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Remarks on generalized rings of quotients, IV

By

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It is known that if R' is a generalized ring of quotients of a ring R contained in the total quotient ring $T(R)$ of R, then for any ring R'' with $R \subseteqq R'' \subseteqq R'$, R' is a generalized ring of quotients of R''. In other words, if R' is R-flat, then R' is R''-flat, for R', R'' such that $R \subseteqq R'' \subseteqq R' \subseteqq T(R)$. (Corollary 1 to Theorem 1 in [1] and Lemma 2 in [11]).

In this paper, we shall show that the converse, in a sence, to the above is valid if R' is a domain. Precisely speaking, let $R \subseteqq R'$ be rings such that R' is R-flat. Consider the following condition (F):

(F) For any ring R'' with $R \subseteqq R'' \subseteqq R'$, R' is R''-flat.

It is clear that if R' is a field, then R' satisfies (F) for any subring R of R'. We shall show the following:

Assume that R' is a domain. If R' satisfies (F) and if $R' \subsetneqq T(R)$, then R' is a field. (Theorem 2.7 in §2)

In this paper, first we shall give some results on flatness of rings in §1. In §2, we shall prove the main result of this paper and in §3 we shall give some results in the general case. Notation is the same as in [1], [2] and [3]. A pair (R, R') means that R' is a ring and R is a subring of R'.

§1. We shall begin with some results on flatness and on the condition (F).

Lemma 1.1. *Let (R, R') be a pair and let $\mathfrak{a}'$ be an ideal of R' containing a non-zero-divisor. Let $R'' = R + \mathfrak{a}'$, which is a subring of R' containing R such that $T(R'') = T(R')$. Assume that R' is R''-flat. Then we have $R' = R''$ if R' is integral over R or if $R = k$ is a field.*

Proof. If R' is integral over R, then R' is also integral over R'' and we have $R' = R''$ by Corollary 2 to Theorem 1 in [1]. Assume that $R = k$ is a field. Suppose that $R' \neq R''$. Then there is an $r' \in R'$ such that $r' \notin R''$. Since $r' \notin \mathfrak{a}'$, it is easily seen that $(R'' : r') = \mathfrak{a}'$ and $\mathfrak{a}'R' = \mathfrak{a}' \neq R'$ which contradicts Theorem 1 in [1]. Thus we have $R' = R''$.

Lemma 1.2. *Assume that a pair (R, R') satisfies (F). Then*: (1) For any ring R_1 *such that $R \subseteqq R_1 \subseteqq R'$, the pair (R_1, R') also satisfies (F).*

(2) *For any multiplicatively closed subset S of R the pair (R_S, R'_S) also satisfies (F).*
(3) *If R' is integral over R, then for any ring R'_1 such that $R \subseteqq R'_1 \subseteqq R'$, the pair* (R, R'_1) satisfies (F).

Proof. (1) is clear from the definition. (2) follows from the facts that any ring between R_S and R'_S is of the form R''_S where R'' is a ring such that $R \subseteqq R'' \subseteqq R'$ and that if R' is R''-flat then R'_S is R''_S-flat. For (3), let R''_1 be a ring such that $R \subseteqq R''_1 \subseteqq R'_1$. Since R' is R'_1-flat and integral over R'_1, R' is faithfully flat over R'_1. Since R' is R''_1-flat, R'_1 is R-${}''_1$flat, as is easily seen. Thus the pair (R, R'_1) satisfies (F).

§**2.** In this section, we assume that R' is a domain.

Lemma 2.1. *Let (k, R') be a pair of domains such that k is a field. If $R' = k + \mathfrak{a}'$ for any non-zero ideal $\mathfrak{a}'$ of R', then R' is a field.*

Proof. Let a' be a non-zero element of R'. By our assumption, we have $k + a'^2R' = R'$. Hence there are $t \in k$ and $b' \in R'$ such that $a' = t + a'^2b'$, that is, $a'(1 - a'b') = t$. If $t = 0$, then a' is a unit since $a' \neq 0$. If $t \neq 0$, it is clear that a' is a unit since k is a field.

Remark 2.2. It is known that there is a pair (R, R') of domains such that for any non-zero ideal $\mathfrak{a}'$ of R', $R' = R + \mathfrak{a}'$ and R' is not a field (see [12]).

Proposition 2.3. *Assume that a pair (k, R') of domains satisfies (F), where k is a field. Then R' is a field.*

Proof. Since for any non-zero ideal $\mathfrak{a}'$ of R', R' is flat over $k + \mathfrak{a}'$, we have $R' = k + \mathfrak{a}'$ by Lemma 1.1. Then Lemma 2.1 implies that R' is a field.

Corollary 2.4. *Let (R, R') be a pair of domains satisfying (F). Then $R'_S = T(R')$, where $S = R - \{0\}$.*

Proposition 2.5. *Assume that a pair (R, R') of domains satisfies (F). If R' is integral over R and if $R' \neq R$, then R' is a field.*

Proof. If $R' \subseteqq T(R)$, then we have $R' = R$ by Corollary 2 to Theorem 1 in [1]. Therefore R' is not contained in $T(R)$. Suppose that R' is not a field. Then R is not a field since R' is integral over R. Let α be an element of R' not contained in $T(R)$. Replacing α with $a\alpha$ for a suitable $a \in R$, if necessary, we assume that $R[\alpha]$ is a free R-module of a basis $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$ with $n \geqq 2$. By Lemma 1.2. we see that the pair $(R, R[\alpha])$ also satisfies (F). Let b be a non-zero, non-unit element of R. Since $R[\alpha]$ is integral over R, $R[\alpha] = R + bR[\alpha]$ by virtue of Lemma 1.1. Then we have $\alpha = r_0 + br_1\alpha + \cdots + br_{n-1}\alpha^{n-1}$ with $r_i \in R$. Since $\{1, \alpha, \dots, \alpha^{n-1}\}$ are linearly independent over R, we have $br_1 = 1$, hence, b is a unit which is a contradiction. Thus R' is a field.

Corollary 2.6. *Assume that a pair (R, R') of domains satisfies (F) and $R' \subsetneqq T(R)$. If $T(R')$ is algebraic over $T(R)$, then R' is a field.*

Proof. If $T(R) \subseteq R'$, applying Proposition 2.3 to the pair $(T(R), R')$, we see that R' is a field. On the other hand, if R' is integral over R, then Proposition 2.5 implies that R' is a field. Therefore we assume that $T(R) \subsetneqq R'$ and R' is not integral over R. Suppose that R' is not a field. Then there is a prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p}R'_{\mathfrak{p}} \neq R'_{\mathfrak{p}}$. Since the pair $(R_{\mathfrak{p}}, R'_{\mathfrak{p}})$ also satisfies (F) by Lemma 1.2, replacing (R, R') with $(R_{\mathfrak{p}}, R'_{\mathfrak{p}})$, we assume that R is a quasi-local domain with maximal ideal $\mathfrak{p}$ such that $\mathfrak{p}R' \neq R'$. Let $\bar{R}$ be the integral closure of R in R'. Then $R' \subseteq T(\bar{R})$ by our assumption that $T(R')$ is algebraic over $T(R)$. Let p be a non-zero element of $\mathfrak{p}$ and let $R'' = R + p\bar{R}$ which is a subdomain of $\bar{R}$ such that $T(R'') = T(\bar{R}) = T(R')$. Since R is quasi-local with the maximal ideal $\mathfrak{p}$, it is easily seen that R'' is a quasi-local domain with the maximal ideal $\mathfrak{p}'' = \mathfrak{p} + p\bar{R}$. Then, since $\mathfrak{p}''R' = \mathfrak{p}R' \neq R'$, we have $R'' = R'$, as is easily seen, hence $R' = \bar{R}$ which is a contradiction.

Now we shall prove the main result of this paper.

Theorem 2.7. *Assume that a pair (R, R') of domains satisfies (F). If $R' \subsetneqq T(R)$, then R' is a field.*

Proof. It is clear that there is a proper subfield K of $T(R')$ such that $T(R) \subseteq K$ and $T(R')$ is algebraic over K. Let $R_1 = R' \cap K$. Then the pair (R_1, R') satisfies (F) and $T(R')$ is algebraic over $T(R_1)$ since $T(R_1) = K$ by Corollary 2.4. Therefore, applying Corollary 2.6 to the pair (R_1, R'), we see that R' is a field.

§**3.** In this section, we assume that rings are reduced.

Proposition 3.1. *Let (R, R') be a pair of rings such that R' is integral over R. If R' (or R) is a regular ring (in von Neumann's sense throughout this section), then (R, R') satisfies (F).*

Proof. Since every ring R'' between R and R' is regular, the assertion is obvious.

Proposition 3.2. *Let (R, R') be a pair of rings such that R' is regular. If R is integrally closed in R', or, more generally, if every idempotent of R' is contained in R, (R, R') satisfies (F).*

Proof. This follows from [7] (see Footnote 4 in [4]).

Lemma 3.3. *Let k be a field, R' a (reduced) ring containing k. If the pair (k, R') satisfies (F), R' is regular.*

Proof. First we show that $R' = T(R')$. Let α be an element of R' which is not a zero-divisor. Then by Lemma 1.1, $R' = k + \alpha^2 R'$. Hence there are $\beta \in R'$ and $u \in k$ such that $\alpha = u + \alpha^2\beta$, that is, $\alpha(1 - \alpha\beta) = u$, which implies that α is a unit. Thus we see that $R' = T(R')$. Since k is a field, for any γ in R', $T(k[\gamma])$ is semi-simple and contained in R'. Then it is easily seen that R' is regular.

Proposition 3.4. *Assume that a pair (R, R') satisfies (F). If $T(R)$ is regular and is contained in R', then R' is also regular (recall that we assume that R' is reduced).*

Proof. By Lemma 1.2, we may assume that R is regular. Let $\mathfrak{m}'$ be a maximal ideal of R' and $\mathfrak{m}=\mathfrak{m}'\cap R$. Then, again, by Lemma 1.2, the pair $(R_{\mathfrak{m}}, R'_{\mathfrak{m}})$ satisfies (F). Since $R_{\mathfrak{m}}$ is a field, $R'_{\mathfrak{m}}$ is regular by virtue of Lemma 3.3. Then it is clear that R' is regular.

Remark 3.5. From Nagata's theorem in [8], we see that there is a pair (R, R') of rings not satisfying (F) even though R is a field and R' is regular (see Remark in [3]).

Theorem 3.6. *Let (R, R') be a pair such that $T(R)$ is regular and is contained in R'. Then the following are equivalent.*
(1) *The pair (R, R') satisfies (F).*
(2) *R' is regular and for every $\alpha\in R'$, $T(R[\alpha])$ is regular and is contained in R'.*
(3) *R' is regular and for any zero-divisor $\alpha\in R'$, $T(R[\alpha])$ is regular and is contained in R'.*

Proof. (1)$\Rightarrow$(2): By Proposition 3.4, R' is regular. Let α be an element of R'. Since $T(R)\subseteq T(R[\alpha])$, we may assume that R is regular. Then it is easily seen that $R[\alpha]$ is locally a domain, that is, for every maximal ideal $\mathfrak{m}''$ of $R[\alpha]$, $R[\alpha]_{\mathfrak{m}''}$ is an integral domain. Since R' is regular and is flat over $R[\alpha]$, the set of minimal prime ideals of $R[\alpha]$ is compact with respect to Zariski-topology by [9]. Then we see that $T(R[\alpha])$ is regular by [10]. Since R' is $R[\alpha]$-flat, $T(R[\alpha])$ is contained in R'.
(2)$\Rightarrow$(3): Trivial.
(3)$\Rightarrow$(1): Let R'' be an arbitrary ring between R and R'. To prove that R' is flat over R'', it is sufficient to show that $T(R'')$ is regular and is contained in R'. First we show that $T(R'')\subseteq R'$. Let α'' be an element of R'' which is not a zero-divisor in R''. If α'' is a zero-divisor in R', α'' is also a zero-divisor in $R[\alpha'']\subseteq R''$ by our assumption which is a contradiction. Hence α'' is a non-zero-divisor in R' which implies that $T(R'')\subseteq R'$. To see that $T(R'')$ is regular, it is sufficient to show that every element of R'' is expressed as a product of an idempotent and a non-zero-divisor in R'' which is clear from our assumption. (Recall that a ring is regular if and only if every element is expressed as a product of an idempotent and a unit by [6].)

Corollary 3.7. *Let (R, R') be a pair of rings such that R' is regular and flat over R. If R is an integral domain or noetherian, (R, R') satisfies (F) if and only if for any zero-divisor α in R', $T(R[\alpha])$ is contained in R'.*

Proof. Since, in these cases, $T(R[\alpha])$ is regular for any $\alpha\in R'$, the assertion follows from the theorem.

Examples 3.8. Let $R[x]$ be a polynomial ring of a variable x over a regular ring R and let I be an ideal of $R[x]$ such that $I\cap R=(0)$ and $R[x]/I=R[\alpha]$ is reduced. We denote by $C(I)$ the content ideal of I, that is, the ideal of R generated by coefficients of polynomials in I. Then the following are easily shown (see [5]).
(1) If $C(I)=R$, $R[\alpha]$ is regular and integral over R. Therefore the pair $(R, R[\alpha])$ satisfies (F).

(2) If I is a principal ideal generated by $ex-a$ with $e^2=e$ and $ea=a$, then $T(R[\alpha])$ is regular and the pair $(R, T(R[\alpha]))$ satisfies (F).

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