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# Dirichlet finite harmonic differentials with integral periods on arbitrary Riemann surfaces

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## Introduction

In this paper we consider Dirichlet finite harmonic differentials with integral periods on arbitrary Riemann surfaces. From such a differential $\sigma$ on an arbitrarily given Riemann surface $R$, we can construct a mapping $u_\sigma(p)$ (, for the definition, see §1,) from $R$ into $S^1=\{|z|=1\}$, and we can take $u_\sigma^{-1}(t)$ as a "level set" of $\sigma$ for every $t \in S^1$. Such a mapping can be extended continuously onto the Royden's compactification $R^*$ of $R$. Now Theorem 1 in §1 states that for almost all $t$ in $S^1-u_\sigma(\Delta)$ the set $u_\sigma^{-1}(t)$ consists only of (at most countable number of) simple closed (, hence compact) curves in $R$, where $\Delta$ is the harmonic boundary of $R^*$. In particular, if $u_\sigma(\Delta)$ is a set of linear measure zero on $S^1$, then the holomorphic quadratic differential $(-{}^*\sigma+\sqrt{-1}\cdot 1\sigma)^2$ has closed trajectories (in the sense of K. Strebel).

Next Theorem 2 states that if $t_1$ and $t_2$ are contained in the same component of $S^1-u_\sigma(\Delta)$, then the "level sets" $u_\sigma^{-1}(t_1)$ and $u_\sigma^{-1}(t_2)$ have same length with respect to the metric naturally induced by $\sigma$ (, or equivalently, ${}^*\sigma$ has same periods along $u_\sigma^{-1}(t_1)$ and $u_\sigma^{-1}(t_2)$ with suitable orientations).

Definitions and main theorems are stated in §1, and the applications are made to basic differentials and functions such as reproducing differentials for 1-cycles, Green's functions and harmonic measures in §2. Proofs of main theorems are given in §3, and examples are provided in §4.

## §1. Definitions and main results

Let $R$ be an arbitrary Riemann surface and $\Gamma_h(R)$ be the Hilbert space of square integrable *real* harmonic differentials on $R$. We say that a differential $\sigma$ in $\Gamma_h(R)$ has *integral periods* if $\int_c \sigma$ is an integer for every 1-cycle $c$ on $R$, and set

$$\Gamma_{hI}(R)=\{\sigma\in\Gamma_h(R)\colon \sigma \text{ has integral periods}\}.$$

Here note that $\Gamma_{he}(R)$ is clearly contained in $\Gamma_{hI}(R)$. For every $\sigma\in\Gamma_{hI}(R)$ and arbitrarily fixed point $p_0\in R$ and real constant $a_0$,

$$u_\sigma(p)=\exp\left[2\pi\sqrt{-1}\cdot\left(\int_{p_0}^{p}\sigma+a_0\right)\right]$$

is well defined function on $R$ whose values fall in $S^1=\{|z|=1\}$. We call $u_\sigma(p)$ *a circular function for* $\sigma$ (with the base point $p_0$ and the additive constant $a_0$). For the later use, a circular function $u_\sigma(p)$ for a given $\sigma$ need to be determined up to multiplicative factor of modulus one, so the choice of $p_0$ and $a_0$ are inessential.

Now since it holds that

$$|\mathrm{grad}\, u_\sigma(p)|=2\pi|u_\sigma(p)\cdot\sigma|=2\pi|\sigma|$$

(, where $|\sigma|^2=(a(z)^2+b(z)^2)dxdy$ if $\sigma=a(z)dx+b(z)dy$ with a local parameter $z=x+\sqrt{-1}\cdot y$), $u_\sigma(p)$ is a (complex) Dirichlet finite function on $R$, hence can be extended to a continuous function from the Royden's compactification $R^*$ of $R$ into $S^1$, which is denoted by the same $u_\sigma(p)$. For the details of the theory of the Royden's compactification, see [4] or [11]. Let $\Delta$ be the harmonic boundary of $R^*$, and set $E_{u_\sigma}=u_\sigma(\Delta)$, which we call *the set of essential boundary values of* $u_\sigma$. Because $\Delta$ is compact, $E_{u_\sigma}$ is also compact, so $S^1-E_{u_\sigma}$ consists of at most countably many open arcs on $S^1$, each of which we call *a supplementary interval for* $u_\sigma$.

Next, in general, for a meromorphic quadratic differential $\varphi$ on $R$, a trajectory of $\varphi$ is defined as a maximal curve along which $\varphi$ is positive. A trajectory $\gamma$ of $\varphi$ is called *critical* if $\gamma$ tends to a zero or a pole of $\varphi$ in either direction, and *regular* if otherwise. Recall that the number of critical trajectories of $\varphi$ is at most countable, and that a compact regular trajectory is a simple closed curve (i.e, Jordan curve). Now we call a meromorphic quadratic differential $\varphi$ *has closed trajectories* (cf. [12]) if $\varphi\equiv 0$ or the complement of all compact regular trajectories is a set of 2-dimensional measure zero (, i.e. its intersection with every parameter neighbourhood has area measure zero). Note that $\varphi$ has closed trajectories if and only if the complement of all compact regular trajectories has zero area with respect to the metric $|\varphi|$. We set

$CA_1(R)=\{\theta$: $\theta$ is a meromorphic abelian differential whose square has closed trajectories$\}$.

Now for every $\sigma\in\Gamma_h(R)$ we set $\theta_\sigma=-{}^*\sigma+\sqrt{-1}\cdot\sigma$ and call the holomorphic differential *associated with* $\sigma$. Then one of main results can be stated as follows.

**Theorem 1.** *Let $R$ be an arbitrary Riemann surface, $\sigma\in\Gamma_{hl}(R)$, $u_\sigma(p)$ a circular function for $\sigma$ and $E_{u_\sigma}$ the set of essential boundary values of $u_\sigma$. Then for almost every $t$ in $S^1-E_{u_\sigma}$, the set $u_\sigma^{-1}(t)$ consists only of compact regular trajectories of $\theta_\sigma^2$, where $\theta_\sigma$ is the holomorphic differential associated with $\sigma$.*

*In particular, if the linear measure of $E_{u_\sigma}$ is zero, then the holomorphic differential $\theta_\sigma$ associated with $\sigma$ belongs to $CA_1(R)$.*

**Remark.** Even if $\sigma$ has not discrete periods, the differential $\theta_\sigma$ associated with $\sigma$ may belong to $CA_1(R)$. Such examples for compact surfaces are already known. Also note that the set $u_\sigma^{-1}(t)$ generally consists of trajectories and possibly zeros of $\theta_\sigma^2$ for every $t\in S^1$.

Finally for every $\sigma\in\Gamma_{hl}(R)$, let $u_\sigma(p)$, $E_{u_\sigma}$ and $\theta_\sigma$ be as before and $\{I_{u_\sigma}^i\}$ be the set of all supplementary intervals for $u_\sigma(p)$. Then for every $t \in S^1$ we call the set of all trajectories of $\theta_\sigma^2$ contained in $u_\sigma^{-1}(t)$ *the set of level curves of* $\sigma$ *for* $t$ (with respect to $u_\sigma(p)$), and denote by $L_t$. This $L_t$ is nothing but $u_\sigma^{-1}(t)$ deleted all zeros of $\theta_\sigma$ as a point set. Also note that the set $L_t$ depends on the choice of $u_\sigma(p)$, but the whole family $\{L_t\}_{t\in S^1}$ depends only on $\sigma$. For every $t \in S^1$, let $L_t = \bigcup_j c_j$, where every $c_j$ is a trajectory of $\theta_\sigma^2$, and in the sequel we assume that every $c_j$ is oriented so that $\int_{c_j} \theta_\sigma \left(= \int_{c_j} -{}^*\sigma\right)$ is positive. Then the length $m_\sigma(t)$ of $L_t$ is defined by

$$m_\sigma(t) = \sum_j \int_{c_j} \theta_\sigma \left( = \sum_j \int_{c_j} |\theta_\sigma| \right)$$

Here if $L_t$ is the empty set, then we consider that $m_\sigma(t)=0$. And we show the following

**Theorem 2.** *Let $R$ be an arbitrary Riemann surface, $\sigma \in \Gamma_{hI}(R)$, $u_\sigma(p)$ a circular function for $\sigma$, $E_{u_\sigma}$ the set of essential boundary values of $u$ and $\{I_{u_\sigma}^i\}$ the set of all supplementary intervals for $u_\sigma(p)$. Then if $t_1$ and $t_2$ are contained in the same $I_{u_\sigma}^i$, then it holds that*

$$m_\sigma(t_1) = m_\sigma(t_2).$$

*Moreover if the linear measure of $E_{u_\sigma}$ is zero, then it holds that*

$$\|\sigma\|_R^2 = \sum_i a(I_{u_\sigma}^i) \cdot m_i,$$

*where $m_i = m_\sigma(t)$ with some (, hence every) $t$ in $I_{u_\sigma}^i$, $a(I_{u_\sigma}^i) = \alpha_2 - \alpha_1$ when $I_{u_\sigma}^i = \{\exp(2\pi\sqrt{-1}\cdot\alpha)\colon \alpha_1 < \alpha < \alpha_2\}$ for every $i$, and $\|\sigma\|_R$ is the Dirichlet norm of $\sigma$ on $R$.*

The proofs of Theorems 1 and 2 will be given in §3.

## §2. Applications

First for an arbitrary Riemann surface $R$, we set

$A_1S(R) = \Big\{\theta\colon \theta$ is a meromorphic abelian differential on $R$ such that $\theta$ has an expansion as $\left(\frac{a}{z} + \text{regular terms}\right)dz$ with purely imaginary number $a$ at every pole$\Big\}$, and

$A_1S_0(R) = \{\theta \in A_1S(R)\colon \operatorname{Im}\theta$ has $\Gamma_{\{0\}}$-behavior, namely, there is a canonical region $G$ in $R$ such that for every component $U$ of $R - \bar{G}$ we can find a function $f_U$ (of $C^\infty$-class) on $R$ such that $df_U \in \Gamma_{e0}(R)$ and $df_U \equiv \operatorname{Im}\theta$ on $U\}$

Note that if $R$ is compact, then $A_1S(R) = A_1S_0(R)$, and hence the following Proposition is a generalization of [13] Lemma 6.

**Proposition 1.** *Let $R$ be an arbitrary Riemann surface and $\theta \in A_1S_0(R)$. If $\operatorname{Im}\theta$ has integral periods, then $\theta$ belongs to $CA_1(R)$.*

*Proof.* First because $\operatorname{Im}\theta$ has $\Gamma_{\{0\}}$-behavior, there is a canonical region $G$

as in the definition of $A_1S_0(R)$. Here we may assume without loss of generality that all poles of $\theta$ are contained in $G$. Since $G$ is relatively compact, $\theta$ has only a finitely many poles, which are denoted by $\{p_i\}_{i=1}^N$. Next the square $\theta^2$ has such an expansion as $\left(\frac{a}{z^2}+\frac{b}{z}+\text{regular terms}\right)dz^2$ with some real negative $a$ near $p_i$ for every $i$. Hence from the local structure theorem near a pole of order two (cf. [5]) we see that there is a compact regular trajectory, say $\gamma_i$, of $\theta^2$ freely homotopic to the point $p_i$ in $G$ for every $i$.

Now let $D_i$ be the disk in $G$ surrounded by $\gamma_i$ for every $i$ and set $R'=R-\bigcup_{i=1}^N \overline{D}_i$ and $\theta'\equiv\theta$ restricted on $R'$. Then clearly $\operatorname{Im}\theta'$ belongs to $\Gamma_h(R')$ and has integral periods. So we can define a circular function $u'(p)$ for $\operatorname{Im}\theta'$. Here note that the number of connected components of $R'-G$ $(=R-G)$ is finite. And it is easily seen from the assumption that $u'(\Delta'\cap\overline{U})$ is a constant for every component $U$ of $R'-G$, where $\Delta'$ is the harmonic boundary of $R'^*$ and $\overline{U}$ is the closure of $U$ in $R'^*$. (Recall that $\Delta'\cap(R'^*-G)$ can be identified with $\Delta$). Also $u'(\gamma_i)$ is a constant for every $i$, for $\operatorname{Im}\theta=0$ along every $\gamma_i$ from the definition. Thus we conclude that the set $E_{u'}$ of essential boundary values of $u'$ consists of a finite number of points, and the assertion follows from Theorem 1. q.e.d.

Next take a 1-cycle $c$ on an arbitrarily given Riemann surface $R$. Then *the reproducing differential* $\sigma(c)$ on $R$ is characterized by the condition that

$$\int_c \omega=(\omega,\sigma(c)) \qquad \text{for every} \quad \omega\in\Gamma_h(R).$$

For basic facts on the theory of differentials, including reproducing ones, see for example, [3] Ch. V or [7] Ch. 9. Here we recall a standard construction of $*\sigma(c)$. First let $c$ be the homology class of $\sum_{k=1}^m n_k\cdot c_k$ with suitable integers $n_k$ and oriented simple closed curves $c_k$ on $R$. Then take a relatively compact annular neighbourhood $U_k$ of $c_k$ and consider a function $v_k$ of the $C^\infty$-class on $R-c_k$ with a compact support in R such that $v_k\equiv1$ on the right side component of $U_k-c_k$ and $v_k\equiv0$ on the other side. And then it is known that

$$\sum_{k=1}^m n_k\cdot dv_k=*\sigma(c)+df \qquad \text{with} \quad df\in\Gamma_{e0}(R).$$

In particular, it is clear that $*\sigma(c)$ has integral periods.

**Proposition 2.** *Let $R$ and $c$ be as above. Then the holomorphic reproducing differential $\theta_c=\sigma(c)+\sqrt{-1}\cdot*\sigma(c)$ (, i.e. the holomorphic differential associated with $*\sigma(c)$) belongs to $CA_1(R)$.*

*Moreover let $u(p)$ be a circular function for $*\sigma(c)$ and denote by $m(t)$ the length $m_{*\sigma(c)}(t)$ of the set $L_t$ of level curves of $*\sigma(c)$ for $t$ for every $t\in S^1$. Then it holds that*

$$m(t)=\|\sigma(c)\|_R^2 \quad (=\|*\sigma(c)\|_R^2)$$

*for every $t\in S^1$ except for at most single one value.*

*Proof.* Note that $u(p)=\exp\left[2\pi\sqrt{-1}\left(\int_{p_0}^{p}\sum_{k=1}^{m} n_k\cdot dv_k+a_0\right)\right]$ on the harmonic boundary of $R^*$, where $n_k$ and $v_k$ are as in the above construction of $*\sigma(c)$ and $p_0\in R$ and $a_0$ are suitably chosen. So we can see that the set $E_u$ of essential boundary values of $u$ is empty or consists of only one value $\exp(2\pi\sqrt{-1}\cdot a_0)$. Thus the assertions follow from Theorems 1 and 2. q.e.d.

**Remark.** Proposition 2 is essentially due to Accola. See [1] and [2]. (Also cf. [8] and [13].) Note that the first assertion of Proposition 2 is also a corollary of Proposition 1, for $\theta_c$ belongs to $A_1S_0(R)$ (, i.e. it is clear from the construction that $*\sigma(c)$ has $\Gamma_{\{0\}}$-behavior.)

Now we state a special but important case of Propositions 1 and 2.

**Corollary 1** ([2] *and* [10]). *Let $g(p, q)$ be the Green's function on a hyperbolic Riemann surface $R$ ($R\notin O_G$) with the pole $q\in R$. Then the meromorphic differential $\theta_q=-*dg(p, q)+\sqrt{-1}\cdot dg(p, q)$ belongs to $CA_1(R)$.*

*Moreover for every $t>0$, it holds that*

$$\int_{\partial\{p\in R:g(p,q)>t\}}-*dg(p, q)=2\pi.$$

*Proof.* The first assertion follows from Proposition 1, for $\theta_q$ belongs to $A_1S_0(R)$. Anyway, recall that for a sufficiently large $a$, the region $D_a=\{p\in R\colon g(p, q)>a\}$ is a disk in $R$ with an analytic boundary (cf. the proof of Proposition 1), and that on the surface $R'=R-\overline{D_a}$ we have

$$a^{-1}\cdot dg(p, q)=*\sigma(c),$$

where $c$ is a simple closed curve freely homotopic to the boundary of $D_a$ on $R'$. Hence the assertions follows from Proposition 2. (Here the exceptional value 1 for a circular function $u(p)=\exp[2\pi\sqrt{-1}\cdot a^{-1}\cdot g(p, q)]$ for $dg(p, q)$ on $R'$ corresponds to $t=0$ and $a$, hence is never taken in $R'$.) q.e.d

Finally we consider HD-harmonic measures. Here we call a Dirichlet finite positive harmonic function $h(p)$ on $R$ *an HD-harmonic measure* if the greatest harmonic minorant of $h(p)$ and 1-$h(p)$ is identically zero. (Hence obviously $0\leq h(p)\leq 1$.) Recall that $h(p)=\int_{p_0}^{p}*\sigma(c)-a_0$ with suitable $p_0\in R$ and $a_0\in(0, 1)$ is an HD-harmonic measure for every dividing Jordan curve $c$ on $R$.

**Proposition 3.** *Let $R$ be an arbitrary Riemann surface and $h(p)$ an HD-harmonic measure. Then the holomorphic differential $-*dh+\sqrt{-1}\cdot dh$ associated with $dh$ belongs to $CA_1(R)$. And for every $t$ in $(0, 1)$, it holds that*

$$\int_{\{p\in R:h(p)=t\}}*dh=\|dh\|_R^2$$

*where every curve in $\{p\in R\colon h(p)=t\}$ is oriented so that $*dh$ is positive along it.*

*Proof.* Because $u(p)=\exp[2\pi\sqrt{-1}\cdot h(p)]$ is a circular function for $dh$, and the set $E_u$ is clearly $\{1\}$. Thus the assertion follows from Theorems 1 and 2. (Here the exceptional value 1 is never taken in $R$.) q.e.d.

## § 3. The proofs of Theorems

Let $R$ be an arbitrary Riemann surface, $\sigma \in \Gamma_{hI}(R)$, $u_\sigma(p)$ a circular function for $\sigma$ and $\theta_\sigma$ the holomorphic differential associated with $\sigma$. Also we set

$$E_0 = \{u_\sigma(p) \colon p \text{ is a zero of } \theta_\sigma\}.$$

Then obviously $E_0$ is a countable set, and for every $t \in S^1 - E_0$ the set $u^{-1}(t)$ consists only of analytic curves (, hence coincident with $L_t$ defined in §1). We note the following lemma essentially due to Accola (cf. [1]).

**Lemma 1.** *For every $t \in S^1 - E_0$, the set $L_t$ of level curves (with the relative topology induced from that of $R$) is locally connected.*

*Proof.* Take $p_0 \in L_t$ arbitrary, and set $z(p) = \int_{p_0}^{p} \theta_\sigma$, then because $t \in S^1 - E_0$, $z(p)$ is local parameter near $p_0$. Now take a sufficiently small positive $\varepsilon(<1)$ and set $U_\varepsilon = \{p \in R \colon |z(p)| < \varepsilon\}$. If $q \in L_t \cap U_\varepsilon$, let $\beta$ be an arc connecting $p_0$ and $q$ in $U_\varepsilon$. Then $u_\sigma(\beta)$ is an arc in $S^1$ starting from and ending at $t$, which implies that $\operatorname{Im} z(q) = \int_\beta \sigma$ is an integer. Hence we conclude that $L_t \cap U_\varepsilon$ is the arc $\{p \in R \colon \operatorname{Im} z(p) = 0, |\operatorname{Re} z(p)| < \varepsilon\}$, which shows the assertion. q.e.d.

In particular, $L_t$ consists of simple analytic curves not accumulating to any point in $R$ and every non-compact component of $L_t$ tends to the ideal boundary of $R$ in both directions for every $t \in S^1 - E_0$. Here we recall the following crucial lemma due to Y. Kusunoki and S. Mori [6] (; also see [11] III2G).

**Lemma 2.** *Let $G$ be a subregion of $R$ whose relative boundary components are simple analytic curves not accumulating to any point in $R$. And suppose that the closure of $G$ in $R^*$ is disjoint from the harmonic boundary. Then the double $\hat{G}$ of $G$ along the relative boundary of $G$ is parabolic (, i.e. $\hat{G} \in O_G$).*

Now let $E_{u_\sigma} = u_\sigma(\Delta)$ and $\{I_u^i\}$ be as in §1, and fix a supplementary interval $I_u^i$. Then for every interval $I = \{\exp(2\pi\sqrt{-1}\cdot\alpha) \colon \alpha_1 < \alpha < \alpha_2\}$ such that $\exp(2\pi\sqrt{-1}\cdot\alpha_j)$ are not contained in $E_0(j=1,2)$ and $\bar{I}$ is contained in $I_{u_\sigma}^i$, the closure of $u_\sigma^{-1}(I)$ in $R^*$ is disjoint from $\Delta$, hence by Lemma 2 we see that the double $\hat{G}$ of G is parabolic for every component $G$ of $u_\sigma^{-1}(I)$. Also we can exteded $\sigma$ restricted on such a G anti-symmetrically to an element, say $\hat{\sigma}$, in $\Gamma_h(\hat{G})$. (Namely, letting $j(p)$ be the canonical anti-conformal involution fixing the relative boundary of $G$, we define $\hat{\theta} = \theta_\sigma$ on $G$ and $\hat{\theta} = \overline{\theta_\sigma \circ j}$ on $\hat{G} - G$, then $\hat{\sigma}$ is, by definition, $\operatorname{Im} \hat{\theta}$. See, for example, [3] Ch. V. 13.) Here it is clear from the construction that $(1/2(\alpha_2 - \alpha_1)) \cdot \hat{\sigma}$ has integral periods, hence Theorems 1 and 2 can be reduced to the following Lemma, which seems to be interesting in itself.

**Lemma 3.** *Let $R$ be a parabolic Riemann surface, $\theta \in \Gamma_{hI}(R)$ and $u_\sigma(p)$ a circular function for $\sigma$. Then the holomorphic differential $\theta_\sigma$ associated with $\sigma$ belongs to $CA_1(R)$ and for every $t \in S^1$ it holds that*

$$m_\sigma(t) = \|\sigma\|_R^2.$$

Lemma 3 was stated by Accola in [2], but for the sake of completeness we give a proof, which depnds on the following classical result.

**Lemma 4** (cf. [9]). *Let $R$ be a parabolic Riemann surface and $\theta$ be a square integrable holomorphic differential on $R$. Then there is an exhaustion $\{\Omega_n\}_{n=1}^\infty$ of $R$ such that the relative boundary $\partial\Omega_n$ of $\Omega_n$ consists of a finite number of analytic simple closed curves for every $n$, and it holds that*

$$\lim_{n\to\infty}\int_{\partial\Omega_n}|\theta| = 0.$$

*Proof of Lemma* 3. First suppose that $\theta_\sigma$ does not belong to $CA_1(R)$, and $X$ be the complement of the set of all compact regular trajectories of $\theta_\sigma^2$. (Note that $X$ is a closed set, for sufficiently small neighbourhood of a compact regular trajectory is swept out by mutually freely homotopic compact regular trajectories.) Then there is a point $p_0 \in R$ such that the area measure of $X \cap U$ is positive for every parameter neighbourhood $U$ of $p_0$. Here we may assume that $p_0$ is not a zero of $\theta_\sigma$, so $z(p)$ in Lemma 1 is a local parameter near $p_0$ and $X$ near $p_0$ corresponds to a set of lines parallel to the real axis near $z=0$ on the $z$-plane. So $u_\sigma(X \cap J)$ has positive linear measure for every open are $J$ containing $p_0$ along which $^*\sigma = 0$. Because critical trajectories are countable, we can find a measurable set, say $F$, in $u_\sigma(X \cap J)$ such that $F$ has positive linear measure and there is a non-compact regular trajectory $\gamma_t$ intersecting with $J$ for every $t \in F$. In the sequel we fix a relatively compact arc $J$ and measurable set $F$ in $u_\sigma(X \cap J)$ as above.

Now let $\{\Omega_n\}_{n=1}^\infty$ be as in Lemma 4 with $\theta = \theta_\sigma$, namely,

$$\lim_{n\to\infty}\int_{\partial\Omega_n}|\theta_\sigma| = 0.$$

Also we may assume that $J$ is contained in $\Omega_1$. Then since $\gamma_t$ tends to the ideal boundary of $R$ for every $t \in F$ (as noted after Lemma 1), hence should intersect with every $\partial\Omega_n$ for every $t \in F$, we see that $u_\sigma(\partial\Omega_n)$ contains $F$ for every $n$. So it should holds that

$$\int_{\partial\Omega_n}|\theta_\sigma| \geq \int_{\partial\Omega_n}|\sigma| \geq a > 0$$

for every $n$, where $2\pi a$ is the linear measure of $F$. Thus it should holds that $\lim_{n\to\infty}\cdot\int_{\partial\Omega_n}|\theta_\sigma| \geq a > 0$, which is a contradiction, and we conclude that $\theta_\sigma$ belongs to $CA_1(R)$.

Next note that $\sigma$ can not be exact, hence $u_\sigma(R) = S^1$. Now we take any open arc $I = \{\exp(2\pi\sqrt{-1}\cdot\alpha)\colon \alpha_1 < \alpha < \alpha_2\}$ and apply the Green-Stokes' formula to $\theta_\sigma$

on the union $\Omega_n \cap u_\sigma^{-1}(I)$ of regions. Here because the relative boundary, say $\Gamma_n$, $\Omega_n \cap u_\sigma^{-1}(I)$ is piecewise analytic and $\theta_\sigma$ is holomorphic also on $\Gamma_n$, we can apply the Green-Stokes' formula. And we have that

$$0=\int_{\Gamma_n} \theta_\sigma = \int_{u_\sigma^{-1}(t_1)\cap\Omega_n} -{}^*\sigma - \int_{u_\sigma^{-1}(t_2)\cap\Omega_n} -{}^*\sigma + \int_{\partial\Omega_n \cap u_\sigma^{-1}(I)} \theta_\sigma,$$

where $t_j=\exp(2\pi\sqrt{-1}\cdot\alpha_j)$ for each $j=1, 2$. Thus it holds that

$$\left|\int_{L_{t_1}\cap\Omega_n} {}^*\sigma\right| \leq \left|\int_{L_{t_2}\cap\Omega_n} {}^*\sigma\right| + \int_{\partial\Omega_n} |\theta_\sigma|$$
$$\leq m_\sigma(t_2) + \int_{\partial\Omega_n} |\theta_\sigma|.$$

Letting $n$ tends to $+\infty$, we conclude from Lemma 4 that $m_\sigma(t_1) \leq m_\sigma(t_2)$. And by considering $S^1 - \bar{I}$ instead of $I$, we have the converse inequality. Hence we have that $m_\sigma(t_1)=m_\sigma(t_2)$. Since $I$ is arbitrary, $m_\sigma(t)$ is a constant, say $m$, on $S^1$. Finally note that

$$\|\sigma\|_R^2 = \iint_R (-{}^*\sigma)\wedge\sigma = \int_{S^1}\int_{L_t} (-{}^*\sigma)\wedge\left(\frac{dt}{2\pi\sqrt{-1}\cdot t}\right)$$
$$= \int_0^1 m_\sigma(\exp(2\pi\sqrt{-1}\cdot s))\,ds = m.$$

Thus we conclude that $m_\sigma(t)=m=\|\sigma\|_R^2$ for every $t \in S^1$. q.e.d.

Finally we modify the first assertion of Lemma 3 as follows.

**Lemma 5.** *Let $R$, $\sigma$, $u_\sigma(p)$ and $\theta_\sigma$ be as in Lemma 3. And set*

$$E_N = \{u_\sigma(p) \in S^1 \colon p \text{ is on a non-compact regular trajectory of } \theta_\sigma^2\}.$$

*Then the linear measure of $E_N$ is zero.*

*Proof.* Let $\{\Omega_n\}_{n=1}^\infty$ be as in Lemma 4 with $\theta=\theta_\sigma$, and set

$E_N^n = \{u_\sigma(p) \in S^1 \colon p$ is in $\Omega_n$ and on a non-compact regular trajectory of $\theta_\sigma^2\}$.

Then it is clear that $E_N^n \subset E_N^m \subset E_N$ for every $n$ and $m$ with $n \leq m$, and that $E_N$ is the union of all $E_N^n$.

Now suppose that the linear measure of $E_N$ is positive, then there is an $n_0$ such that the linear measure of $E_N^{n_0}$ is greater than a positive constant $a_0$. Note that the linear measure of $E_N^n$ is also greater than $a_0$ for every $n \geq n_0$, and similarly as in the proof of Lemma 3, we can show that

$$\lim_{n\to\infty} \int_{\partial\Omega_n} |\theta_\sigma| \geq \frac{1}{2\pi} a_0$$

which is a contradiction. q.e.d.

*Proof of Theorems* 1 *and* 2. Let $R$, $\sigma$, $u_\sigma(p)$, $E_{u_\sigma}$ and $\theta_\sigma$ be as in Theorem 1, and $\{I^i_{u_\sigma}\}$ be the set of all supplementary intervals for $u_\sigma$. Also let $E_N$ be as in Lemma 5, namely,

$$E_N=\{u_\sigma(p)\in S^1: p \text{ is on a non-compact regular trajectory of } \theta_\sigma^2\},$$

then we see from Lemma 5 that $E_N\cap I^i_{u_\sigma}$ is a set of zero linear measure for every $i$. In other words, $u_\sigma^{-1}(t)$ consists only of compact regular trajectories of $\theta_\sigma^2$ for almost every $t$ in $S^1-E_{u_\sigma}$ $(=\bigcup_i I^i_{u_\sigma})$, which is the first assertion of Theorem 1. If the linear measure of $E_{u_\sigma}$ is zero, then the linear measure of $E_N$ itself is zero. On the other hand, we can show similarly as in the proof of Lemma 3 that, if $\theta_\sigma$ does not belongs to $CA_1(R)$, then the linear measure of $E_N$ should be positive. Thus we conclude that $\theta_\sigma$ belongs to $CA_1(R)$.

Next Lemma 3 implies that if $t_1$ and $t_2$ are contained in the same interval, say $I^i_{u_\sigma}$, then it holds that $m_\sigma(t_1)=m_\sigma(t_2)$, which we denote by $m_i$ (as in Theorem 2), and that for every $I^i_{u_\sigma}$ it holds that

$$\|\sigma\|^2_{u_\sigma^{-1}(I^i_{u_\sigma})}=a(I^i_{u_\sigma})\cdot m_i$$

where $a(I^i_{u_\sigma})$ is as in Theorem 2 and $\|\sigma\|_{u_\sigma^{-1}(I^i_{u_\sigma})}$ is the Dirichlet norm of $\sigma$ on the open set $u_\sigma^{-1}(I^i_{u_\sigma})$ in $R$. And if the linear measure of $E_{u_\sigma}$ is zero, then we can easily seen that the Dirichlet norm of $\sigma$ on the set $R-\bigcup_i u_\sigma^{-1}(I^i_{u_\sigma})$ $(=u_\sigma^{-1}(E_{u_\sigma}))$ is zero, hence we have that

$$\|\sigma\|^2_R=\sum a(I^i_{u_\sigma})\cdot m_i.$$

Thus we have shown Theorem 2. q.e.d.

## §4. Examples

First we construct the following

**Example 1.** There is an HB-harmonic measure $u(p)$ on a surface $R$ (, i.e. the greatest harmonic minorant of $u(p)$ and $1-u(p)$ is the constant zero,) such that $du$ does not belong to $\Gamma_h(R)$, but the holomorphic differential $\theta=-{}^*du+\sqrt{-1}\cdot du$ associated with $du$ belongs to $CA_1(R)$.

*Construction.* Let $R=\{|z|<1\}-\bigcup_{n=1}^{\infty} I_n$, where $I_n=[r_{2n-1}, r_{2n}]$ with an increasing sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers converging to 1, and $u(p)$ the uniquely determined harmonic function on $R$ whose boundary value is 1 on $\bigcup_{n=1}^{\infty} I_n$ and 0 on $\{|z|=1\}-\{1\}$. Then we can see that all zeros of the holomorphic differential $\theta$ associated with $du$ are contained in the positive real axis and there is exactly one simple zero, say $s_n$, of $\theta$ in $(r_{2n}, r_{2n+1})$ for every $n$.

Now by choosing $\{r_n\}_{n=1}^{\infty}$ sufficiently scarcely (, for example, choose $\{r_n\}_{n=1}^{\infty}$ inductively so that $(r_{n-1}-r_n)/(1-r_n\cdot r_{n+1})$ converges to 1 as $n$ tends to $+\infty$), we can assume that $du\notin\Gamma_h(R)$ and $u(s_n)$ converges to zero as $n$ tends to $+\infty$. Then $u^{-1}(c)$ consists only of compact connected components for every $c$ in (0, 1), which implies that $\theta$ belongs to $CA_1(R)$.

**Remark.** Similarly as above we can construct an unbounded Dirichlet finite harmonic function $u(p)$ on a surface $R$ such that the holomorphic differential associated with $du$ belongs to $CA_1(R)$.

Also recall that the typical example of (unbounded and Dirichlet infinite) harmonic function $u(p)$ such that the differential associated with $du$ belongs to $CA_1(R)$ is the Green's function or the so-called Evans-Selberg potential on a surface $\bar{R}$ considered as a harmonic function on $R=\bar{R}-\{p_0\}$, where $p_0$ is the only singularity of the function.

Next we show by an example that, even for a (non-dividing) simple closed curve $c$, every set $L_t$ $(t\in S^1)$ of level curves of the conjugate of the reproducing differential $\sigma(c)$ may be non-compact.

**Example 2.** Let $R_0=\{|z|<+\infty\}-(-\infty, -2]\cup[-1/2, 1/2]\cup[2, +\infty)$, and for every $n$ set $I_n=[r_{2n-1}, r_{2n}]$ and $J_n=[-1/r_{2n-1}, -1/r_{2n}]$, where $1<r_1$ and $r_n$ converging increasingly to 3/2. Next set $R=R_0-\bigcup_{n=1}^{\infty}(I_n\cup J_n)-\{3/2, -2/3\}$ and let $u(z)$ be the uniquely determined harmonic function whose boundary value is 1 on $[-1/2, 1/2]\cup(\bigcup_{n=1}^{\infty}I_n)$ and 0 on $(-\infty, -2]\cup[2, +\infty)\cup(\bigcup_{n=1}^{\infty}J_n)$. Here by taking $\{r_n\}_{n=1}^{\infty}$ so that $r_{2n-1}-r_{2n}$ converges sufficiently rapidly to zero as $n$ tends to $+\infty$, we may assume that $du\in\Gamma_h(R)$ and $u(s_n)$ converges to a value, say $d$, less than 1/2, where $s_n$ is the unique simple zero of the holomorphic differential $\theta$ associated with $du$ on $(r_{2n}, r_{2n+1})$ for every $n$. Also note that $u(-1/z)\equiv 1-u(z)$. Hence $-1/s_n$ is the unique simple zero of $\theta$ on $(-1/r_{2n}, -1/r_{2n+1})$ and $u(-1/s_n)$ converges to $1-d$ $(>1/2)$. In particular, we can see that $u^{-1}(t)$ is non-compact and consists of infinitely many (compact) connected components for every $t\in S^1$. On the other hand it is clear that $du={}^*\sigma(c)$ with the simple closed curve $c$ on $R$ which separates $[-1/2, 1/2]\cup(\bigcup_{n=1}^{\infty}I_n)\cup\{3/2\}$ from $(-\infty, -2]\cup[2, +\infty)\cup(\bigcup_{n=1}^{\infty}J_n)\cup\{-2/3\}$.

Finally let $\hat{R}$ be the double of $R$ along all non-degenerate boundary components of $R$ and $\hat{\sigma}$ be the anti-symmetric extention of $du$ on $R$. Then it is clear that $\hat{\sigma}=2\cdot{}^*\sigma(\hat{c})$, where $\hat{c}$ is the non-dividing simple closed curve on $R$ corresponding to $c$, and that the set $L_t$ of level curves of ${}^*\sigma(\hat{c})$ is non-compact and consists of infinitely many compact connected components for every $t\in S^1$.

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