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Leray-Volevich's system and Gevrey class

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§1. Introduction

We consider the Cauchy problem for hyperbolic systems with multiple characteristics of constant multiplicity. Let Ω be a band $[0, T]\times R^n$ in R^{n+1}. We consider the following equations in Ω,

$$\sum_{q=1}^{N} a_q^p(x, D)u^q(x)=f^p(x), \qquad p=1, \cdots, N, \tag{1.1}$$

where $x=(x_0, x_1, \cdots, x_n)=(x_0, x')\in\Omega$ and $a_q^p(x, D)$ differential operators of order m_q^p of which coefficients are in the Gevrey class $\gamma^s(\Omega)(s\geqq 1)$.

We use the notation as follows,

$$D=(D_0, \cdots, D_n), \qquad D_k=-\sqrt{-1}\frac{\partial}{\partial x_k},$$

$$\alpha=(\alpha_0, \cdots, \alpha_n), \qquad \alpha_k \text{ integers},$$

$$D^\alpha=D_0^{\alpha_0}D_1^{\alpha_1}\cdots D_n^{\alpha_n}, \qquad |\alpha|=\sum\alpha_k,$$

$$\xi=(\xi_0, \xi_1, \cdots, \xi_n); \text{ dual variables of } x,$$

and $\gamma_s(\Omega)$ consists of all functions f such that there exists positive constants C and A satisfying for any α,

$$|D^\alpha f(x)|\leqq CA^{|\alpha|}|\alpha|!^s, \qquad x\in\Omega.$$

We correspond the polynomial $a_q^p(x, \xi)$ in ξ to a differential operator $a_q^p(x, D)$. We denote by $\mathring{a}_q^p(x, \xi)$ the homegeneous part of degree m_q^p of $a_q^p(x, \xi)$. We define the total order m of $\{a_q^p(x, D)\}$ such that

$$m=\max_\pi \sum_{p=1}^{N} m_{\pi(p)}^p,$$

where π runs over all permutations of $[1, \cdots, N]$. Then it follows from Volevich's lemma [16] that there exists a pair of integes $\{t_p, s_p\}$, $p=1, \cdots, N$, such that

$$\begin{aligned} &m_q^p\leqq t_q-s_p, \qquad (p, q)\in[1, \cdots, N]^2, \\ &m=\sum_{p=1}^{N}(t_p-s_p), \end{aligned} \tag{1.2}$$

where for convinience we define $m_q^p=-\infty$ if $a_q^p(x, D)\equiv 0$. We denote by $\hat{a}(x, \xi)$ the homogeneous part of degree m of det $(a_q^p(x, \xi))$. We call $\hat{a}(x, \xi)$ the characteristic polynomial of the system $a_q^p(x, D)$. The pair of weights $\{t_p, s_q\}$ satisfing the property (1.2) is not uniquely determined for $\{m_q^p\}$. But the weights $\{t_p, s_q\}$ is fixed from now on. We call the Leray-Volevich's system of weights $\{t_p, s_q\}$, a system of differential operators $a_q^p(x, D)$ of which orders satisfies the property (1.2).

We assume that the intial plane $\{x_0=0\}$ is not characteristic with respect to $a(x, \xi)$, that is, $\hat{a}(x, \xi)\neq 0$ for $\xi=(1, 0, \cdots, 0)$ and $x\in\Omega$. For the equations (1, 1) of the Leray-Volevich's system of weights $\{t_p, s_q\}$, we can give t_p numbers of the intial deta (cf. [5]),

$$D_0^h u^p|_{x_0=0}=w_h^p(x'), \qquad h=0, 1, \cdots, t_p-1. \tag{1.3}$$

If $s_p>0$, the deta $\{f^p, w_h^p\}$ must satisfy the following compatibility conditions,

$$f^p-\sum_{q=1}^N a_q^p(x, D)w^q=O(x_0^{s_p}), \qquad p=1, \cdots, N, \tag{1.4}$$

where $\{w^p\}$ are the functions in $\gamma_s(\Omega)$ such that

$$D_0^h w^p|_{x_0=0}=w_h^p, \qquad h=0, 1, \cdots, t_p-1, \quad p=1, \cdots, N. \tag{1.5}$$

The compatibility conditions (1.4) do not depend on the choice of $\{w^p\}$ satisfying (1.5). We assume that the characteristic polynomial $\hat{a}(x, \xi)$ of $\{a_q^p(x, D)\}$ is of constant multiplicity, that is,

$$\hat{a}(x, \xi)=\prod_{l=1}^d (\xi_0-\lambda^{(l)}(x, \xi'))^{m^{(l)}}, \tag{1.6}$$

where $m^{(l)}$ are constant integers and

$$\inf_{\substack{\Omega\times\{|\xi'|=1\}\\ l\neq j}} |\lambda^{(l)}(x, \xi')-\lambda^{(j)}(x, \xi')|\neq 0.$$

Then we note that the characteristic roots $\lambda^{(l)}(x, \xi')$ are in $\gamma_s(\Omega\times R^n-0)$ and in paticular analytic in ξ'. It follows from Matsuura's lemma [13] that we can factorize

$$\hat{a}(x, \xi)=a_1(x, \xi)^{\nu_1}a_2(x, \xi)^{\nu_2}\cdots a_r(x, \xi)^{\nu_r},$$

where each $a_j(x, \xi)$ and the product $a_1(x, \xi)a_2(x, \xi)\cdots(a_r(x, \xi))$ are strictly hyperblic polynomials. To study the Cauchy problem (1.1) and (1.3) with the compatibility conditions (1.4), we reduce a Leray-Valevich's system to a system with a diagonal principal part by a transformation of unknown functions. To do so, we introduce the cofactor operator of a system $\{a_q^p\}$. We denote by $h_q^p(x, \xi)$ the homogeneous part of degree t_q-s_p of $a_q^p(x, \xi)$ that is,

$$h_q^p(x, \xi)=\begin{cases} \hat{a}_q^p(x, \xi), & \text{if } m_q^p=t_q-s_p, \\ 0, & \text{if } m_q^p<t_q-s_p. \end{cases}$$

Denote by $G_q^p(x, \xi)$ the cofactor of $h_q^p(x, \xi)$. Then the degree of $G_q^p(x, \xi) \leqq m-(t_q-s_p)$ and

$$\sum_{r=1}^{N} h_r^p(x, \xi) G_q^r(x, \xi)=\delta_q^p a(x, \xi) .$$

Hence we have

$$\sum_{r=1}^{N} a_r^p(x, D) G_q^r(x, D)=\delta_q^p a(x, D)-b_p^q(x, D) ,$$

where $a(x, D)=a_1(x, D)^{\nu_1} \cdots a_r(x, D)^{\nu_r}$, each $a_i(x, \xi)$ and its products $\prod_{i=1}^{\nu} a_i(x, \xi)$ are strictly hyperbolic polynomials, and

$$\text{order } b_q^p(x, D) \leqq m-1+s_q-s_p . \tag{1.7}$$

We call the above system of differential operators a Leray-Volevich's system with diagonal principal part of constant multiplicity of order m. In the equations (1.1), we put

$$u^p(x)=\sum_{q=1}^{N} G_q^p(x, D) v^q(x), \qquad p=1, \cdots, N .$$

Then we can see that it is sufficient to solve (1.1) and (1.3) that we can solve the following Cauchy problem (c. f. [17])

$$\begin{cases} a(x, D) v^p(x)-\sum_{q=1}^{N} b_q^p(x, D) v^q(x)=f^p(x), & p=1, \cdots, N . \\ D_0^h v^p|_{x_0=0}=g_h^p(x'), & h=0,1, \cdots, m-1 . \end{cases} \tag{1.8}$$

Our aim is to construct a fundamental solution for the system (1.8). We factorize the principal part $\hat{a}(x, \xi)$ of $a(x, D)$ as follows

$$\hat{a}(x, \xi)=\prod_{l=1}^{d}(\xi_0-\lambda^{(l)}(x, \xi'))^{m^{(l)}} .$$

Denote by $\phi^{(l)}$ a phase function associated with $\lambda^{(l)}$, that is, $\phi_{x_0}^{(l)}=\lambda^{(l)}(x, \phi_{x'}^{(l)})$, $\phi_{x'}^{(l)} \neq 0$. Let $m_q^{p(l)}$ be integers satisfying for each (p, q, l),

$$e^{-i \rho \phi^{(l)}} b_q^p(x, D)(e^{i \rho \phi^{(l)}} f)=O(\rho^{m_q^{p(l)}}), \qquad \rho \rightarrow \infty . \tag{1.9}$$

That it follows from (1.7) that we have

$$m_q^{p(l)} \leqq m-1+s_q-s_p . \tag{1.10}$$

We define the rational numbers $q^{(l)}$ as

$$q^{(l)}=\max_{\pi} \sum_{p=1}^{N} m_{\pi(p)}^p / N+m^{(l)}-m , \qquad l=1, \cdots, d , \tag{1.11}$$

where π stands for a parmutation of $[1, \cdots, N]$. Then using again the Volvich's lemma, we can find the rational numbers $\{n_p^{(l)}\}$ such that

$$m_q^{p(l)} \leqq m-m^{(l)}+q^{(l)}+n_q^{(l)}-n_p^{(l)} .$$

Remark. In the case of $q^{(l)}=0(l=1, \cdots, d)$, we can solve the Cavchy problem (1.8) in the sense of C^{∞}—class (c. f. [9]). When $q^{(l)}\neq 0$, we can not solve (1.8) in C^{∞}-class. Then we treat the Cauchy problem (1.8) in the Gevrey class $\gamma_s(\Omega)$.

We return to (1.9) and expand it in a power of ρ,

$$e^{i\rho\phi^{(l)}}b_q^p(e^{i\rho\phi^{(l)}}f)=\sum_{k=0}^{m_q^{p(l)}}\rho^{m_q^{p(l)}-k}b_{qk}^{p(l)}(x, D)f, \tag{1.12}$$

where $b_{p\,k}^{p(l)}(x, D)$ is a differential operator and denote by $d_{q\,k}^{p(l)}$ it's order and without loss of generality we may put

$$m_q^{p(l)}=m-m^{(l)}+q^{(l)}+n_q^{(l)}-n_p^{(l)}.$$

Then (1.7) implies

$$m_q^{p(l)}-k+d_{q\,k}^{p(l)}\leqq m-1+s_q-s_p. \tag{1.13}$$

We define

$$d_k^{(l)}=\max_{\pi}\sum_{p=1}^{N}d_{\pi(p)\,k}^{p\,(l)}/N,$$

$$\kappa^{(l)}=\inf_{q^{(l)}-k>0}\frac{m^{(l)}-d_k^{(l)}}{q^{(l)}-k}.$$

Then from (1.13) we have

$$d_k^{(l)}\leqq m^{(l)}-1-q^{(l)}+k,$$

which implies

$$\kappa^{(l)}\geqq\inf_k\frac{q^{(l)}+1-k}{q^{(l)}-k}=\frac{q^{(l)}+1}{q^{(l)}}>1. \tag{1.14}$$

Moreover we note that (1.18) and valevich's lemma imply

$$\begin{aligned} d_{q\,k}^{p(l)}&\leqq d_k^{(l)}+s_q^{(l)}-s_p^{(l)} \\ &\leqq m^{(l)}-\kappa^{(l)}(q^{(l)}-k)+s_q^{(l)}-s_p^{(l)}, \end{aligned} \tag{1.15}$$

where $s_p^{(l)}=s_p-n_p^{(l)}$. The number $\kappa^{(l)}$ given by (1.14) is same one which is introduced in [7], [10] and [3] in the case of $N=1$. We call the fundamental solution of the Cauchy problem (1.8) a distribution satisfying

$$\begin{cases} a(x, D)K^p(x, y)=\sum_{q=1}^{N}b_q^p(x, D)K^q(x, y), & p=1, \cdots, N, \\ D_0^hK^p|_{x_0=y_0}=\delta(x'-y')\delta_{m-1}^h, & h=0, 1, \cdots, m-1. \end{cases} \tag{1.16}$$

When we regard $K^p(x_0, y_0)$ as an operator from $\gamma_s(R_{y'}^n)$ to $\gamma_s(R_{x'}^n)$, we write $K(x_0; y_0)$, that is,

$$(K^p(x_0; y_0)u)(x')=\int_{R^n}K^p(x, y)u(y')dy',$$

for $u\in\gamma_s(R^n)$.

Theorem 1.1. *Let $a(x, D)\delta_q^p+b_q^p(x, D)$ be a Leray-Volevich's system with diagonal principal part of constant multiplicity of order m and the order of b_q^p*

satisfies (1.7) *for the weight* (s_p, s_q). *We assume that the order* $d_{q}^{p(l)}{}_{k}$ *of the operator* $b_{q}^{p(l)}{}_{k}$ *given in* (1.12) *satisfies* (1.15) *and assume*

$$s_p^{(l)} - s_q^{(l)} \leqq q^{(l)}, \qquad \textit{if } q^{(l)} \neq 0 \tag{1.17}$$

for any (p, q). *Then we can construct the fundamental solution* $K^p(x_0\,;\,y_0)$ *of* (1.16) *as follows,*

$$K^p(x_0\,;\,y_0) = W^p(x_0\,;\,y_0) + \int_{y_0}^{x_0} W^p(x\,;\,t) F^p(t\,;\,y_0)\,dt\,, \tag{1.18}$$

where

$$W^p(x_0\,;\,y_0) = \sum_{l=1}^{d} \int e^{i\phi^{(l)}(x,y;\xi')} w^{p(l)}(x,\,y_0\,;\,\xi')\,d\xi'\,,$$

and $\phi^{(l)}$ *the phase function associated with* $\lambda^{(l)}$ *such that* $\phi_{x_0}^{(l)} = \lambda^{(l)}(x, \phi_{x'}^{(l)})$, $\phi^{(l)} = \langle x'-y', \xi' \rangle$ *at* $x_0 = y_0$, *and* $F^p(x_0\,;\,y_0)$ *is a solution of the integral equation,*

$$F^p(x_0\,;\,y_0) = -R^p(x_0\,;\,y_0) - \int_{y_0}^{x_0} R^p(x_0\,;\,t) F^p(t\,;\,y_0)\,dt\,, \tag{1.19}$$

here

$$\begin{aligned} R^p(x_0\,;\,y_0) &= a(x, D) W^p(x_0\,;\,y_0) - \sum_{q=1}^{N} b_q^p(x, D) W^q(x_0, y_0) \\ &= \sum_{l=1}^{d} \int e^{i\phi^{(l)}} r^{p(l)}(x,\,y_0\,;\,\xi')\,d\xi'\,, \end{aligned}$$

and the amplitude functions $w^{p(l)}(x, y_0\,;\,\xi')$ *and* $r^{p(l)}(x, y_0\,;\,\xi')$ *are estimated by*

$$|D_x^\alpha D_{\xi'}^\beta w^{p(l)}| \leqq C_1 A_1^{|\alpha+\beta|} \exp\{A_2 |x_0 - y_0|\, |\xi'|^{1/\kappa(l)}\} |\xi'|^{m_p^{(l)} - |\beta| + \alpha_0/\kappa^{(l)}} |\alpha+\beta|!^s\,, \tag{1.20}$$

$$\begin{aligned} |D_x^\alpha D_{\xi'}^\beta r^{p(l)}| \leqq{}& C_1 A_1^{|\alpha+\beta|+\mu} \exp\{A_2 |x_0 - y_0|\, |\xi'|^{1/\kappa(l)}\} |\xi'|^{m_p^{(l)} - |\beta| - \mu + \alpha_0/\kappa(l)} \\ &\times |\alpha+\beta|!^s \mu!^{m^{(l)}(s-1)+\kappa(l)s} \end{aligned} \tag{1.21}$$

for $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$ *and* $\mu = 1, 2, 3, \cdots$, *and* $m_p^{(l)} = m^{(l)} - m - n_p^{(l)} + \max_q n_q^{(l)}$.

§ 2. Asymptotic solution of fundamental solution

We note that the distribution $\delta(x'-y')$ is represented by

$$\delta(x'-y') = \frac{1}{(2\pi)^n} \int_{R^n} e^{i\langle x'-y', \xi' \rangle} d\xi'\,.$$

Then we can obtain the asymptotic solutions of (1.16) by integrating in ξ' the following solutions

$$\begin{aligned} &a(x, D) u^p(x, y\,;\,\xi') = \sum_{q=1}^{N} b_q^p(x, D) u^q(x, y\,;\,\xi')\,, \qquad p = 1, \cdots, N\,, \\ &D_0^h u^p|_{x_0 = y_0} = \frac{1}{(2\pi)^n} e^{i\langle x'-y', \xi' \rangle} \delta_{m-1}^h\,, \qquad h = 0, 1, \cdots, m-1\,, \end{aligned} \tag{2.1}$$

where δ_j^i is Kronener's delta. We seek for $u^p(x, y, \xi')$ as forms

$$u^p(x, y; \xi')=\sum_{k=0}^{\infty} u_k^p(x, y; \xi')$$

where u_0^p is a solution such that

$$\text{(2.2)} \qquad \begin{cases} a(x, D)v=0\,, \\ D_0^h v|_{x_0=y_0}=\dfrac{1}{(2\pi)^n} e^{i<x'-y', \xi'>}\delta_{m-1}^h\,, \qquad h=0, 1, \cdots, m-1\,, \end{cases}$$

and for $k\geqq 1$, u_k^p satisties

$$\text{(2.3)}_k \qquad \begin{cases} a(x, D)u_k^p=\sum_{q=1}^{N} b_q^p(x, D)u_{k-1}^q\,, \\ D_0 u_k^p|_{x_0=y_0}=0\,, \qquad h=0, 1, \cdots, m-1\,. \end{cases}$$

We construct an asymptotis solution of (2.2) as follows

$$v(x, y; \xi')=\sum_{l=1}^{d}\sum_{j=0}^{\infty} e^{i\phi^{(l)}}|\xi'|^{-(m-m^{(l)})j}v_j^{(l)}(x, y; \xi')\,,$$

$\phi^{(l)}(x, y, \xi')$ is the phase function associated with $\lambda^{(l)}$ such that

$$\text{(2.4)} \qquad \begin{cases} \phi_{x_0}^{(l)}=\lambda^{(l)}(x, \phi_{x'}^{(l)}) \\ \phi^{(l)}|_{x_0=y_0}=\langle x'-y', \xi'\rangle\,. \end{cases}$$

We note that $\phi^{(l)}(x, y, \xi')$ is a homogenous function of degree one in ξ'. For a differential operator $P(x, D)$ of order m and for a function ϕ we define the defferential operators $\sigma_\mu(P, \phi)$ such that for $\rho>0$

$$e^{-i\rho\phi}P(x, D)(e^{i\rho\phi}f)=\sum_{\mu=0}^{m}\rho^{m-k}\sigma_\mu(P, \phi)f\,.$$

Then the principal part of $\sigma_\mu(p, \phi)$ is given by

$$\text{(2.5)} \qquad \sum_{|\xi|=\mu}\frac{1}{\alpha!}\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\hat{P}(x, \phi_x)D^{\alpha}\,,$$

where $\hat{P}(x, \xi)$ is the principal part of $P(x, D)$.

Lemma 2.1 (c. f. [2]). *Let $P(x, D)$ be a defferential operator of order m. Assume that for a phase function $\phi^{(l)}$ associated with $\lambda^{(l)}$*

$$\sigma_\mu(P, \phi^{(l)})=0\,, \qquad \textit{for } \mu=0, 1, \cdots, r-1\,.$$

Then we have

$$\sigma_r(P, \phi^{(l)})=\sum_{j=0}^{r} a_j(x)H^{(l)}(x, D)^j\,,$$

where

$$H^{(l)}(x, D)=D_0-\sum_{j=1}^{n}\lambda_{\xi_j}^{(l)}(x, \phi_{x'}^{(l)})D_j\,.$$

Now we return to (2.2). Noting that $a(x, D)$ is a product of strictly hyperbolic operators, we have

$$e^{-i\varphi^{(l)}}a(x, D)(e^{i\varphi^{(l)}}f)=\sum_{l=0}^{m-m^{(l)}}\rho^{m-m^{(l)}-\mu}\sigma_{\mu+m^{(l)}}(a, \tilde{\varphi}^{(l)})f\,,$$

where $\rho=|\xi'|$ and $\tilde{\varphi}^{(l)}=\varphi^{(l)}/|\xi'|$. In particular, we have by Lemma 2.1,

$$\sigma_{m^{(l)}}(a, \tilde{\varphi}^{(l)})=\sum_{j=0}^{m^{(l)}}a_q^{(l)}(x)H^{(l)}(x, D)^j\,, \tag{2.6}$$

$$(a_{m^{(l)}}^{(l)}\neq 0)\,.$$

Hence inserting $v(x, y\,; \xi')$ into (2.2) we obtain

$$a(x, D)v(x, y, \xi')=\sum_{l=1}^{d}e^{i\varphi^{(l)}}\sum_{j=1}^{\infty}\sum_{\mu=0}^{m^{(l)}}\sigma_{m^{(l)}+\mu}(a, \tilde{\varphi}^{(l)})\rho^{-\mu-j}v_j^{(l)}\,.$$

Therefore we have

$$\sum_{\mu=0}^{m-m^{(l)}}\sigma_{m^{(l)}+\mu}(a, \tilde{\varphi}^{(l)})v_{j-p}^{(l)}=0\,, \qquad j=0, 1, \cdots, \quad l=1, 2, \cdots, d\,. \tag{2.7}$$

From the initial condition of (2.2) it follows

$$\begin{aligned}D_0^h v|_{x_0=y_0}&=e^{i<x'-y',\xi'>}\sum\sigma_\mu(D_0^h, \tilde{\varphi}^{(l)})v_j^{(l)}\rho^{h-\mu-(m-m^{(l)})-j}\\&=e^{i<x'-y',\xi'>}\sum\sigma_\mu(D_0^h, \tilde{\varphi}^{(l)})v_{j+m^{(l)}-\mu}^{(l)}\rho^{h-m-j}\\&=\frac{1}{(2\pi)^n}e^{i<x'-y',\xi'>}\delta_{m-1}^h\,.\end{aligned}$$

Hence

$$\sum_{l=1}^{d}\sum_{\mu=0}^{h}\sigma_\mu(D_0^h, \tilde{\varphi}^{(l)})v_{j+m^{(l)}-\mu}^{(l)}=\begin{cases}1\,, & h=m-1,\ j=-1\\ 0\,, & \text{otherwise}\,.\end{cases} \tag{2.8}$$

Noting that the principal part of $\sigma_\mu(D_0^h, \tilde{\varphi}^{(l)})$ is $\binom{h}{\mu}(\lambda^{(l)})^{h-\mu}D_0^\mu$, we can solve (2.8) as the linear equations of $\{D_0^h v_{j+m^{(l)}-h}^{(l)}\}_{h=0,1,\cdots,m^{(l)}-1,\,l=1,\cdots,d}$. For, the determinant of Van der Monde $\left\{\binom{h}{\mu}(\lambda^{(l)})^{h-\mu}\right\}$ $(h=0, 1, \cdots, m-1,\ \mu=0, 1, \cdots, m^{(l)}-1,\ l=1, \cdots, d)$ does not vanish. Therefore we obtain $l=0, 1, \cdots, d$, $\mu=0, 1, \cdots, m^{(l)}-1$,

$$\begin{aligned}D_0^\mu v_{j+m^{(l)}-\mu}^{(l)}=\sum_{h=0}^{m-1}c_h^{\mu(l)}f_j^h+\sum_{l'=1}^{d}\Big\{&\sum_{\mu'=1}^{m-1}N_{l',\mu'-1}^{K(l)}v_{j+m^{(l)}-\mu}^{(l)}\\&+\sum_{\mu'=m^{(l')}}^{m}N_{l',\mu'}^{\mu(l)}v_{j+m^{(l')}-\mu'}^{(l')}\Big\}\,, \qquad j\geqq m^{(l)}\,,\end{aligned} \tag{2.9}$$

where $f_j^h=1$ if $h=m-1$ and $j=1$, $f_j^h=0$ if otherwise, and $N_{l',\mu'}^{M(l)}$ are differential operators of order μ' with respect to D_0 and $\{c^{\mu(l)}_h\}$ is the inverse matrix of $\left\{\binom{h}{\mu}(\lambda^{(l)})^{h-\mu}\right\}$.

Thus we can determine $\{v_j^{(l)}\}$ successively by (2.7) and (2.9). We define u_0^p by,

$$u_0^p=\sum_{l=1}^{d} e^{i\psi(l)}\sum_{j=0}^{\infty}\rho^{-m+m(l)+\bar{n}_p^{(l)}-j}u_{0\,j}^{p(l)}, \tag{2.10}$$

where $\bar{n}_p^{(l)}=\max_q n_q^{(l)}-n_p^{(l)}$,

$$u_{0)q}^{p,l)}=\begin{cases} v_{j-\bar{n}_p^{(l)}}^{(l)}, & j\geqq\bar{n}_p^{(l)} \\ 0, & j<\bar{n}_p^{(l)}. \end{cases}$$

Next we seek for u_k^p the solution of $(2.3)_k$ as

$$u_k^p(x, y;\xi')=\sum_{l=1}^{d}\sum_{j=0}^{\infty}e^{i\psi(l)}\rho^{-m+m(l)+\bar{n}_p^{(l)}+kq(l)-j}u_{k,j}^{p(l)}. \tag{2.11}$$

Then inserting u_{k-1}^p into $(2.3)_k$, we have by virtue of (1.12),

$$\sum_{q=1}^{N}b_q^p(x, D)u_{k-1}^q=\sum e^{i\psi(l)}\rho^{\bar{n}_p^{(l)}+kq(l)-\mu-j}b_{q\,\mu}^{p(l)}(x, D)u_{k-1,j}^{(l)}.$$

On the other hand we have

$$a(x, D)u_k^p=\sum e^{i\psi(l)}\sigma_{m(l)+\mu}(a, \tilde{\phi}^{(l)})\rho^{\bar{n}_p^{(l)}+kq(l)-\mu-j}u_{k,j}^{p(l)}.$$

Therefore we obtain

$$\sum_{\mu=0}^{m-m(l)}\sigma_{m(l)+\mu}(a, \tilde{\phi}^{(l)}u_{k,j-\mu}^p=\sum_{q=1}^{N}\sum_{\mu=0}^{m_q^{p(l)}}b_{q\,\mu}^{p(l)}(x, D)u_{k-1,j}^{q(l)}. \tag*{$(2.12)_{k,j}$}$$

As the intial condition we have

$$D_0^h u_k^p|_{x_0=y_0}=\sum e^{i<x'-y',\xi>}\rho^{h-\mu-j}\sigma_\mu(D_0^h, \tilde{\phi}^{(l)})u_{k,m_p^{(l)}+kq(l)+j}^{p(l)}=0,$$

which irnplies that

$$\sum_{l=1}^{d}\sum_{\mu=0}^{h}\sigma_\mu(D_0^h, \tilde{\phi}^{(l)})u_{k,m_p^{(l)}+kq(l)+j-\mu}^{p(l)}=0.$$

Hence we have analogously to (2.9)

$$\begin{aligned} D_0^\mu u_{k,m_p^{(l)}+kq(l)+j-\mu}^{p(l)}=\sum_{l'=1}^{d}\{&\sum_{\mu'=1}^{m-1}N_{l',\mu-1}^{\mu(l)}u_{k,m_p^{(l')}+kq(l')+j-\mu'}^{p(l)} \\ &+\sum_{\mu'=m(l')}^{m-1}N_{l',\mu'}^{\mu(l)}u_{l,m_p^{(l')}+kq(l')+j-\mu'}^{p(l')}\}, \end{aligned} \tag*{$(2.13)_{k,j}$}$$

Thus we can seek for $u_{k,j}^{p(l)}$ by solving $(2.12)_{k,j}$ and $(2.13)_{k,j}$. We reduce $(2.12)_{k,j}$ to simple forms by a canonical transformation $x'=\hat{x}'(z, \xi')$, $x_0=z_0$,

$$\begin{aligned} &\frac{d}{dz_0}\hat{x}'(z_0z')=-\lambda_{\xi'}^{(l')}(z_0, \hat{x}', \hat{\xi}') \qquad &&\frac{d\hat{\xi}'}{dz_0}=\lambda_{x'}^{(l)}(z_0, \hat{x}, \hat{\xi}') \\ &\hat{x}'(0, z')=z' &&\hat{\xi}'(0, z')=\xi'. \end{aligned} \tag{2.14}$$

Then we have for any function $f(x)$

$$(H^l(x, D_x)f)_{x=(z_0, x')}=D_{z_0}(f(z_0, x'(z_0, z'))).$$

Hence it follows from (2.6) that

$$\sigma_{m(l)}(a, \tilde{\varphi}^{(l)})|_{x=(z_0, x')}=\sum_{j=0}^{m(l)} a_j^{(l)}(z)D_0^j(\equiv A_{m(l)}^{(l)}(z, \xi', D_0)).$$

We put

$$U_{k,j}^{p(l)}(z)=u_{k,j}^{p(l)}|_{x=(z_0, z')},$$

$$A_j^{(l)}(z, \xi', D_z)=\sigma(a, \psi^{(l)})|_{x=(z_0, z')}$$

$$B_{qj}^{p(l)}(z, \xi', D_z)=b_{qj}^{p(l)}(x, D)|_{x=(z_0, z')}.$$

Then $(1.12)_{k,j}$ is reduced to

$$(2.15)_{k,j} \qquad A_{m(l)}^{(l)}(z, \xi', D_0)U_{k,j}^{p(l)}=F_{k,j}^{p(l)}+G_{k,j}^{p(l)},$$

where

$$F_{k,j}^{p(l)}=\sum_{\mu=1}^{m-m(l)} A_{m(l)+\mu}^{(l}(z, \xi', D)U_{k,j-\mu}^{p(l)},$$

$$G_{k,j}^{p(l)}=\sum_{q=1}^{N}\sum_{\mu=0}^{m_p^{(l)}} B_{q,\mu}^{p(l)}(z, \xi', D)U_{k-1,j-\mu}^{q(l)},$$

and

$$(2.17) \qquad \begin{aligned} &\text{order} \quad A_j^{(l)}(z, D)\leqq j, \\ &\text{order} \quad B_{q\mu}^{p(l)}\leqq d_\mu^{(l)}+s_q^{(l)}-s_q^{(l)} \\ &\qquad\leqq m^{(l)}-\kappa^{(l)}(q^{(l)}-\mu)+s_q^{(l)}-s_p^{(l)}. \end{aligned}$$

As the initial conditions, we have by $(2.12)_{k,j}$

$$(2.18)_{k,j} \qquad D_0^\mu U_{k,m_p^{(l)}+kq^{(l)}+j-\mu}^{p(l)}=\sum_{l'=1}^{d}\{\sum_{\mu'=1}^{m-1} M_{l',\mu'-1}^{(l)}U_{k,m(l')+kq(l')+j-\mu'}^{p(l')}$$
$$+\sum_{\mu'=m(l)}^{m-1} M_{l',\mu'}^{\mu(l)}U_{k,m(l')+kq(l')+j-\mu'}^{p(l')}\}$$

where $M_{l,\mu}^{\mu(l)}=N_{l,\mu}^{\mu(l)}|_{x=(z_0,\hat{x}')}$ is a differential operator of order μ. Then we have

Theorem 2.1. *There exist positive constants C, A and δ independendent of k, j, α and β such that for any j, k, α and β, $|z_0-y_0|\leqq\delta$, $|\xi'|=1$, $z'\in K$, a compact set in R^n,*

$$|D_z^\alpha D_{\xi'}^\beta U_{k,j}^{p(l)}(z, y_0, \xi')|\leqq C^{j+k+1}A^{|\alpha|+|\beta|+j-k\kappa(l)q(l)-s_p^{(l)}}$$
$$\times\sum_\mu \frac{(|z_0-y_0|A)}{\mu!}[|\alpha+\beta|+j\kappa^{(l)}-k\kappa^{(l)}q^{(l)}+\mu-s_p^{(l)}]|^s,$$

where the summation in μ is from $[m^{(l)}-\alpha_0]_+$ to $[jm^{(l)}+k\kappa^{(l)}q^{(l)}+[m^{(l)}-\alpha_0]_{+1}$

$+s_p^{(l)}]$, *and* $[h]$ *stands for the largest integer* $\leqq h$, *and*

$$[h]_+=\begin{cases}[h], & h>0\\ 0\ , & h\leqq 0,\end{cases}$$

$$[h]_{+1}=\begin{cases}[h], & h>1\\ 1\ , & h\leqq 1.\end{cases}$$

Therefore, noting that $u_{k,j}^{p(l)}(x, y_0, \xi')$ is homogeneous of degree zero, we obtain

Theorem 2.2. *The solutions* $u_{k,j}^{p(l)}(x, y_0, \xi')$ *of* $(2.12)_{k,j}$ *and* $(2.13)_{k,j}$ *satisfies*

$$|D_x^\alpha D_{\xi'}^\beta u_{k,j}^{p(l)}(x, y_0, \xi')| \leqq C_1^{j+k+1} A_1^{|\alpha|+|\beta|+j-k\kappa^{(l)}q^{(l)}-s_p^{(l)}} |\xi'|^{-|\beta|}$$

$$\times \sum_\mu \frac{(|x_0-y|A_1)^\mu}{\mu!}[|\alpha+\beta|+j\kappa^{(l)}-k\kappa^{(l)}q^{(l)}+\mu-s_p^{(l)}]!^s$$

for $|x_0-y_0|\leqq\delta, \xi'\in R^n\backslash 0$, *where* $\mu=[m^{(l)}-\alpha_0]_+, \cdots, [jm^{(l)}+k\kappa^{(l)}q^{(l)}+[m^{(l)}-\alpha_0]_{+1}+s_p^{(l)}]$, *and* C_1 *and* A_1 *are independent of* j, k, α *and* β.

§ 3. Successive estimates of asymptotic solution

We start with a lemma to be used often in our reasonning which proof refers to [14] and [8].

Lemma 3.1. *Let* p_1 *and* p_2 *be non negative integers,* $\gamma>1$ *and* $s\geqq 1$. *For any multi integer* $\alpha=(\alpha_1, \cdots, \alpha_n)$, *we have*

$$\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}\gamma^{-|\alpha'|}(|\alpha'|+p_1)!^s(|\alpha''|+p_2)!^s \tag{3.1}$$

$$\leqq \frac{\gamma}{\gamma-1}(|\alpha|+p_1+p_2)!^s,$$

where $\displaystyle\binom{\alpha}{\alpha'}=\frac{\alpha_1!}{(\alpha_1-\alpha_1')!\alpha_1'!}\cdots\frac{\alpha_n!}{(\alpha_n-\alpha_n')!\alpha_n'!}$.

Leibniz formula and Lemma 3.1 imply,

Lemma 3.2. *Let* $P(x, D)=\sum_{|\alpha|\leqq d}a_\alpha(x)D^\alpha$ *be a differential operator in* R^n, p_1 *and* p_2 *non negative integers and* $\gamma>1$. *Assume the coefficients of* $P(x, D)$ *satisfy*

$$|D^\alpha a_\beta(x)|\leqq C_0(\gamma^{-1}A)^{|\alpha|}(|\alpha|+p_1)!^s, \tag{3.2}$$

$$|D^\alpha u(x)|\leqq CA^{|\alpha|}(|\alpha|+p_2)!^s, \tag{3.3}$$

for $x\in K$ *a compact set in* R^n. *Then we have*

$$|D^\alpha P(x, D)u(x)|\leqq CC_0n_dA^{|\alpha|}(|\alpha|+p_1+p_2)!^s, \qquad x\in K,$$

where $\bar{n}_d=(n^d-1)(n-1)^{-1}(\gamma-1)^{-1}$.

Lemma 3.3. *Let* $X_j(x, D)=\sum_{i=1}^{n} a_{ji}(x)D_i+a_{j0}(x)$ *be first order differential operators* $(j=1, \cdots, N)$. *Assume that the coefficients* $a_{ji}(x)$ *of* $X_j(x, D)$ *and* $u(x)$ *satisfy* (3.2) *and* (3.3) *with* $p_1=0$ *and* $p_2=p$ *respectively. Then we have*

$$|D^\alpha X_{j_1}X_{j_2}\cdots X_j u(x)|\leqq C(C_0\bar{n}_1)^l A^{|\alpha|+l}(|\alpha|+\cdots+p)!^s \quad ,$$

for $x\in K$ *and for* $(j_1, \cdots, j_l)\subset[1, 2, \cdots, N]$, *where* $\bar{n}_1=(n+1)(\gamma-1)^{-1}\gamma$.

Let $\psi=(\psi_1(y), \cdots, \psi_n(y))$ be a mapping from R^m to R^n. Noting that

$$D_y^\alpha(u(\psi(y))=(X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_m^{\alpha_m}u)(\psi(y))\,,$$

where $X_j=\sum_{l=1}^{n}\frac{\partial\psi_l}{\partial y_j}\frac{\partial}{\partial x_l}+\frac{\partial}{\partial y_j}$, we obtain by virtue of Lemma 3.3.

Lemma 3.4. *Assume that* $\psi_l(y)$ *satisfy*

$$|D^\alpha\psi_l(y)|\leqq C_0A_0^{|\alpha|}|\alpha|\,!^s\,, \qquad l=1, \cdots, m\,,$$

for $y\in K_1$ *a compact set in* R^m. *Then for* $u(x)$ *satisfying* (3.3), *if* $A>A_0$, *we have*

$$|D_y^\alpha(u\psi(y)|\leqq C(2^sC_0\bar{m}_1A_0+1)^{|\alpha|}A^{|\alpha|}(|\alpha|+p)!^s\,,$$

for $y\in K_1$, *where* $\bar{m}_1=(m+1)\left(\frac{A}{A_0}-1\right)^{-1}\frac{A}{A_0}$.

We consider an ordinary differential equation in y_0 with parameters $y'=(y_1, \cdots, y_n)$,

$$\text{(3.4)}\qquad \begin{aligned}&\sum_{j=0}^{m} a_j(y)D_0^j u(y)=f(y)\,, \qquad (a_m(y)=1)\,,\\ &D_0^h u|_{y_0=0}=u_h(y')\,, \qquad h=0, 1, \cdots, m-1\,.\end{aligned}$$

We assume that the coefficients satisfy

$$\text{(3.5)}\qquad |D^\alpha a_j(y)|\leqq C_0A_0^{|\alpha|}|\alpha|\,!^s\,,$$

for $y\in K$, a compact set in R^{n+1}, and that $f(y)$ and $u_k(y')$ are estimated by respectively,

$$\text{(3.6)}\qquad |D^\alpha f(y)|\leqq CA^{m+|\alpha|}\sum_{\mu=0}^{m_0}\frac{(A|y_0|)^\mu}{\mu!}(|\alpha|+m+\mu+p)!^s\,, \qquad y\in K\,,$$

$$\text{(3.7)}\qquad |D^\alpha u_h(y')|\leqq CA^{|\alpha|+h}(|\alpha|+h+p)!^s\,, \qquad y'\in K\cap\{y_0=0\}\,.$$

We denote by $[h]$ the largest integer which does not exceed h and

$$[h]_+=\begin{cases}[h]\,, & h>0\\ 0 & h\leqq 0\end{cases}, \qquad [h]_{+1}=\begin{cases}[h]\,, & h>1\\ 1 & h\leqq 1\end{cases}.$$

Then we have

Proposition 3.5. *Assume that $f(y)$ and $u_h(y)$ satisfy* (3.6) *and* (3.7) *respectively. Then if $A>2^{s+2}C_0A_0$, the solution $u(y)$ of the equation* (3.4) *can be estimated by*

$$|D^\alpha u(y)|\leqq C\hat{C}A^{|\alpha|}\sum_{\mu=0}^{m_0+[m-\alpha_0]+1}\frac{(A|y_0|)^\mu}{\mu!}(|\alpha|+\mu+p)!^s, \tag{3.8}$$

for $y\in K$, where $\hat{C}=m\frac{A}{2^{s+2}C_0A_0}\left(\frac{A}{2^{s+2}C_0A_0}-1\right)^{-2}$.

Proof. We reduce (3.4) to the first order system as follows,

$$\begin{aligned} &w_i=D_0^{i-1}u\,, \qquad i=1,\cdots,m\,,\\ &D_0w_i=w_{i+1}\,, \qquad i=1,\cdots,m-1\,,\\ &D_0w_m=D_0^m u=-\sum_{j=0}^{m-1}a_iw_{i+1}\,. \end{aligned} \tag{3.9}$$

Denote

$$M=\begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & & 0 & 1 & \\ -a_0 & \cdots & & & -a_{m-1} & \end{pmatrix},\qquad w=\begin{pmatrix} w_1\\ \\ w_m\end{pmatrix},\qquad F=\begin{pmatrix}0\\0\\f\end{pmatrix}.$$

Then we can rewrite (3.9),

$$D_0w=Mw+F.$$

Putting $S=\exp\left\{\int_0^{y_0}M(t,\,y')dt\right\}$, we can represent

$$w(y)=S(y)\left\{w(0,\,y')+\int_0^{y_0}S(t,\,y')^{-1}F(t,\,y')dt\right\}. \tag{3.10}$$

Denote by $a_q^{p(\pm)}(y)$ the $(p,\,q)$-ellement of $S(y)^{\pm1}$. Then by virtue of Lemma 3.3 with $A=2A_0$ and $\gamma=2$,

$$|D^\alpha a_q^p(y)|\leqq(2^{S+2}C_0A_0^2)^{|\alpha|}|\alpha|\,!^s, \qquad y\in K.$$

Therefore using again Lemma 3.2, we obtain from (3.6),

$$\begin{aligned} |D^\alpha S^{-1}F|&=\max_i|D^\alpha(a_i^{m(-)}(y)f(y)|\\ &\leqq\frac{C\gamma_1}{\gamma_1-1}A^{m+|\alpha|}\sum_{\mu=0}^{m_0}\frac{(A|y_0|)^\mu}{\mu!}(|\alpha|+m+\mu+p)!^s, \end{aligned} \tag{3.11}$$

where we have assumed that $\gamma_1=A(2^{s+2}C_0A_0^2)^{-1}>1$. We define

$$Tu(y)=\sqrt{-1}\int_0^{y_0}u(t,\,y')dt.$$

Then we have

$$D_0^i u = TD_0^{i+1}u + D_0^i u(0, y'), \qquad i=0, 1, \cdots.$$

Hence we obtain

$$u(y) = T^{m-1}D_0^{m-1}u + \sum_{i=0}^{m-2} \frac{y_0^i}{i!} D_0^i u(0, y'). \tag{3.12}$$

On the other hand the m-th component of w is given by (3.10) as follows,

$$w_m = D_0^{m-1}u(y) = \sum_{i=1}^{m} a_i^{m(+)}(y)\{D_0^{i-1}u(0, y') + Ta_i^{m(-)}f\}.$$

Hence we have from (3.4) and (3.12)

$$u(y) = T^{m-1}\{\sum_{i=1}^{m} a_i^{m(+)}(y)(u_{i-1}(y') + Ta_i^{m(-)}f)\} + \sum_{i=1}^{m-2} \frac{y_0^i}{i!} u_i(y'). \tag{3.13}$$

For $\alpha=(\alpha_0, \beta)$, $\alpha_0 \leqq m-1$,

$$D^\alpha u = T^{m-1-\alpha_0} \sum \binom{\beta}{\beta'} D^{\beta'} a_i^{m(+)}\{D^{\beta''}u_{i-1} + TD^{\beta''}(a_i^{m(-)}f)\} + \sum_{i=\alpha_0}^{m-2} \frac{y_0^{i-\alpha_0}}{(i-\alpha_0)!} D^\beta u_i.$$

Hence from (3.6), (3.11) and Lemma 3.2 we have

$$\begin{aligned}
|D^\alpha u| \leqq & \left(\frac{1}{\sqrt{-1}}T\right)^{m-1-\alpha_0} \Big\{ \frac{m\gamma_1}{\gamma_1-1} C(|\beta|+p+m-1)!^s A^{|\beta|+m-1} \\
& + \left(\frac{\gamma_1}{\gamma_1-1}\right)^2 \left(\frac{1}{\sqrt{-1}}T\right) CA^{|\beta|+m} \sum_{\mu=0}^{m_0} \frac{|y_0 A|^\mu}{\mu!} (|\beta|+p+m+\mu)!^s \\
& + \sum_{i=\alpha_0}^{m-2} \frac{|y_0|}{(i-\alpha_0)!} A^{|\beta|+i} (|\beta|+i+p)!^s \Big\} \\
\leqq & C\hat{C}A^{|\alpha|} \sum_{\mu=0}^{m+m_0-\alpha} \frac{|y_0 A|^\mu}{\mu!} (|\alpha|+\mu+p)!^s,
\end{aligned}$$

where $\hat{C} = m\gamma_1^2(\gamma_1-1)^{-2} \geqq \max(1, m\gamma_1(1-\gamma_1)^{-1}, \gamma_1^2(\gamma_m-1)^{-2})$. Next let be $\alpha=(\alpha_0, \alpha')$, $\alpha_0 \geqq m$ and put $\beta=(\alpha_0-m+1, \alpha')$. Then from (3.13) we have

$$\begin{aligned}
D^\alpha u &= D^\beta\{\sum a_i^{m(+)}u_{i-1} + Ta_i^{m(-)}f\} \qquad (3.14) \\
&= \sum_{\substack{\beta'+\beta''=\beta \\ \beta_0''=0}} \binom{\beta}{\beta'} \{D^{\beta'} a_i^{m(+)} D^{\beta''} u_{i-1}(y') + \sum_{\beta'+\beta''=\beta} \binom{\beta}{\beta'} D^{\beta'} a_i^{m(+)} D^{\beta''}(Ta_i^{m(-)}f).
\end{aligned}$$

By virtue of (3.11) we obtain

$$|D^{\beta''}Ta_i^{m(-)}f| \begin{cases} \leqq \dfrac{C\gamma_1}{\gamma_1-1} A^{|\beta''|-1+m} \displaystyle\sum_{\mu=0}^{m_0} \frac{|y_0 A|^\mu}{\mu!} (|\beta''|-1+m+\mu+p)!^s, & \text{if } \beta_0'' \neq 0 \\ \leqq \dfrac{C\gamma_1}{\gamma_1-1} A^{|\beta''|+m-1} \displaystyle\sum_{\mu=1}^{m_0+1} \frac{|y_0 A|^\mu}{\mu!} (|\beta''|-1+m+\mu+p)!^s, & \text{if } \beta_0''=0. \end{cases}$$

Hence we have from (3.14) and (3.7)

$$|D^\alpha u| \leqq C \sum_{\substack{\beta'+\beta''=\beta \\ \beta'_0=0}} m\binom{\beta}{\beta'}(\gamma_1^{-1}A)^{|\beta'|}|\beta'|!^s A^{|\beta''|+m-1}(|\beta''|+m-1+p)!^s$$

$$+\sum_{\beta'_0\neq 0}\binom{\beta}{\beta'}(\gamma_1^{-1}A)^{|\beta'|}|\beta'|!^s \sum_{\mu=0}^{m_0}\frac{\gamma_1 C}{\gamma_1-1}\frac{|\mu_0 A|^\mu}{\mu!}A^{|\beta''|-1+m}(|\beta''|-1+m+\mu+p)!^s$$

$$+\sum_{\beta'_0=0}\binom{\beta}{\beta'}(\gamma_1^{-1}A)^{|\beta'|}|\beta'|!^s \sum_{\mu=1}^{m_0+1}\frac{\gamma_1 C}{\gamma_1-1}A^{|\beta''|-1+m}\frac{|y_0 A|^\mu}{\mu!}(|\beta''|-1+m+\mu+p)!^s$$

$$\leqq C\hat{C}A^{|\alpha|}\sum_{\mu=0}^{m_0+1}\frac{|y_0A|^\mu}{\mu!}(|\alpha|+\mu+p)!^s.$$

Thus we complete the proof of Proposition 3.5. We can prove the following theorem analogously

Proposition 3.6. *We assume that the initial data* $u_h=0$, $h=0, 1, \cdots, m-1$ *in* (3.4) *and* $f(y)$ *satisfied for any* α *and* $y\in K$

$$|D^\alpha f(y)| \leqq CA^{|\alpha|+m}\sum_{\mu=[k_0-m-\alpha_0]_+}^{m_0+k_0}\frac{|y_0A|^\mu}{\mu!}(|\alpha|+m+\mu-p_0)!^s,$$

where m_0, k_0 *and* p *are non negative integers. Then the solution* $u(y)$ *of* (3.4) *is estimated by*

$$|D^\alpha u(y)| \leqq C\check{C}A^{|\alpha|}\sum_{\mu=[k_0-\alpha_0]_+}^{m_0+k_0+[m-\alpha_0]+1}\frac{|y_0A|^\mu}{\mu!}(|\alpha|+\mu-p_0)!^s, \qquad y\in K,$$

where $\check{C}=m\gamma_1^2(\gamma_1-1)^{-2}$, $\gamma_1=A(2^{2+s}C_0A_0)^{-1}>1$ *and* $h!=0$, *if,* $h<0$.

Now we shall prove Theorem 2.1 and Theorem 2.2. We return to the equations $(2.15)_{k,j}$. We estimate $U_{k,j}^{p(l)}(z, y_0, \xi')$ successively by use of Proposition 3.5 and Proposition 3.6. We may assume that the coefficients of $A_j^{(l)}(z, \xi', D_z)$ and $B_{q,\mu}^{(l)}(z, \xi', D_z)$ satisfy (3.5), if we replace the variables (z_0-y_0, z', ξ') by y and $K=K_z\times S_{\xi'}^{n-1}$ where K_z is a compact ret in R^{n+1} and $S_{\xi'}^{n-1}$ is a unite sphere in R^n. Here we note that $U_{k,j}^{p(l)}$ and the coefficients of $A_j^{(l)}$ and $B_{q,\mu}^{(l)}$ are homogeneous degree zero in ξ'. Then we can prove that there are positive constants C and A such that

$$(3.16)_{k,j} \qquad |D^\alpha U_{k,j}^{p(l)}| \leqq C(C_0\hat{C})^{j+k+1}A^{|\alpha|+j-k\kappa^{(l)}q^{(l)}-s_p^{(l)}}$$
$$\times\sum_{\mu}\frac{|y_0A|^\mu}{\mu!}[|\alpha|+j\kappa^{(l)}-k\kappa^{(l)}q^{(l)}+\mu-s_p^{(l)}]!^s,$$

where the summation in μ is from $[k\kappa^{(l)}q^{(l)}+s_p^{(l)}-\alpha_0]_+$ to $[jm^{(l)}+k\kappa^{(l)}q^{(l)}+[m^{(l)}-\alpha_0]_{+1}+s_p^{(l)}]$.

We shall prove $(3.16)_{k,j}$ by induction on k and j. At first we estimate $U_{0,j}^{p(l)}$ by induction on j. We recall the definition of $U_{0,j}^{p(l)}$, that is,

$$U_{0,j}^{p(l)}=u_{0,j}^{p(l)}|_{x=(z_0,\hat{x}')}=\begin{cases} v_{j-\bar{n}_p^{(l)}}^{(l)}|_{x=(z_0,\hat{x}')}, & j\geqq \bar{n}_p^{(l)} \\ 0, & j<\bar{n}_p^{(l)}\end{cases}$$

where $\bar{n}_p^{(l)}=\max_q n_q^{(l)}-n_p^{(l)}$ and $v_j^{(l)}$ is the solution of (2.7) and (2.9). Put

$$V_j^{(l)}=v_j^{(l)}|_{x=(z_0,\hat{x}'(z,\xi'))}.$$

Then we shall prove

$$(3.17)_j \qquad |D^\alpha V_j^{(l)}|\leqq C(C_0C)^{j+1}A^{|\alpha|+j}\sum_{\mu=0}^{jm^{(l)}+[m^{(l)}-\alpha_0]+1}\frac{|y_0A|^\mu}{\mu!}(|\alpha|+j+\mu)!^s$$

from which it follows that $U_{0,j}^{p(l)}$ satisfies $(3.16)_{0,j}$, if we choose $s_p^{(l)}\leqq\bar{n}_p^{(l)}$ for each p and l. From (2.7) we have

$$(3.18) \qquad A_{m^{(l)}}^{(l)}(y, D_0)V_j^{(l)}=F_j^{(l)},$$

where $F_j^{(l)}=\sum_{\mu=1}^{m-m^{(l)}}A_{m^{(l)}+\mu}^{(l)}(y, D)V_{j-\mu}^{(l)}(y)$, and we replaced (z_0-y_0, z', ξ') to y. We note that $A_{m^{(l)}+\mu}^{(l)}$ is a differential operator of order $m^{(l)}+\mu$ and in paticular

$$A_{m^{(l)}}^{(l)}(y, D_0)=\sum_{\mu=0}^{m^{(l)}}a_\mu^{(l)}(y)D_0^\mu, \qquad (a_{m^{(l)}}^{(l)}\neq 0).$$

We may assume that the coefficients of $A_{\mu+m^{(l)}}^{(l)}=\sum_{|\alpha|\leq\mu+m^{(l)}}a_{\mu\alpha}^{(l)}(y)D^\alpha$ satisfy (3.5).

At first we shall estimate $V_0^{(l)}$ which is a solution,

$$A_{m^{(l)}}^{(l)}(y, D_0)V_0^{(l)}=0,$$

and as the initial condition we have from (2.9)

$$D_0^\mu V_0^{(l)}|_{y_0=0}=0, \qquad \mu\leqq m^{(l)}-2$$

$$D_0^{m^{(l)}-1}V_0^{(l)}|_{y_0=0}=C_{m-1}^{m^{(l)}-1(l)}$$

where we may assume that $C_{m-1}^{m^{(l)}-1(l)}$ satisfies (3.7) with $p=0$, if we choose C and A suitably. Hence by virtue of Proposition 3.5 with $p=0$, $f=0$, $m_0=0$ and $m=m^{(l)}$, we can obtain $(3.17)_0$. Next we assume that $(3.17)_i$, $i=0, \cdots, j-1$ are valid. Then in (3.18) we have

$$F_j^{(l)}=\sum_{i=1}^{m-m^{(l)}}\sum_{|\beta|\leq m^{(l)}+i}a_{j\beta}^{(l)}(y)D^\beta V_{j-i}^{(l)},$$

Hence the hypothesis of induction and (3.5) imply

$$(3.19) \qquad |D^\alpha F_j^{(l)}|\leqq\sum\binom{\alpha}{\alpha'}|D^{\alpha'}a_{i\beta}^{(l)}||D^{\beta+\alpha'}V_{j-i}|$$

$$\leqq C(C_0C_1)(C_0\hat{C})^jA^{|\alpha|+m^{(l)}+j}\sum_{\mu=0}^{jm}\frac{|y_0t|^\mu}{\mu!}(|\alpha|+m^{(l)}+j-\mu)!^s,$$

where

$$C_1=m(n^m-1)\gamma_0(n-1)^{-1}(\gamma_0-1)^{-1}, \qquad \gamma_0=A/A_0>1.$$

As the initial condition we have from (2.9)

$$D_0^{\mu} V_{j-\mu}^{(l)}=\sum_{h=1}^{m-1} C_h^{\mu(l)} f_{j-m(l)+\mu}^{h}+\sum_{l'=1}^{d} \sum_{\mu'=1}^{m-1} M_{l',\mu'}^{\mu(l)} V_{j-\mu'+m(l')-m(l)}^{(l')}$$

$$+\sum_{l'=1}^{d} \sum_{\mu'=m(l')}^{m} M_{l',\mu'}^{\mu(l)} V_{j-\mu'+m(l')-m(l)}^{(l')},\ (y_0=0),$$

where $M_{l',\mu'}^{\mu(l)}=N_{l',\mu'}^{\mu(l)}|_{x=(z_0,x')}$. If we assume that $m^{(1)}\geqq m^{(2)}\geqq \cdots \geqq m^{(d)}$, we have

$$V_j^{(1)}|_{y_0=0}=\sum_h C_h^{0(l)} f_{j-m(1)}^{h}+\sum_{l'=1}^{d} \sum_{\mu'=1}^{m-1} M_{l',\mu'}^{0(1)} V_{j-\mu'+m(l')-m(1)}^{(l')}|_{y_0=0}.$$

Hence by the assumption of induction we obtain

$$|D'^{\alpha} V_j^{(1)}|_{y_0=0}\leqq C(C_0C_1)(C_0\hat{C})^j A^{|\alpha|+j}(|\alpha|+j)!^s$$

where we note that $f_j^h=1$, $(j=-1,\ h=m-1)$ and $f_j^h=0$ (otherwise), and the coefficients of $M_{l,\mu'}^{\mu(l)}$ satisfy (3.5) and $C_h^{\mu(1)}$ is estimated by

$$|D'^{\alpha} C_h^{\mu(l)}|\leqq CA^{|\alpha|}|\alpha|!^s.$$

By induction of μ we obtain.

$$|D_0 V_j^{(1)}|_{y_0=0}\leqq C(C_0C_1)(C_0\hat{C})^j A^{|\alpha|+j+\mu}(|\alpha|+j+\mu)!^s.$$

for $\mu=0, 1, \cdots, m^{(1)}-1$. Moreover by induction on l we have

$$|D_0^{\mu} V_j^{(l)}|_{y_0=0}\leqq C(C_0C_1)(C_0\hat{C})^j A^{|\alpha|+j+\mu}(|\alpha|+j+\mu)!^s,$$

for $\mu=0, 1, \cdots, m^{(l)}-1$ and $l=1, 2, \cdots, d$. Therefore by virute of Proposition 3.5 we obtain $(3.17)_j$, if we choose $\hat{C}$ and A such that $\hat{C}\geqq C_1 m\gamma_1^2/(\gamma_1-1)^2$, $\gamma_1=A(2_0^{2+s}A_0)^{-1}>1$.

Next we shall prove $(3.16)_{k,j}$, $k\geqq 1$ by induction in k and j. We assume $(3.17)_{k-1,j,j=0,1,\dots}$. We shall at first prove $(3.16)_{k,0}$. From $(2.15)_{k,0}$ we have

$$A_{m(l)}^{(l)} U_{k,0}^{p(l)}=G_{k,0}^{p(l)},$$

where

$$G_{k,0}^{p(l)}=\sum_{q=1}^{N} B_{q,0}^{p(l)} U_{k-1,0}^{q(l)}=\sum_{q=1}^{N} \sum_{t=0}^{d_q^{p(l)}{}_0} B_{q,0,t}^{p(l)} D_0^{d_q^{p(l)}{}_0-t} U_{k-1,0}^{q(l)}.$$

Noting that $d_q^{p(l)}{}_0\leqq m^{(l)}-\kappa^{(l)}q^{(l)}+s_q^{(l)}-s_p^{(l)}$ and that the coefficients of $B_{q,0}^{p(l)}$ satisfy (3.5), we have

$$|D^{\alpha} G_{k,0}^{p(l)}|\leqq \sum\binom{\alpha}{\alpha'}|D^{\alpha'} B_{q,0,t}^{p(l)}|\,|D_0^{d_{q,0}^{p(l)}-t} D^{\alpha''} U_{k-1,0}^{p(l)}|$$

$$\leqq C\sum\binom{\alpha}{\alpha'} C_0(C_0\hat{C})^k A_0^{|\alpha'|}|\alpha'|!^s A^{|\alpha''|+d_{q,0}^{p(l)}-t-(k-1)\kappa^{(l)}q^{(l)}-s_q^{(l)}} \frac{|y_0A|^{\mu}}{\mu!}$$

$$\times[|\alpha''|+d_{q,0}^{p(l)}-t-(k-1)\kappa^{(l)}q^{(l)}+\mu-s_q^{(l)}]!^s,$$

$$\leqq C\sum_{\mu}\frac{C_0\gamma_0}{\gamma_0-1}(C_0\hat{C})^k A^{|\alpha|+m^{(l)}-k\kappa^{(l)}q^{(l)}-s_q^{(l)}-t} \frac{|y_0A|^{\mu}}{\mu!}$$

$$\times[|\alpha|+m^{(l)}-k\kappa^{(l)}q^{(l)}-t+\mu-s_q^{(l)}]!^s,$$

where the summation in μ is from $[(k-1)\kappa^{(l)}q^{(l)}+s_q^{(l)}-(d_{p,0}^{p(l)}-t+\alpha_0)]_+$ to $[(k-1)\kappa^{(l)}q^{(l)}+[m^{(l)}-\alpha_0-d_{q,0}^{p(l)}+t]_{+1}+s_q^{(l)}]$, $\gamma_0=A/A_0>1$ and $h!=0$, if $h<0$. It follows from the assumption (1.17) and (1.14)

$$
\begin{aligned}
(3.20)\qquad & s_q^{(l)}-\kappa^{(l)}q^{(l)}+[m^{(l)}-\alpha_0-d_{q,0}^{p(l)}+t]_{+1}-t \\
&\leqq s_p^{(l)}-\alpha_0\leqq s_p^{(l)}, \qquad \text{if } m^{(l)}-\alpha_0-d_{q,0}^{p(l)}+t>0 \\
&\leqq s_q^{(l)}-\kappa^{(l)}q^{(l)}+1-t\leqq s_p^{(l)}, \qquad \text{if } m^{(l)}-\alpha_0-d_{q,0}^{p(l)}+t\leqq 0 .
\end{aligned}
$$

Hence replacing μ to $\mu+t$, we obtain

$$
\begin{aligned}
|D^\alpha G_{k,0}^{p(l)}| &\leqq CC_0\frac{\gamma_0}{\gamma_0-1}(C_0\hat{C})^k A^{|\alpha|+m^{(l)}-k\kappa^{(l)}q^{(l)}-s_p^{(l)}} \\
&\quad\times\sum_{q,\mu,t}\frac{|y_0A|^\mu|y_0|^t}{(\mu+t)!}[|\alpha|+m^{(l)}-k\kappa^{(l)}q^{(l)}-s_q^{(l)}+\mu]!^s \\
&\leqq CC_0C_2(C_0\hat{C})^k A^{|\alpha|+m^{(l)}-k\kappa^{(l)}q^{(l)}-s_q^{(l)}} \\
&\quad\times\sum_{\mu=[k\kappa^{(l)}q^{(l)}+s_p^{(l)}-m^{(l)}-\alpha_0]_+}^{k\kappa^{(l)}q^{(l)}+s_p^{(l)}}\frac{|y_0A|^\mu}{\mu!}[|\alpha|+m^{(l)}-k\kappa^{(l)}q^{(l)}-s_q^{(l)}+\mu]!^s,
\end{aligned}
$$

where $C_2=\gamma_0Nm(1+\delta^m)/\gamma_0-1$. As the initial condition of $U_{k,0}^{p(l)}$ it follows from $(2.18)_{k,0}$ that

$$D_0^hU_{k,0}^{p(l)}|_{y_0=0}=0, \qquad h=0, 1, \cdots, m^{(l)}-1 .$$

Therefore we can apply Proposition 3.6 to $U_{k,0}^{p(l)}$, putting $k_0=k\kappa^{(l)}q^{(l)}+s_p^{(l)}$, $m=m^{(l)}$, $m_0=0$ and $p_0=k\kappa^{(l)}q^{(l)}+s_p^{(l)}$. Thus we obtain $(3.16)_{k,0}$ if we choose $\hat{C}$ and A such that

$$
\begin{aligned}
&\hat{C}\geqq m^{(l)}\gamma_1^2(\gamma_1-1)^{-2}C_2 , \\
&\gamma_1=A(2^{s+2}C_0A_0)^{-1}>1 , \\
&\gamma_0=A/A_0>1 .
\end{aligned}
$$

Next we assume that $(3.16)_{k,i}$ are valid for $i=0, 1, \cdots, j-1$. Then we shall estimate $F_{k,j}^{p(l)}$ and $G_{k,j}^{p(l)}$ in $(2.15)_{k,j}$. We have by (2.61) and (2.17)

$$
\begin{aligned}
|D^\alpha F_{k,j}^{p(l)}| &\leqq \sum_{\alpha'+\alpha''=\alpha}\sum_{i=1}^{m-m^{(l)}}\sum_{|\beta|\leqq m^{(l)}+i}|D^{\alpha'}A_{m^{(l)}+i,\beta}^{(l)}||D^{\beta+\alpha''}U_{k,j-i}^{p(l)}| \\
&\leqq \sum_{\alpha',i,\beta}\binom{\alpha}{\alpha'}C_0A_0^{|\alpha'|}|\alpha'|!^sC(C_0\hat{C})^{j-i+k+1}A^{|\beta+\alpha''|+j-\mu-k\kappa^{(l)}q^{(l)}-s_p^{(l)}} \\
&\quad\times\sum_\mu\frac{|y_0A|^\mu}{\mu!}[|\alpha''+\beta|+\kappa^{(l)}(j-i)-k\kappa^{(l)}q^{(l)}-s_p^{(l)}+\mu]!^s \\
&\leqq C(C_0C_3)(C_0\hat{C})^{j+k}A^{|\alpha|+m^{(l)}+j-k\kappa^{(l)}q^{(l)}-s_p^{(l)}} \\
&\quad\times\sum_\mu\frac{|y_0A|^\mu}{\mu!}[|\alpha|+m^{(l)}+j\kappa^{(l)}-k\kappa^{(l)}q^{(l)}-s_p^{(l)}+\mu]!^s
\end{aligned}
$$

where the sommation in μ is from $[k\kappa^{(l)}q^{(l)}+s_p^{(l)}-m^{(l)}-\alpha_0]$ to $[jm^{(l)}+k\kappa^{(l)}q^{(l)}+s_p^{(l)}]$ and $C_3=((n+1)^m-1)\gamma_0 m/n(\gamma_0-1)$. Next we shall estimate $G_{k,j}^{p(l)}$. From (2.16) and (2.17) we have

$$G_{k,j}^{p(l)}=\sum_{q=1}^{N}B_{q,0}^{p(l)}U_{k-1,j}^{q(l)}+\sum_{q=1}^{N}\sum_{i=1}^{m_q^{p(l)}}B_{q,i}^{p(l)}U_{k-1,j-i}^{q(l)},$$

$$B_{q,i}^{p(l)}=\sum_{|\beta|\leq d_{q,i}^{p(l)}}B_{q,i,\beta}^{p(l)}(y)D^{\beta},$$

where

(3.21) $$d_{q,i}^{p(l)}\leqq m^{(l)}-\kappa^{(l)}(q^{(l)}-i)+s_q^{(l)}-s_p^{(l)}$$

and in partcular we note that $B_{q,0}^{p(l)}$ are differential operators only in y_0. Hence

$$|D^{\alpha}(B_{q,0}^{p(l)}U_{k-1,j}^{q(l)})|\leqq\sum_{\substack{\alpha'+\alpha''=\alpha\\ 0\leq t\leq d_{q,0}^{p(l)}}}\binom{\alpha}{\alpha'}|D^{\alpha'}B_{q,0t}^{p(l)}|D_0^{d_{q,0}^{p}-t}D^{\alpha''}U_{k-1,j}^{p(l)}|$$

$$\leqq\sum\binom{\alpha}{\alpha'}C_0A_0^{|\alpha'|}|\alpha'|!^sC(C_0\hat{C})^{j+k}A^{|\alpha''|+j-(k-1)\kappa^{(l)}q^{(l)}s_q^{(l)}+d_{q,0}^{p(l)}-t}$$

$$\times\sum_{\mu}\frac{|y_0A|^{\mu}}{\mu!}[|\alpha''|+d_{q,0}^{p(l)}-t+j\kappa^{(l)}-(k-1)\kappa^{(l)}q^{(l)}-s_q^{(l)}+\mu]!^s,$$

where the summation in μ is from $[(k-1)\kappa^{(l)}q^{(l)}+s_q^{(l)}-t-\alpha_0'']_+$ to

$$[jm^{(l)}+(k-1)\kappa^{(l)}q^{(l)}+s_q^{(l)}+[m^{(l)}-t-\alpha_0'']_{+1}].$$

Taking account of (3.20) and (3.21), we have analogously to $G_{k,0}^{p(l)}$,

$$|D^{\alpha}(B_{q,0}^{p(l)}U_{k-1,j}^{q(l)})|\leqq C(C_0C_4)(C_0\hat{C})^{j+k}A^{|\alpha|+m^{(l)}+j-k\kappa^{(l)}q^{(l)}-s_p^{(l)}}$$

$$\times\sum_{\mu}\frac{|y_0A|^{\mu}}{\mu!}[|\alpha|+m^{(l)}+j\kappa^{(l)}-k\kappa^{(l)}q^{(l)}-s_p^{(l)}+\mu]!^s,$$

where $\mu=[k\kappa^{(l)}q^{(l)}+s_p^{(l)}-m^{(l)}-\alpha_0]_+,\cdots,[jm^{(l)}+k\kappa^{(l)}q^{(l)}+s_q^{(l)}]$, and $C_4=m(1+\delta^m)\gamma_0/\gamma_0-1$. Next for $i\geqq1$, we have

$$|D^{\alpha}(B_{q,i}^{p(l)}U_{k-1,j-i}^{q(l)})|\leqq\sum\binom{\alpha}{\alpha'}|D^{\alpha'}B_{q,i}^{p(l)}||D^{\beta+\alpha''}U_{k-1,j-i}^{q(l)}|$$

$$\leqq\sum C_0A_0^{|\alpha'|}|\alpha'|!^sC(C_0\hat{C})^{j-i+k}A^{|\beta+\alpha''|+j-i-(k-1)\kappa^{(l)}q^{l)}}$$

$$\times\sum_{\mu}\frac{|y_0A|^{\mu}}{\mu!}[|\beta+\alpha''|+\kappa^{(l)}(j-i)-(k-1)\kappa^{(l)}q^{(l)}-s_q^{(l)}+\mu]!^s,$$

where the summation in μ is from $[(k-1)\kappa^{(l)}q^{(l)}+s_q^{(l)}-\beta_0-\alpha_0'']_+$ to

$$[(j-i)m^{(l)}+(k-1)\kappa^{(l)}q^{(l)}+s_q^{(l)}+[m^{(l)}-\beta_0-\alpha_0'']_{+1}].$$

Noting that $[(k-1)\kappa^{(l)}q^{(l)}+s_q^{(l)}-\beta_0-\alpha_0]_+\geqq[k\kappa^{(l)}q^{(l)}+s_p^{(l)}-\alpha_0]_+$ and

$$[(j-i)m^{(l)}+(k-1)\kappa^{(l)}q^{(l)}+s_q^{(l)}+[m^{(l)}-\beta_0-\alpha_0'']_{+1}]$$
$$\leqq[jm^{(l)}+k\kappa^{(l)}q^{(l)}+s_p^{(l)}],$$

we obtain

$$|D^\alpha G_{q,j}^{p(l)}|\leqq \sum_{q=1}^{N}(|D^\alpha(B_{q,0}^{p(l)}U_{k-1,j}^{q(l)})|+\sum_{i=1}^{m_q^{p(l)}}|D^\alpha(B_{q,i}^{p(l)}U_{k-1,j-i}^{q(l)})|)$$

$$\leqq C(C_0C_5)(C_0\hat{C})^{i+k}A^{|\alpha|+m^{(l)}+j-k\kappa^{(l)}q^{(l)}-s_p^{(l)}}$$

$$\times\sum_{\mu}\frac{|y_0A|^\mu}{\mu!}[|\alpha|+m^{(l)}+j\kappa^{(l)}-k\kappa^{(l)}q^{(l)}-s_p^{(l)}+\mu]!^s$$

where $\mu=[k\kappa^{(l)}q^{(l)}+s_p^{(l)}-m^{(l)}-\alpha_0]_+,\cdots,[jm^{(l)}+k\kappa^{(l)}q^{(l)}+s_p^{(l)}]$ and $C_5=\gamma_0Nm(1+\delta^m)(n^m-1)(n-1)^{-1}(\gamma_0-1)^{-1}$. As the intial condition we have by $(2.18)_{k,j}$

$$|D^hU_{k,j}^{p(l)}|_{y_0=0}=0,\qquad h=0,1,\cdots,m^{(l)}-1.$$

Thus we can apply Proposition 3.6 to $U_{k,j}^{p(l)}$ and we obtain $(3.16)_{k,j}$, if we choose $\hat{C}$ and A such that $\hat{C}=2m\gamma_1^2(\gamma_1-1)^{-2}\max_{1\leq i\leq 5}C_i$, $\gamma_1=A(2^{+2}C_0A_0)^{-1}>1$ and $\gamma_0=AA_0^{-1}>1$.

§ 4. Construction of Fundamental solution

In the previous sections we have sought for the asymptotic solutions for (2.1). In the present section we shall construct the fundamental solution by use of the solutions of (2.2) and $(2.3)_k$. In the expression (2.11) we denote in brief $|\xi'|^{m_p^{(l)}+kq^{(l)}-j}u_{k,l}^{p(l)}$ by $w_{k,j}^{p(l)}(x, y_0, \xi')$ which is homogeneous degree $m_p^{(l)}+kq^{(l)}-j$ in ξ', where $m_p^{(l)}=-m+m^{(l)}+\bar{n}_p^{(l)}$. Hence we have from Theorem 2.2,

$$(4.1)\qquad |D_x^\alpha D_{\xi'}^\beta w_{k,j}^{p(l)}(x, y_0, \xi')|$$
$$\leqq CC_1^jC_2^{-k}A_1^{|\alpha|}|x_0-y_0|^{[k\kappa^{(l)}q^{(l)}-\alpha_0]_+}|\alpha+\beta|!^s$$
$$\times|\xi'|^{m_p^{(l)}+kq^{(l)}-j-|\beta|}j!^{\kappa^{(l)}s+m^{(l)}(s-1)}k!^{-\kappa^{(l)}q^{(l)}},$$

where $C_1=C_0\hat{C}A_1$ and $C_2=A_1^{\kappa^{(l)}}(C_0\hat{C})^{-1}$. Hence it follows from Boutet de Monvel and Kree [1] that we can find $w_k^{p(l)}(x, y_0, \xi')$ for each k such that

$$(4.2)\qquad |D_x^\alpha D_{\xi'}^\beta(w_k^{p(l)}-\sum_{j=0}^{M-1}w_k^{p(l)})|$$
$$\leqq CC_2^{-k}k!^{-\kappa^{(l)}q^{(l)}}|x_0-y_0|^{[k\kappa^{(l)}q^{(l)}-\alpha_0]_+}|\alpha+\beta|!^s$$
$$\times A^{|\alpha+\beta|+M}|\xi'|^{m_p^{(l)}+kq^{(l)}-|\beta|-M}M!^{\kappa^{(l)}s+m^{(l)}(s-1)}$$

where $M=1,2,3,\cdots$ and the constant A is dependent only of C_1 and A_1. We define

$$w^{p(l)}(x, y_0, \xi')=\sum_{k=0}^{\infty}w_k^{p(l)}(x, y_0, \xi'),$$

which converges uniformly by (4.1), and put

$$r^{p(l)}(x, y_0, \xi')=e^{-i\phi(l)}a(x, D)e^{i\phi(l)}w^{p(l)}-\sum_{q=1}^{N}e^{-i\phi(l)}b_q^p(x, D)e^{i\phi(l)}w^{q(l)}.$$

It is then clear that $w^{p(l)}$ and $r^{p(l)}$ satisfy (1.20) and (1.21) respectively, according to (4.1) and (4.2). Here our aim is to prove the existence of the solution for the integral equation (1.19).

Lemma 4.1 (c.f. [5]). *There exists a positive constant* C *such that for any positive number* ρ

$$\inf_{j\in N}\frac{j!}{\rho^j}\leqq C\sqrt{\rho+2}\,e^{1-\rho}.$$

Noting that $\sqrt{\rho+2}\leqq 2e^{\rho/2}$, we have from the above lemma

$$\inf_{M\in N}(A_0^{-1}|\xi'|)^{-M}M!^{\kappa^{(l)}s+m^{(l)}(s-1)}\leqq Ce^{-\frac{1}{2}\left(\frac{|\xi'|}{A_1}\right)^{\frac{1}{\kappa^{(l)}s+m^{(l)}(s-1)}}}.$$

Hence by (1.21) we have the estimate

$$\begin{aligned}(4.3)\qquad &|D_x^\alpha D_{\xi'}^\beta r^{p(l)}|\\ &\leqq CA_1^{|\alpha+\beta|}\exp\left\{|x_0-y_0|A_2|\xi'|^{1/\kappa^{(l)}}-\frac{1}{2}\left(\frac{|\xi'|}{A_1}\right)^{(\kappa^{(l)}s+m^{(l)}(s-1))^{-1}}\right\}\\ &\quad\times|\alpha+\beta|!^s|\xi'|^{m_p^{(l)}-|\beta|+\alpha_0/\kappa^{(l)}}.\end{aligned}$$

We define

$$R^{p(l)}(x_0, y_0)u(x')=\iint e^{i\phi^{(l)}(x,y,\xi')}r^{p(l)}(x, y_0, \xi')u(y')d\xi'dy'$$

which is an operator from $\gamma_s(R^n)$ to $\gamma_s(R^n)$, where $\phi^{(l)}$ is a phase function. Moreover we can see more precisely,

Proposition 4.2. *Assume that* $u\in\gamma_s(R^n)$ *satisfies*

$$|D_{x'}^\alpha u(x')|\leqq CA^{|\alpha|}|\alpha|!^s \qquad \text{for } x'\in R^n,$$

When $s<\kappa^{(l)}$, *there exist positive constants* $\hat{C}$, $\hat{\gamma}$ *and* $\hat{A}$ *such that*

$$|D_{x'}^\alpha R^{p(l)}[x_0, y_0]u(x')|\leqq\hat{C}C(e^{\hat{\gamma}|x_0-y_0|}A)^{|\alpha|}|\alpha|!^s$$

for $x'\in R^n$, $|x_0-y_0|\leqq\delta$ *and* $A>\hat{A}$. *When* $s=\kappa^{(l)}$, *there exist,* $\hat{C}$, $\hat{A}$ *and* $\hat{\gamma}$ *such that*

$$|D_{x'}^\alpha R^{p(l)}[x_0, y_0]u(x')|\leqq\hat{C}(e^{|x_0-y_0|\gamma}A)^{|\alpha|}|\alpha|!^s,$$

for $x'\varepsilon\ R^n$, $|x_0-y_0|\leqq\min\{\delta, \hat{\delta}A^{-1/\kappa^{(l)}}\}$ *and* $A>\hat{A}$.

Proof. The phase function $\phi^{(l)}$ can be decomposed as

$$\phi^{(l)}(x, y, \xi')=\langle\theta^{(l)}(s, y_0, \xi')+x'-y', \xi'\rangle,$$

where $\theta^{(l)}=(\theta_1^{(l)}, \cdots, \theta_n^{(l)})$ satisfies

$$(4.4)\qquad |D_{x'}^\alpha D_{\xi'}^\beta\theta_i^{(l)}|\leqq\gamma_1|x_0-y_0|A_1^{|\alpha+\beta|}|\alpha+\beta|!^s|\xi'|^{-|\beta|}.$$

Then we can write

$$R^{p(l)}[x_0, y_0]u(x')=\iint e^{-i<y',\xi'>}r^{p(l)}(x, y_0, \xi')u(\theta^{(l)}+y')dy'd\xi .$$

Hence

$$D_{x'}^{\alpha}(R^{p(l)}[x_0, y_0]u(x'))$$

$$=\sum_{\alpha'+\alpha''=\alpha}\iint e^{-i<y',\xi'>}\binom{\alpha}{\alpha'}D_{x'}^{\alpha'}r^{p(l)}D_{x'}^{\alpha''}(u(\theta^{(l)}+x'+y'))dyd\xi' ,$$

Noting that

$$D_{x'}^{\alpha''}(u(\theta^{(l)}+x'+y'))=(X_1^{\alpha''_1}\cdots X_n^{\alpha''_n}u)(\theta^{(l)}+x'+y') ,$$

where $X_i=\sum_{j=1}^{n}\frac{\partial}{\partial x_i}\theta_j^{(l)}\frac{\partial}{\partial y_j}+\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_i}$, we have

$$D_{x'}^{\alpha}R^{p(l)}[x_0, y_0]u(x') \tag{4.5}$$

$$=\sum_{\alpha'}\binom{\alpha}{\alpha'}\iint e^{i\phi^{(l)}}D_{x'}^{\alpha'}r^{p(l)}(X_1^{\alpha''_1}X_2^{\alpha''_2}\cdots X_n^{\alpha''_n}u)(y')dy'd\xi' .$$

For $x'\in R^n$ fixed, there exist, a compact set K in R^n such that for $y'\notin K$

$$\phi_{\xi'}^{(l)}=\theta^{(l)}+x'-y'\neq 0 .$$

Hence we can find a first a first order differential operator $L^{(l)}(x, \xi', y, D_{\xi'})$ such that

$$L^{(l)}(e^{i\phi^{(l)}})=(1+|x'-y'|^2)^{1/2}e^{i\phi^{(l)}} , \qquad \text{for } y\notin K , \tag{4.6}$$

that is,

$$L^{(l)}=(1+|x'-y'|^2)^{1/2}\{\sum_{j=1}^{n}\phi_{\xi_j}^{(l)2}\}^{-1}\sum_{j=1}^{n}\phi_{\xi_j}^{(l)}\frac{\partial}{\partial\xi_j} .$$

$$=\sum_{j=1}^{n}a_j^{(l)}(s, y, \xi')\frac{\partial}{\partial\xi_j} .$$

Then the coefficients $a_j^{(l)}$ of $L^{(l)}$ satisfy

$$|D_{x'}^{\alpha}a_j^{(l)}|\leqq C_1A_1^{|\alpha|}|\alpha|!^s , \qquad y\notin K , \tag{4.7}$$

if we choose C_1 and A_1 suitablly. Let $\Psi(y')$ be a function of $\gamma_{\hat{s}}(R^n)(\hat{s}=\hat{\kappa}^{(l)}$ if $s=\kappa^{(l)}$ and $s<\hat{s}<\kappa^{(l)}$, if $s<\kappa^{(l)})$ such that $\Psi=1$ on K, supp Ψ compact and

$$|D_{y'}^{\alpha}\Psi(y')|\leqq CA_1^{|\alpha|}|\alpha|!^{\hat{s}} \tag{4.8}$$

for $y'\in R^n$. Then we decompose (4.5)

$$D_{x'}^{\alpha}R^{p(l)}[x_0, y_0]u(x')$$

$$=\sum\binom{\alpha}{\alpha'}\iint e^{i\phi^{(l)}}D_{x'}^{\alpha'}r^{p(l)}(1-\Psi)(X_1^{\alpha''_1}\cdots X_n^{\alpha''_n}u)(y')dy'd\xi$$

$$+\sum\binom{\alpha}{\alpha'}\iint e^{i\phi^{(l)}}D_{x}^{\alpha'}r^{p(l)}\Psi(X_1^{\alpha''_1}\cdots X_n^{\alpha''_n}u)(y')dy'd\xi'$$

$$=I_1+I_2 .$$

We at first estimate I_1. By (4.6) have

$$I_1=\Sigma\binom{\alpha}{\alpha'}\iint e^{i\phi^{(l)}}(1-\Psi)(1+|x'-y'|^2)^{-\frac{n+1}{2}}$$

$$\times(L^{(l)*})^{n+1}D_{x'}^{\alpha'}r^{p(l)}(X_1^{\alpha_1''}\cdots X_n^{\alpha_n''}u(y')dyd\xi' .$$

We can write

$$(L^{(l)*})^{n+1}D_{x'}^{\alpha'}r^{p(l)}(x,\ y_0,\ \xi')(X_1^{\alpha_1''}\cdots X_n^{\alpha_n''}u)(y)$$

$$=\sum_{|\beta'+\beta''|\leqq n+1}a_{\beta',\beta''}(x,\ y_0,\ \xi')D_{\xi'}^{\beta'}D_{x'}^{\alpha'}r^{p(l)}D_{\xi'}^{\beta''}(X_1^{\alpha_1''}\cdots X_n^{\alpha_n''}u)(y')$$

where $a_{\beta'\beta''}(x,\ y_0,\ \xi')$ satisfies (4.7), if we choose A_1 and C suitablly. Moreover we have

$$D_{\xi'}^{\alpha''}(X_1^{\alpha_1''}\cdots X_n^{\alpha_n''}u)(y)=(Y_1Y_2\cdots Y_{|\alpha''|}u)(y)$$

where

$$Y_j=\sum_{k=1}^{n}D_{\xi'}^{\beta^{(j)}}\frac{\partial}{\partial x_{i(j)}}\theta_k^{(l)}\frac{\partial}{\partial y_k}+\frac{\partial}{\partial x_{i(j)}}+\frac{\partial}{\partial y_{i(j)}},$$

where $i(j)\varepsilon\ [1,\ \cdots,\ n]$ and $\beta^{(j)}=(\beta_1^{(j)},\ \cdots\ \beta_n^{(j)})$, $|\beta^{(j)}|\leqq n+1$. Noting that the coefficients of Y_j satisfy (4.4), we obtain by virtue of the lemma A.1 (It's proof will be given in the appendix.), for $y'\varepsilon\ \mathrm{supp}\,(1-\Psi)$,

$$(4.9)\qquad |D_{y'}^{\alpha}Y_1\cdots Y_{|\alpha''|}u)(y')|\leqq CC_2(e^{|x_0-y_0|\gamma}A)^{|\alpha''|}A^{|\alpha|}(|\alpha|+|\alpha'|)!^s ,$$

$$\leqq CC_2(e^{|x_0-y_0|\gamma}A)^{|\alpha''|}A^{|\alpha|}(1+\varepsilon)^{s|\alpha''|}\left(1+\frac{1}{\varepsilon}\right)^{s|\alpha''|}|\alpha|\,!^s|\alpha''|\,!^s ,$$

for any $\varepsilon>0$, where we used the inequality

$$(|\alpha|+|\alpha''|)!\leqq(1+\varepsilon)^{|\alpha|}\left(1+\frac{1}{\varepsilon}\right)^{|\alpha''|}|\alpha|\,!|\alpha''|\,! ,\qquad \varepsilon>0 .$$

We choose later on $\varepsilon=\gamma_0|x_0-y_0|$. We put

$$F_{\alpha''}(\xi')=\int e^{-i<y',\xi'>}(1-\Psi(y'))(Y_1\cdots Y_{|\alpha''|}u)(y')(1+|x'-y'|^2)^{-\frac{n+1}{2}}dy' .$$

Then we have

$$\xi'^{\alpha}F_{\alpha''}(\xi')=\int e^{-i<y',\xi'>}D_{y'}^{\alpha}((1-\Psi)(Y_1\cdots Y_{|\alpha''|}u)(y')(1+|x'-y'|^2)^{-\frac{n+1}{2}})dy'$$

Hence from (4.8) and (4.9)

$$|\xi'^{\alpha}F_{\alpha''}|\leqq CC_3(e^{|x_0-y_0|\gamma}(1+\varepsilon)^sA)^{|\alpha''|}|\alpha''|\,!^s\left(\frac{(1+\varepsilon)^sA}{\varepsilon^s}\right)^{|\alpha|}|\alpha|\,!^{\hat{s}} .$$

for any α. Therefore from Lemma 4.1 it follows that

(4.10) $$|F_{\alpha''}(\xi')|\leqq CC_4(e^{|x_0-y_0|\gamma}(1+\varepsilon)^s A)^{|\alpha''|}|\alpha''|!^s\exp\left\{-\left(\frac{\varepsilon^s|\xi'|}{(1+\varepsilon)^s A}\right)^{1/\hat{s}}\right\}.$$

$$\leqq CC_4(e^{|x_0-y_0|\hat{\gamma}}A)^{|\alpha''|}|\alpha''|!^s\exp\left\{-\frac{|x_0-y_0|\gamma_0}{(1+\varepsilon)A^{1/s}}|\xi'|^{1/\hat{s}}\right\}$$

where we put $\varepsilon=\gamma_0|x_0-y_0|$ and $\hat{\gamma}=\gamma+\gamma_0 s$. We choose γ_0 such that

$$|x_0-y_0|A_2|\xi|^{1/\kappa^{(l)}}-\frac{|x_0-y_0|\gamma_0}{(1+|x_0-y_0|\gamma_0)A^{1/s}}|\xi'|^{1/\hat{s}}\leqq 0 \qquad (|\xi'|\to\infty).$$

When $s<\kappa^{(l)}$, the above is valid. When $s=\kappa^{(l)}$, for $|x_0-y_0|A_2A^{1/s}\leqq 1/2$, we may put $\gamma_0=2A^{1/s}$. Then we can estimate from (4.3) and (4.11)

$$|I_1|\leqq\sum\binom{\alpha}{\alpha'}\int_{|\beta'+\beta''|\leq n+1}\sum|a_{\beta'\beta''}||D_{\xi'}^{\beta'}D_{x'}^{\alpha'}r^{p(l)}||F_{\alpha''}|d\xi'$$

$$\leqq CC_5(e^{|x_0-y_0|\gamma_1}A)^{|\alpha|}|\alpha|!^s,$$

for $|x_0-y_0|\leqq\delta$, ($|x_0-y_0|\leqq\min(\delta, (2A_2A^{\frac{1}{\hat{s}}})^{-1})$, if $s=\kappa^{(l)}$). For I_2, we put

$$G_{\alpha''}=\int e^{-i<y',\xi'>}(\Psi X_1^{\alpha_1''}\cdots X_n^{\alpha_n''}u)(y')dy'.$$

Then noting that supp Ψ is compact, we can prove analogously that $G_{\alpha''}$ satisfies (4.10) and therefore I_2 also satisfies (4.4).

Now we shall construct a solution $F^p[x_0, y_0]$ of the integral equation (1.19). We define inductively

$$F_0^p[x_0, y_0]=R^p[x_0, y_0]=\sum_{l=1}^{d}R^{p(l)}[x_0, y_0],$$

$$F_j^p[x_0, y_0]=\int_{y_0}^{x_0}R^p[x_0, t]F_{j-1}^p[t, y_0]dt, \qquad j=1, 2, \cdots.$$

Then we have

Proposition 4.3. *Let $v(x')$ be in $\gamma_s(R^n)$ and satisfies*

(4.11) $$|D^\alpha v(x')|\leqq CA^{|\alpha|}|\alpha|!^s, \qquad x'\in R^n.$$

Then when $s<\inf_l\kappa^{(l)}$, there exists positive constants $\hat{C}$, $\hat{C}_0$, $\hat{\gamma}$ and $\hat{A}$ independent of j and α such that

$(4.12)_j$ $$|D^\alpha F_j^p[x_0, y_0]v(x')|\leqq\hat{C}C\frac{(\hat{C}_0|x_0-y_0|)^j}{j!}(e^{\gamma|x_0-y_0|}A)^{|\alpha|}|\alpha|!^s,$$

for $|x_0-y_0|\leqq\hat{\delta}$ and $A>\hat{A}$. When $s=\inf\kappa^{(l)}=\hat{\kappa}$, moreover there exists $\hat{\delta}>0$ such that $F_j^p[x_0, y_0]$ satisfies $(4.1)_j$ for $|x_0-y_0|\leqq\inf(\delta, \hat{\delta}A^{-1/\kappa})$ and $A>\hat{\gamma}$.

Proof. We shall prove $(4.12)_j$ by induction. For $j=0$ it follows from Proposition 4.2 that $(4.12)_0$ is valid. Assume that $(4.12)_{j-1}$ is valid. Then again applying Proposition 4.2 to $u=F_{j-1}^{p}[t, y_0]v$, we obtain

$$|R^p[x_0, t]F_j^p[t, y_0]v| \leqq \hat{C}C\frac{(\hat{C}_0(t-y_0))^{j-1}}{(j-1)!}(e^{\gamma(x_0-t)}e^{\gamma(t-y_0)}A)^{|\alpha|}|\alpha|!^s$$

which we integrate in t, we have $(4.12)_j$.

We define

$$F^p[x_0, y_0]=\sum_{j=0}^{\infty}F_j^p[x_0, y_0]$$

which is evidently a solution of (1.19) and satisfies

$$|D_v F^p[x_0, y_0]v(x')| \leqq \hat{C}Ce^{C_0|x_0-y_0|}(e^{\gamma|x_0-y_0|}A)^{|\alpha|}|\alpha|!^s .$$

Thus we have prove that the fundamental solution $K^p[x_0, y_0]$ is given by (1.18) and satisfies

$$|D^\alpha F^p[x_0, y_0]v(x')| \leqq \hat{C}C(e^{\gamma|x_0-y_0|}A)^{|\alpha|}|\alpha|!^s .$$

for $|x_0-y_0| \leqq \delta$ and $A>\hat{A}$ ($|x_0-y_0| \leqq \inf(\delta, \hat{\delta}A^{1/\kappa})$ if $s=\hat{\kappa}$) where v satisfies (4.11).

Appendix

Here we shall prove the estimate (4.9), which is sharper one than Lemma 3.3. Let $X_j=\sum_{i=1}^{n}a_{ij}(x)\frac{\partial}{\partial y_i}+\frac{\partial}{\partial y_j}+\frac{\partial}{\partial x_j}$ $(j=1, 2, \cdots, n)$ be first order differential operators. We assume that

$$\text{(A.1)} \qquad |D_x^\alpha a_{ji}(x)| \leqq \varepsilon_0 A_0^{|\alpha|}|\alpha|!^s, \qquad x\in K,\ i=j=1, \cdots, n ,$$

where K is a compact set in R^n and that $u(y)$ satisfies

$$\text{(A.2)} \qquad |D_y^\alpha u(y)| \leqq CA^{|\alpha|}|\alpha|!^s, \qquad y\in K_1 .$$

Then we have

Lemma A.1. *Assume that $A>A_0$. Then there exist positive constants $\hat{C}$ and γ such that*

$$|D_y^\alpha X_{j_1}X_{j_2}\cdots X_{j_k}u(y)| \leqq C\hat{C}((1+\gamma\varepsilon_0)A)^k A^{|\alpha|}(|\alpha|+k)!^s ,$$

for $p=1, 2, \cdots$, and $x\in K$, $y\in K_1$, where $(j_1, \cdots, j_k)\subset[1, \cdots, n]$.

Proof. We assume that

$$|D_x^\beta D_y^\alpha X_{j_1}\cdots X_{j_k}u(y)| \leqq C\sum_{j=1}^{k}C_{|\beta|,j}^{(k)}A^{|\alpha|+j}(|\alpha|+j)!^s ,$$

$$C_{|\beta|,j}^{(k)}=\sum_{i=0}^{k-j}b_{j,i}^{(k)}A_1^{|\beta|+i}(|\beta|+i)!^s+d_j^{(k)},\ (A_1>A_0) ,$$

where $b_{j,i}^{(k)}$ and $d_k^{(k)}$ are positive constants. We shall prove that there exists a positive constant γ such that for any positive integer k

$$(\mathrm{A}.3)_k \qquad \begin{aligned} &b_{j,i}^{(k)} \leqq (1+\gamma\varepsilon_0)^j \frac{k!}{i!j!}, \qquad \begin{aligned}&i=0,1,\cdots,k-j,\\ &j=1,\cdots,k,\end{aligned}\\ &d_j^{(k)} \leqq \frac{k!}{j!}, \qquad j=1,\cdots,k. \end{aligned}$$

We shall at first derive the recursive relation of $C_{|\beta|,j}^{(k)}$. We have

$$\begin{aligned}
&|(D_x^\beta D_y^\alpha X_{j_1})(X_{j_2}\cdots X_{j_k}u)|\\
&=\left|D_x^\beta D_y^\alpha\Big(\sum a_{j_1 i}(x)\frac{\partial}{\partial y_i}+\frac{\partial}{\partial y_{j_1}}+\frac{\partial}{\partial x_{j_1}}\Big)X_{j_2}\cdots X_{j_k}u\right|\\
&\leqq \sum\binom{\beta}{\beta'}|D_x^{\beta'}a_{j_1 i}|\left|D_x^{\beta''}D_y^\alpha\frac{\partial}{\partial y_i}X_{j_2}\cdots X_{j_k}u\right|\\
&\quad+\left|D_x^\beta D_y^\alpha\frac{\partial}{\partial y_{j_1}}X_{j_2}\cdots X_{j_k}u\right|+\left|D_x^\beta\frac{\partial}{\partial x_{j_1}}D_y^\alpha X_{j_2}\cdots X_{j_k}u\right|\\
&\leqq C\sum\binom{\beta}{\beta'}n\varepsilon_0 A_0^{|\beta'|}|\beta'|!^s\sum_{j=1}^{k-1}C_{|\beta''|,j}^{(k-1)}A^{|\alpha|+1+j}(|\alpha|+1+j)!^s\\
&\quad+C\sum_{j=1}^{k-1}C_{|\beta|,j}^{(k-1)}A^{|\alpha|+1+j}(|\alpha|+1+j)!^s\\
&\quad+C\sum_{j=1}^{k-1}C_{|\beta|+1,j}^{(k-1)}A^{|\alpha|+j}(|\alpha|+j)!^s\\
&=C\Big\{n\varepsilon_0\sum_{j=2}^{k}\Big(\sum_{\beta'}\binom{\beta}{\beta'}A_0^{|\beta'|}|\beta'|!^s C_{|\beta''|,j-1}^{(k-1)}\Big)A^{|\alpha|+j}(|\alpha|+j)!^s\\
&\quad+\sum_{j=2}^{k}C_{|\beta|,j-1}^{(k-1)}A^{|\alpha|+j}(|\alpha|+j)!^s\\
&\quad+\sum_{j=1}^{k-1}C_{|\beta|+j,1}^{(k-1)}A^{|\alpha|+j}(|\alpha|+j)!^s\Big\}.
\end{aligned}$$

Hence we obtain for $k\geqq 2$

$$(\mathrm{A}.4) \qquad C_{|\beta|,1}^{(k)}=C_{|\beta|+1,1}^{(k-1)}$$

$$(\mathrm{A}.5) \qquad C_{|\beta|,j}^{(k)}=C_{|\beta|+1,j}^{(k-1)}+C_{|\beta|,j-1}^{(k-1)}+n\varepsilon_0\sum_{\beta'+\beta''=\beta}\binom{\beta}{\beta'}A_0^{|\beta'|}|\beta'|!^s C_{|\beta''|,j-1}^{(k-1)}, \qquad j=2,\cdots,k-1,$$

$$(\mathrm{A}.6) \qquad C_{|\beta|,k}^{(k)}=C_{|\beta|,k-1}^{(k-1)}+n\varepsilon_0\sum_{\beta'}\binom{\beta}{\beta'}A_0^{|\beta'|}C_{|\beta''|,k-1}^{(k-1)}.$$

From (A.4) we have

$$\begin{aligned}
\sum_{i=0}^{k-1}b_{1,i}^{(k)}A_1^{|\beta|+i}(|\beta|+i)!^s+d_1^{(k)}&=\sum_{i=0}^{k-1-1}b_{1,i}^{(k-1)}A_1^{|\beta|+1+i}(|\beta|+1+i)!^s+d_1^{(k-1)}\\
&=\sum_{i=1}^{k-1}b_{1,i-1}^{(k-1)}A_1^{|\beta|+i}(|\beta|+i)!^s+d_1^{(k-1)}.
\end{aligned}$$

Hence we obtain for $k \geqq 2$,

$$b_{1,0}^{(k)}=0$$

$$b_{1,i}^{(k)}=b_{1,i-1}^{(k-1)}, \qquad i=1, \cdots, k-1,$$

$$d_1^{(k)}=d_1^{(k-1)}.$$

For $k=1$ noting that

$$\begin{aligned}
|D_x^\beta D_y^\alpha X_{j_1} u(y)| &\leqq \sum \left| D_x^\beta a_{j_1 i}(x) \frac{\partial}{\partial y_i} D_y^\alpha u(y) \right| + \left| D_x^\beta D_y^\alpha \frac{\partial}{\partial y_{j_1}} u(y) \right| \\
&\leqq \begin{cases} n\varepsilon_0 C A_0^{|\beta|} |\beta|!^s A^{|\alpha|+1}(|\alpha|+1)!^s, & (|\beta| \neq 0) \\ C(n\varepsilon_0+1) A_0^{|\alpha|+1}(|\alpha|+1)!^s, & (|\beta|=0) \end{cases} \\
&\leqq C(n\varepsilon_0 A_1^{|\beta|} |\beta|!^s+1)(|\alpha|+1)!^s, \qquad (A_1 > A_0),
\end{aligned}$$

we have

$$(\mathrm{A}.7)_1 \qquad \begin{aligned} &b_{1,0}^{(1)}=\gamma\varepsilon_0 \\ &d_1^{(1)}=1. \end{aligned}$$

where $\gamma=nA_1/(A_1-A_0)$. Hence we have

$$(\mathrm{A}.7)_k \qquad \begin{aligned} &b_{1,i}^{(k)}=0, \qquad i=0, \cdots, k-2, \\ &b_{1,k-1}^{(k)}=1, \\ &d_1^{(k)}=1. \end{aligned}$$

For $j \geqq 2$, we have from (A.5)

$$\begin{aligned}
C_{|\beta|,j}^{(k)} \leqq & \sum_{i=0}^{k-1-j} b_{j,i}^{(k-1)} A_1^{|\beta|+1+i}(|\beta|+1+i)!^s + d_j^{(k-1)} \\
& + \sum_{i=0}^{k-1-j+1} b_{j-1,i}^{(k-1)} A_1^{|\beta|+i}(|\beta|+i)!^s + d_{j-1}^{(k-1)} \\
& + n\varepsilon_0 \sum_{\beta'+\beta''=\beta} \binom{\beta}{\beta'} A_0^{|\beta'|} |\beta'|!^s \{ \sum_{i=0}^{k-1-j+1} b_{j-1,i}^{(k-1)} A_1^{|\beta''|+i}(|\beta''|+i)!^s + d_{j-1}^{(k-1)} \} \\
\leqq & \sum_{i=0}^{k-j} b_{j,i-1}^{(k-1)} A_1^{|\beta|+i}(|\beta|+i)!^s + d_j^{(k-1)} \\
& + \sum_{i=0}^{k-j} b_{k-1,i}^{(k-1)} A_1^{|\beta|+i}(|\beta|+i)!^s + d_{j-1}^{(k-1)} \\
& + \gamma\varepsilon_0 \sum_{i=0}^{k-j} b_{j-1,i}^{(k-)} A_1^{|\beta|+i}(|\beta|+i)!^s \\
& + \gamma\varepsilon_0 d_{j-1}^{(k-1)} A_1^{|\beta|} |\beta|!^s,
\end{aligned}$$

where we used Lemma 3.1 and $\gamma=nA_1/(A_1-A_0)$. Hence we obtain

$$b_{i,0}^{(k)}=\gamma\varepsilon_0 d_{j-1}^{(k-1)}+b_{j-1,0}^{(k-1)}(\gamma\varepsilon_0+1)$$

$$\text{(A.8)}\qquad b_{j,i}^{(k)}=b_{j,i-1}^{(k-1)}+b_{j-1,i}^{(k-1)}(\gamma\varepsilon_0+1)\,,\qquad i=1,\cdots,k-j\,,$$

$$d_j^{(k)}=d_j^{(k-1)}+d_{j-1}^{(k-1)}\,,\qquad j=2,\cdots,k-1\,.$$

From (A.6) we have

$$C_{|\beta|,k}^{(k)}\leqq(b_{k-1,0}^{(k-1)}(\gamma\varepsilon_0+1)+\gamma\varepsilon_0 d_{k-1}^{(k-1)})A_1^{|\beta|}|\beta|\,!^s+d_{k-1}^{(k-1)},$$

from which it follows

$$b_{k,0}^{(k)}=b_{k-1,0}^{(k-1)}(\gamma\varepsilon_0+1)+\gamma\varepsilon_0 d_{k-1}^{(k-1)}$$

$$\text{(A.1)}\qquad d_k^{(k)}=d_{k-1}^{(k-1)}=d_1^{(1)}=1\,.$$

We shall prove by induction on k that the $b_{j,i}^{(k)}$ and $d_j^{(k)}$ given by (A.7) and (A.8) satisfy $(A.3)_k$. It is evident from $(A.7)_1$ that $(A.3)_1$ is valid. Assume that $(A.7)_p$ $(p=1,\cdots,k-1)$ are valid. Then by virtue of (A.8) we have for $j\leqq k-1$,

$$d_j^{(k)}\leqq\frac{(k-1)\,!}{j\,!}+\frac{(k-1)\,!}{(j-1)\,!}=\frac{k\,!}{j\,!}\frac{j+1}{k}\leqq\frac{k\,!}{j\,!}\,,$$

$$\begin{aligned}b_{j,0}^{(k)}&\leqq\gamma\varepsilon_0\frac{(k-1)\,!^s}{(j-1)\,!^s}+(1+\gamma\varepsilon_0)^{j-1}\frac{(k-1)\,!^s}{(j-1)\,!^s}(\gamma\varepsilon_0+1)\\&\leqq(1+\gamma\varepsilon_0)^j\frac{k\,!^s}{j\,!^s}\Big(\frac{\gamma\varepsilon_0 j}{(1+\gamma\varepsilon_0)^j k}+\frac{j}{k}\Big)\\&\leqq(1+\gamma\varepsilon_0)^j\frac{k\,!^s}{j\,!^s}\Big(\frac{j+1}{k}\Big)\\&\leqq(1+\gamma\varepsilon_0)^j\frac{k\,!^s}{j\,!^s}\,,\end{aligned}$$

where we used the inequality $\gamma\varepsilon_0 j\leqq(1+\gamma\varepsilon_0)^j$, and

$$\begin{aligned}b_{j,i}^{(k)}&\leqq(1+\gamma\varepsilon_0)^j\frac{(k-1)\,!}{j\,!(i-1)\,!}+(1+\gamma\varepsilon_0)\frac{j(k-1)\,!}{(j-1)\,!i\,!}\\&=(1+\gamma\varepsilon_0)^j\frac{k\,!}{j\,!i\,!}\frac{i+j}{k}\leqq(1+\gamma\varepsilon_0)^j\frac{k\,!}{j\,!i\,!}\,.\end{aligned}$$

It is evident from (A.9) that $d_k^{(k)}$ satisfies $(A.3)_k$ and

$$b_{k,0}^{(k)}=(1+\gamma\varepsilon_0)^k-1\leqq(1+\gamma\varepsilon_0)^k\,.$$

Thus we have proved $(A.3)_k$. Hence we have

$$\begin{aligned}|D_y^\alpha X_{j_1}\cdots X_{j_k}u(y)|&\leqq C\sum_{j=1}^k C_{0,j}^{(k)}A^{|\alpha|+j}(|\alpha|+j)\,!^s\\&\leqq C\sum_{j=1}^k\Big\{\sum_{i=1}^{k-j}(1+\gamma\varepsilon_0)^j\frac{k\,!}{j\,!i\,!}A_1^i i\,!^s+\frac{k\,!}{j\,!}\Big\}\quad A^{|\alpha|+j}(|\alpha|+j)\,!^s\,,\\&\leqq C\sum_{j=1}^h\binom{k}{j}(1+\gamma\varepsilon_0)^j A^{|\alpha|+j}(|\alpha|+j)\,!^s\sum_{i=0}^{k-j}\binom{k-j}{i}A_1^i(k-j-i)\,!i\,!^s\,,\end{aligned}$$

which and Lemma 3.1 imply our lemma, if we choose $\hat{C}$ suitably and A_1 such that $A>A_1>A_0$.

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