \phi_j(s,\ s\ ;\ x,\ \xi)=x\cdot\xi\ . \end{cases} \tag{5.8}$$

There is a positive constant ε $(\leqq\varepsilon_1)$, independent of s and j, such that the solution of (5.8) exists on $I\times\boldsymbol{R}^n=[t_\circ-\varepsilon,\ t_\circ+\varepsilon]\times\boldsymbol{R}^n$ for arbitrary $s\in I$ and arbitrary $t_\circ\in I_\circ$. (See, for example, H. Kumano-go [13].)

For a while, we omit the variable s. We write ∇ instead of ∇_x and ∂_t instead of $\partial/\partial t$. Let us set

$$h_j(t\ ;\ y,\ x,\ \xi)\equiv h_j(t,\ s\ ;\ y,\ x,\ \xi)=\phi_j(t\ ;\ y,\ \xi)-\phi_j(t\ ;\ x,\ \xi)-(y-x)\cdot\nabla\phi_j(t\ ;\ x,\ \xi)\ . \tag{5.9}$$

By L. Hörmander [6], [7], we have

$$\begin{aligned} \mathrm{e}^{-i\phi_j}\sigma(\mathcal{D}^jW^j)&\sim\sum_{k,m}\mathcal{D}_{-k}^j(t,\ x\ ;\ \partial_t\phi_j,\ \nabla\phi_j)w_{-m}^j(t\ ;\ x,\ \xi)+\sum_m D_tw_{-m}^j(t\ ;\ x,\ \xi) \\ &\quad+\sum_{k,m}(1/\alpha\,!)\mathcal{D}_{-k}^{j\,(\alpha)}(t,\ x\ ;\ \nabla\phi_j)(w_{-m}^j(t\ ;\ y,\ \xi)\mathrm{e}^{ih_j(t;y,x,\xi)})_{(\alpha)}|_{y=x}\ , \\ &\qquad\qquad\qquad\qquad (k\geqq-1,\ m\geqq-1,\ |\alpha|\geqq1) \\ &\sim\mathcal{D}_1^j(t,\ x\ ;\ \partial_t\phi_j,\ \nabla\phi_j)w_1^j(t\ ;\ x,\ \xi)+D_tw_1^j(t\ ;\ x,\ \xi) \\ &\quad+\sum_{i=1}^{n}\mathcal{D}_1^{j\,(i)}(t,\ x\ ;\ \nabla\phi_j)D_{x_i}w_1^j(t\ ;\ x,\ \xi)+\mathcal{D}_0^j(t,\ x\ ;\ \nabla\phi_j)w_1^j(t\ ;\ x,\ \xi) \end{aligned} \tag{5.10}$$

$$+\sum_{|\alpha|=2}(\sqrt{-1}/\alpha!)\mathcal{D}_1^{j(\alpha)}(t,\ x\ ;\nabla\phi_j)\phi_{j(\alpha)}(t\ ;\ x,\ \xi)w_1^j(t\ ;\ x,\ \xi)$$

$$+\mathcal{D}_1^j(t,\ x\ ;\partial_t\phi_j,\ \nabla\phi_j)w_0^j(t\ ;\ x,\ \xi)+\cdots.$$

Here, $\mathcal{D}_1^j(t,\ x\ ;\partial_t\phi_j,\ \nabla\phi_j)=\begin{bmatrix}0 & -\varepsilon_j(t,\ x\ ;\nabla\phi_j)\\ 0 & 0\end{bmatrix}$ in $(\omega\cap\overline{\Omega_\nu^j})\times\Gamma$, but, regarding that the above equality is valid on $I_1\times\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash\{O\}$, we solve (5.10)$\sim$0, which is equivalent to (5.6). By virtue of (5.1$'$) and (5.3), the initial data of V is given by the following:

$$\text{(5.11)}\qquad \begin{cases} v_1(s\ ;\ x,\ \xi)=0\ , \quad v_0(s\ ;\ x,\ \xi)=\zeta(x)\eta(\xi)\mathcal{B}_0^{-1}(s,\ x\ ;\ \xi)\ ,\\ v_{-m}(s\ ;\ x,\ \xi)=\mathcal{B}_0^{-1}(s,\ x\ ;\ \xi)G_m(s,\ x,\ \xi\ ;\ v_{-l}\ ;\ -1\leqq l\leqq m-1)\ ,\end{cases}$$

where $V(s\ ;\ x,\ y,\ \xi)\sim\sum_{m=-1}^{\infty}v_{-m}(s\ ;\ x,\ \xi)e^{ix\cdot\xi-iy\cdot\xi}$, and G_m is a function of $s,\ x,\ \xi$ and v_{-l}, $(-1\leqq l\leqq m-1)$. (5.5) and (5.11) bring the following:

$$\text{(5.12)}\qquad \begin{cases} w_1^j(s\ ;\ x,\ \xi)=0\ , \quad w_0^j(s\ ;\ x,\ \xi)=\zeta(x)\eta(\xi)\ (\mathcal{G}_0^{j,\nu})^{-1}(\mathcal{B}_0^{-1})^j\ ,\\ w_{-m}^j(s\ ;\ x,\ \xi)=(\mathcal{G}_0^{j,\nu})^{-1}\tilde{G}_m^j(s,\ x,\ \xi\ ;\ w_{-l}\ ;\ -1\leqq l\leqq m-1)\ ,\end{cases}$$

where $(\mathcal{B}_0^{-1})^j$ is $\begin{array}{c} \qquad\qquad\ \ {\scriptstyle 2j-1}\ \ {\scriptstyle 2j} \\ \begin{bmatrix}0\cdots0 & 1 & 0 & 0\cdots0\\ 0\cdots0 & 0 & 1 & 0\cdots0\end{bmatrix}\end{array}\mathcal{B}_0^{-1}$ and where $\tilde{G}_m^j$ is of size $2\times N$ and is a function of $s,\ x,\ \xi$ and w_{-l}^j, $(-1\leqq l\leqq m-1)$. From now on, we omit the suffix j.

(i) *The case of* $\varepsilon\equiv0$ *in* $(\omega\cap\Omega_\nu)\times\Gamma$.

We regard thet $\varepsilon\equiv0$ on $I\times\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash\{O\}$, then (5.10)$\sim$0 is equivalent to

$$\text{(5.13)}\qquad \begin{aligned}&D_tw_{-m}-\sum_{i=1}^{n}\lambda^{(i)}D_{x_i}w_{-m}+\mathcal{D}_0w_{-m}-\sum(\sqrt{-1}/\alpha!)\lambda^{(\alpha)}\phi_{(\alpha)}w_{-m}\\ &=F_m(t,\ x,\ \xi\ ;\ \phi\ ;\ w_{-l}\ ;\ -1\leqq l\leqq m-1)\ ,\quad (m\geqq-1),\end{aligned}$$

where F_m is a function of $t,\ x,\ \xi,\ \phi$ and w_{-l}, $(-1\leqq l\leqq m-1)$. (5.13) is solvable with the initial data (5.12). Obviously, $w_1(t\ ;\ x,\ \xi)\equiv0$, and $\mathrm{supp}_x w_{-m}(t\ ;\ x,\ \xi)$ is contained in $\tilde{K}_4=\{(t,\ x)\,|\,\mathrm{dist}(x,\ K_4)\leqq\bar{\lambda}_{\max}\cdot(t-s)\}$ because the support in x of $w_{-m}(s\ ;\ x,\ \xi)$ is contained in K_4 and the support of the right-hand side is contained in $\tilde{K}_4$, $(m\geqq0)$[1]. Therefore,

$$\mathrm{supp}\, w_{-m}(t\ ;\ x,\ \xi)\subseteqq\tilde{K}_4\times\Gamma\subset I\times K_5\times\Gamma.$$

(ii) *The case of* $\varepsilon\neq0$ *in* $(\omega\cap\Omega_\nu)\times\Gamma$.

$\mathcal{D}_1(t,\ x\ ;\partial_t\phi,\ \nabla\phi)$ and $\mathcal{D}_0(t,\ x\ ;\nabla\phi)$ have the forms

1) (5.13) has the propagation speed $\lambda_{\max}(t,x)$ $(\leqq\bar{\lambda}_{\max})$, since $\lambda_{\max}(t,x)=\max_{\xi,j}|\Gamma_\xi\lambda_j(t,x\,;\xi)|$, where ξ and j run over the unit sphere in $\boldsymbol{R}_\xi^n$ and $1\leqq j\leqq s$, respectively.

$$\begin{bmatrix} 0 & -\varepsilon(t,\ x\ ;\ \nabla\phi) \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} d_1(t,\ x\ ;\ \nabla\phi) & d_2(t,\ x\ ;\ \nabla\phi) \\ 0 & d_4(t,\ x\ ;\ \nabla\phi) \end{bmatrix},$$

respectively, in $(\omega\cap\Omega_\nu)\times\Gamma$. We extend them in $I\times\boldsymbol{R}^n\times\boldsymbol{R}^n\backslash\{O\}$, holding the above forms. We also extend $\mathcal{D}_{-j}(t,\ x\ ;\ \xi)$ there, which is smooth and homogeneous of degree $-j$ in ξ, $(j\geqq 1)$. Let $\mathcal{D}_{-1}(t,\ x\ ;\ \nabla\phi)$ be $\begin{bmatrix} * & * \\ \rho(t,\ x\ ;\ \nabla\phi) & * \end{bmatrix}$ and let us put $w_{-m}={}^t(w^1_{-m},\ w^2_{-m})$, where w^i_{-m} is of size $1\times N$, $(i=1,\ 2)$. Then, (5.10)$\sim$0 is equivalent to the following:

$$(5.14)_1 \qquad D_t w^1_{-m+1}-\sum_{i=1}^n \lambda^{(i)} D_{x_i} w^1_{-m+1}-\sum_{|\alpha|=2}(\sqrt{-1}/\alpha\,!)\lambda^{(\alpha)}\phi_{(\alpha)} w^1_{-m+1}+d_1 w^1_{-m+1}-\varepsilon w^2_{-m}$$
$$=F^1_{m-1}(t,\ x\ ;\ \phi\ ;\ w^i_{-l},\ w^2_{-m+1}\ ;\ i=1,\ 2,\ -1\leqq l\leqq m-2),$$

$$(5.14)_2 \qquad D_t w^2_{-m}-\sum_{i=1}^n \lambda^{(i)} D_{x_i} w^2_{-m}-\sum_{|\alpha|=2}(\sqrt{-1}/\alpha\,!)\lambda^{(\alpha)}\phi_{(\alpha)} w^2_{-m}+d_4 w^2_{-m}+\rho w^1_{-m+1}$$
$$=F^2_m(t,\ x\ ;\ \phi\ ;\ w^i_{-l},\ w^2_{-m+1}\ ;\ i=1,\ 2,\ -1\leqq l\leqq m-2),$$

where F^1_{m-1} and F^2_m are the functions of t, x, ϕ, w^i_{-l} and w^2_{-m+1} $(i=1,\ 2,$ $-1\leqq l\leqq m-2)$, $(m\geqq 0)$. Let us set $w_{\langle m\rangle}={}^t(w^1_{-m+1},\ |\xi|w^2_{-m})$, $w^2_1\equiv 0$ and $F_{\langle m\rangle}={}^t(F^1_{m-1},\ |\xi|F^2_m)$. $(5.14)_1$ and $|\xi|\times(5.14)_2$ are equivalent to

$$(5.15) \qquad D_t w_{\langle m\rangle}-\sum_{i=1}^n \lambda^{(i)} D_{x_i} w_{\langle m\rangle}$$
$$+\begin{bmatrix} d_1-\Sigma(\sqrt{-1}/\alpha\,!)\lambda^{(\alpha)}\phi_{(\alpha)} & -\varepsilon|\xi|^{-1} \\ \rho|\xi| & d_4-\Sigma(\sqrt{-1}/\alpha\,!)\lambda^{(\alpha)}\phi_{(\alpha)} \end{bmatrix} w_{\langle m\rangle}$$
$$=F_{\langle m\rangle}, \qquad (m\geqq 0).$$

(5.15) is inductively solvable with the initial data (5.12). Obviously, the support of $w_{\langle m\rangle}(t\ ;\ x,\ \xi)$ is contained $\tilde{K}_4\times\Gamma$ $(\subset I\times K_5\times\Gamma)$.

5.5° Determination of V^j on $I\times K_5\times\boldsymbol{R}^n\backslash\{O\}$.

We get the solution of (5.10), which depends on $\mathcal{G}^\nu$, in the previous section. However, since the validity of $\mathcal{G}^\nu$ is guaranteed only on $(\omega\cap\overline{\Omega_\nu})\times\Gamma$, the existence of the solution of (5.4′) is guaranteed only on the shaped portions $\overline{M_k}$ and $\overline{M_l}$ in the figure 8 by the relation (5.5) and the propagation speed $\bar{\lambda}_{\max}$. Let us set on $\overline{M_\nu}$ $(\nu=k,\ l)$,

$$(5.16) \qquad e^{-i\phi}v^\nu(t\ ;\ x,\ \xi)\sim\Sigma(1/\alpha\,!)\mathcal{G}^\nu_{-m}{}^{(\alpha)}(t,\ x\ ;\ \nabla\phi)[w_{-l}(t\ ;\ y,\ \xi)e^{ih(t;y,x,\xi)}]_{(\alpha)}|_{y=x},$$

and

$$(5.17) \qquad v^\nu(t\ ;\ x,\ \xi)\sim\sum_{m=-1}^{\infty} v^\nu_{-m}(t\ ;\ x,\ \xi),$$

where v^ν_{-m} is homogeneous of degree $-m$ in ξ. Here, all of the derivatives of

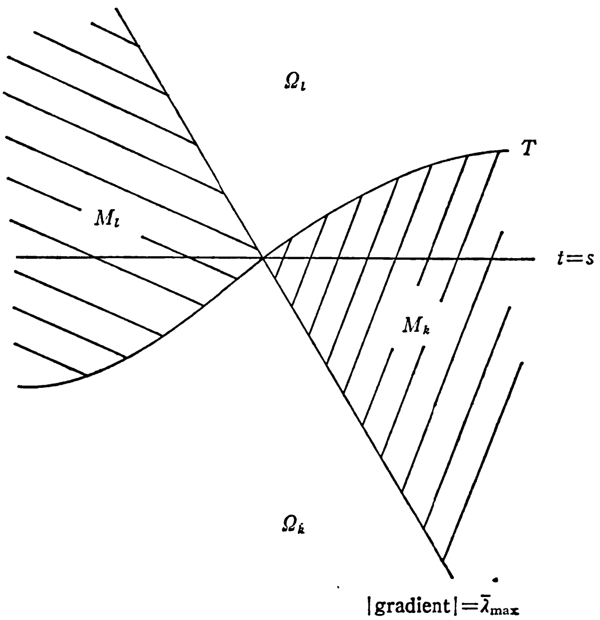

Figure 8.

$v^{\nu}_{-m}(t\,;x,\xi)$ $(\nu=k,l)$ coincide each other on $(T\cap\{s\}\times\boldsymbol{R}^n)\times\boldsymbol{R}^n\backslash\{O\}$ because $v^{\nu}_{-m}(t\,;x,\xi)$ satisfies the relation (5.11) on $(\overline{M_\nu}\cap\{s\}\times\boldsymbol{R}^n)\times\boldsymbol{R}^n\backslash\{O\}$ and the relation (5.4′) on $\overline{M_\nu}\times\boldsymbol{R}^n\backslash\{O\}$. Let us set

$$v^{j}_{-m}(t\,;x,\xi)=v^{j,\nu}_{-m}(t\,;x,\xi)\quad\text{on}\quad T\cap\overline{M_\nu},\quad(m\geqq-1,\ \nu=k,l). \tag{5.18}$$

We solve once more the equation (5.4′) with the initial data (5.18) in $\overline{\Omega_\nu}\times\boldsymbol{R}^n\backslash\{O\}$ $(\nu=k,l)$, regarding $T\times\boldsymbol{R}^n\backslash\{O\}$ as the initial surface and reducing (5.4′) to (5.6). Here, we need start, in general, from $v^j_2(t\,;x,\xi)$, that is,

$$V^j(t\,;x,y,\xi)\sim\sum_{m=-2}^{\infty}v^{j}_{-m}(t\,;x,\xi)e^{i\phi_j(t;x,\xi)-iy\cdot\xi}, \tag{5.19}$$

because $v^j_1(t\,;x,\xi)$ may not vanish on $T\times\boldsymbol{R}^n\backslash\{O\}$. We put

$$I_j={}^t\begin{array}{c}\begin{matrix} & 2j-1 & 2j & \end{matrix}\\ \begin{bmatrix}0\cdots0 & 1 & 0 & 0\cdots0\\ 0\cdots0 & 0 & 1 & 0\cdots0\end{bmatrix}\end{array},\quad(1\leqq j\leqq r),$$

$I_j={}^t(0\cdots0\ \overset{j+r}{1}\ 0\cdots0)$, $(r+1\leqq j\leqq s)$, and $\tilde{v}_j=I_jv_j(t\,;x,\xi)$. Let us set

$$e^{-i\phi_j}u^j(t\,;x,\xi)\sim\sum(1/\alpha!)\mathscr{B}_{-m}{}^{(\alpha)}(t,x\,;\nabla\phi)[\tilde{v}^j_{-l}(t\,;y,\xi)e^{ih_j(t;y,x,\xi)}]_{(\alpha)}|_{y=x}, \tag{5.20}$$

where m, l, and α run over $m\geqq0$, $l\geqq-2$ and $|\alpha|\geqq0$, respectively. Here, we take a symbol u^j $(\equiv u^{j,i})$ whose asymptotic expansion is given by (5.20) and whose support is contained $I\times K_5\times\Gamma_i$. Thus, we get the solution $U(t,s\,;x,y,D_y)$

of (5.1);

$$U(t, s; x, y, D_y)=\sum_i \sum_{j=1}^{s} u^{j,i}(t; x, D_y)_{\phi_j}, \tag{5.21}$$

where $\mathrm{supp}_{(t,x)}U(t, s; x, y, \xi)\subseteqq I\times K_5$ and the order of U may be 2.

5.6° Construction of the fundamental matrix.

Let us construct the fundamental matrix $\tilde{U}(t, s; x, y, D_y)$, which satisfies

$$\begin{cases} P(t, x; D_t, D_x)\cdot\tilde{U}(t, s; x, y, D_y)=0, \\ \tilde{U}(s, s; x, y, D_y)=\zeta(x)I, \end{cases} \tag{5.22}$$

by the same method as in H. Kumano-go [13]. (See also Ch. Tsutsumi [22].)

Let us put

$$P(t, x; D_t, D_x)\cdot U(t, s; x, y, D_y)=R(t, s; x, y, D_y)\equiv R(t, s)\in S^{-\infty}, \tag{5.23}$$

$$R_1(t, s)=-\sqrt{-1}\,R(t, s), \tag*{(5.24)$_1$}$$

and inductively,

$$R_k(t, s)=\int_s^t R_1(t, \tau; x, x', D_{x'})\cdot R_{k-1}(\tau, s; x', y, D_y)d\tau \qquad (k\geqq 2). \tag*{(5.24)$_k$}$$

At last, we set

$$\tilde{R}(t, s)=\sum_{k=1}^{\infty} R_k(t, s). \tag{5.25}$$

The right-hand side of (5.25) converges and satisfies

$$\tilde{R}(t, s)=-\sqrt{-1}\,R(t, s)-\sqrt{-1}\int_s^t R(t, \tau; x, x', D_{x'})\cdot\tilde{R}(\tau, s; x', y, D_y)d\tau, \tag{5.26}$$

and

$$\mathrm{supp}_{(t,x)}\tilde{R}(t, s)\subseteqq\tilde{K}_4. \tag{5.27}$$

Now, we set

$$U^\infty=\int_s^t U(t, \tau; x, x', D_{x'})\cdot\tilde{R}(\tau, s; x', y, D_y)d\tau, \tag{5.28}$$

and

$$\tilde{U}=U+U^\infty. \tag{5.29}$$

Then, $\tilde{U}(t, s; x, y, D_y)$ satisfies

$$\begin{cases} P\cdot\tilde{U}=\sqrt{-1}(1-\zeta(x))\tilde{R}(t, s; x, y, D_y), \\ \tilde{U}(s, s; x, y, D_y)=\zeta(x)I, \end{cases} \tag{5.30}$$

and

$$\mathrm{supp}_{(t,x)}\tilde{U}(t, s; x, y, D_y)\subset I\times K_5. \tag{5.31}$$

Let us take $u_{\circ}(x)$ in $\mathcal{D}(K_{\circ})$ and $f(t, x)$ in $\mathcal{E}_t(I; \mathcal{D}(K_{\circ}))$, and let us set

(5.32) $$\tilde{u}(t, x)=\tilde{U}(t, t_{\circ}; x, y, D_y)u_{\circ}(y)+\sqrt{-1}\int_{t_{\circ}}^{t}\tilde{U}(t, s; x, y, D_y)f(s, y)ds.$$

$u(t, x)$ satisfies the following:

(5.30′) $$\begin{cases} P\tilde{u}=f+\sqrt{-1}(1-\zeta)\left\{\tilde{R}u_{\circ}+\int_{t_{\circ}}^{t}\tilde{R}f\,d\,s\right\}\equiv\tilde{f}, \\ \tilde{u}(t_{\circ}, x)=u_{\circ}(x), \end{cases}$$

where $\operatorname{supp}\tilde{f}\subset I\times[K_{\circ}\cup(\overline{K_3^c}\cap K_5)]$.

By virtue of Proposition 5.2 and (5.31), we see the following:

(5.33) $$\operatorname{supp}\tilde{u}(t, x)\subset I\times[K_1\cup\overline{K_2^c}\cap K_5)].$$

Let us take $\gamma(x)\in C_0^{\infty}(K_2)$ which satisfies $\gamma(x)\equiv 1$ on K_1, and let us set

(5.34) $$u(t, x)=\gamma(x)\tilde{u}(t, x).$$

$u(t, x)$ satisfies (1) and (2) because the differential operator does not widen the support of the functions. Therefore, $\gamma(x)\tilde{U}(t, s; x, y, D_y)$ is the exact fundamental matrix for $u_{\circ}(x)$ in $\mathcal{D}(K_{\circ})$ and $f(t, x)$ in $\mathcal{E}_t(I; \mathcal{D}(K_{\circ}))$. Q.E.D.

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