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Isogenous tori and the class number formulae

By

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Introduction

T. Ono and J.-M. Shyr generalized Dedekind's class number formulae to a class number formula of an algebraic torus T defined over $\mathbf{Q}$ (cf. [7], [10]). From this generalized class number formula, they obtained a relation between the relative class number of two isogenous tori and their Tamagawa numbers and q-symbols of several maps induced by an isogeny of them (cf. Lemma 1). Here q-symbols of α is defined as follows. Let A, B be commutative groups and α be a homomorphism $A \to B$. If Ker α and Cok α are both finite, that is, α is admissible, we define the q-symbol of α by putting

$$q(\alpha) = \frac{[\mathrm{Cok}\ \alpha]}{[\mathrm{Ker}\ \alpha]},$$

where $[X]$ denotes the order of a finite group X.

Let F be an algebraic number field of finite degree over $\mathbf{Q}$ and T be an algebraic torus defined over F. $h(T)$ denotes the class number of T. Consider the following exact sequence of algebraic tori defined over F

$$0 \longrightarrow R_{K/F}{}^{(1)}(G_m) \longrightarrow R_{K/F}(G_m) \longrightarrow G_m \longrightarrow 0,$$

where K is a finite extension of F and $R_{K/F}$ is the Weil functor of restricting the field of definition from K to F. As a generalization of the formula of Gauss on the genera of binary quadratic forms, T. Ono defined a new arithmetical invariant $E(K/F)$ by putting

$$E(K/F) = h(R_{K/F}(G_m))/(h(R_{K/F}{}^{(1)}(G_m)) \cdot h(G_m)).$$

In [9], he obtained a formula of $E(K/F)$ expressed in terms of cohomological invariants for K/F. He also defined another invariant $E'(K/F)$, and in [5], we briefly announced similar formula for $E'(K/F)$ when K/F is finite normal. In [6], using I. T. Adamson's non-normal cohomology, we announced that one could generalize these formulae of $E(K/F)$ and $E'(K/F)$ for any finite extension K/F.

In this paper, we shall prove these announced results of [5] and [6] in §1. In §2, we shall show another class number formula for a biquadratic extension K/F. In §3, we shall show this formula implies some class number

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formula of Dirichlet for biquadratic extensions $\mathbf{Q}(\sqrt{q}, \sqrt{-1})/\mathbf{Q}$ (q is a prime number). Hence, the formula may be viewed as a generalization of this formula of Dirichlet.

§1. Following [9], we shall start by recalling the definition of the class number of a torus. Let F be an algebraic number field of finite degree and $F_{\mathfrak{p}}$ be the completion of F at a place $\mathfrak{p}$ of F. When $\mathfrak{p}$ is non-archimedean, we denote the ring of $\mathfrak{p}$-adic integers by $O_{\mathfrak{p}}$.
Then $U_F = \prod_{\mathfrak{p}:\text{archimedean}} F_{\mathfrak{p}}^{\times} \times \prod_{\mathfrak{p}:\text{non-archimedean}} O_{\mathfrak{p}}^{\times}$ is the unit group of the idele group $F_A^{\times}$. Here, for a ring A, $A^{\times}$ denotes the multiplicative group consisting of all the invertible elements of A. Let T be an algebraic torus defined over F. $T(F)$ denotes the group of F-rational points of T and $T(F_{\mathfrak{p}})$ denotes the group of $F_{\mathfrak{p}}$-rational points of T. We denote the character module of T by $\hat{T} = \mathrm{Hom}(T, G_m)$, where G_m is the multiplicative group of the universal domain. Let $\hat{T}_0$ be the integral dual of $\hat{T}$. Then, for the case when $\mathfrak{p}$ is non-archimedean, $T(O_{\mathfrak{p}})$ the unique maximal compact subgroup of $T(F_{\mathfrak{p}})$ is isomorphic to $\hat{T}_0 \otimes O_{\mathfrak{p}}^{\times}$. The adelization of T over F shall be written $T(F_A)$. Then the unit group of $T(F_A)$ is defined by $T(U_F) = \prod_{\mathfrak{p}} T(O_{\mathfrak{p}})$, where $\mathfrak{p}$ runs all the places of F and $T(O_{\mathfrak{p}}) = T(F_{\mathfrak{p}})$ when $\mathfrak{p}$ is archimedean. We define the class group of T over F by putting

$$C(T) = T(F_A)/(T(U_F)\cdot(T(F)).$$

We call the order $[C(T)]$ the class number of the torus T and denote it by $h(T)$. Let K be a finite extension of F and $R_{K/F}$ be the Weil functor restricting the field of definition from K to F. Then, from the definition of the class group of tori, we have $C(G_m) \cong F_A^{\times}/U_F F^{\times}$ and $C(R_{K/F}(G_m)) \cong K_A^{\times}/U_K K^{\times}$. Hence the class numbers $h(G_m)$ and $h(R_{K/F}(G_m))$ are usual class numbers of algebraic number fields h_F and h_K, respectively. Let T, T^* be the tori defined over F and λ: $T \to T^*$ be an isogeny defined over F. λ induces the following natural homomorphisms

$$\hat{\lambda}(F): \hat{T}^*(F) \longrightarrow \hat{T}(F),$$
$$\lambda(O_{\mathfrak{p}}): T(O_{\mathfrak{p}}) \longrightarrow T^*(O_{\mathfrak{p}}),$$
$$\lambda(O_F): T(O_F) \longrightarrow T^*(O_F).$$

Here $\hat{T}(F)$ denotes the submodule of $\hat{T}$ consisting of all the rational characters of T defined over F. In this situation, we have the following key lemma.

Lemma 1 (cf. [7] or [10]). *With the notations as above, we have*

$$\frac{h(T)}{h(T^*)} = \frac{\tau(T)\prod_{\mathfrak{p}} q(\lambda(O_{\mathfrak{p}}))}{\tau(T^*)\, q(\lambda(O_F))\, q(\hat{\lambda}(F))},$$

where $\tau(T)$, $\tau(T^*)$ are the Tamagawa nombers of T, T^*.

Let $\gamma\colon T \to T$ be a F-isogeny of T. Then, from this lemma, the following corollary is obvious.

Corollary 1. *For any F-isogeny $\gamma\colon T \to T$, we have*

$$1 = \frac{\prod q(\lambda(O_{\mathfrak{p}}))}{q(\lambda(O_F))\, q(\hat{\lambda}(F))}.$$

Consider the following exact sequence of algebraic tori defined over F

$$0 \longrightarrow T' \xrightarrow{\alpha} T \xrightarrow{\mu} T'' \longrightarrow 0, \tag{1}$$

where α and μ are defined over F. Maschke's theorem states that every rational representation of a finite group is completely reducible. Hence, one can take a homomorphism $\beta\colon T \to T'$ defined over F such that $\lambda = \beta \times \mu\colon T \to T' \times T''$ and $\gamma = \beta \cdot \alpha\colon T' \to T'$ are F-isogenies. From Lemma 1, we have the equality

$$\frac{h(T)}{h(T')\, h(T'')} = \frac{\tau(T)}{\tau(T')\, \tau(T'')} \times \frac{\prod_{\mathfrak{p}} q(\lambda(O_{\mathfrak{p}}))}{q(\hat{\lambda}(F))\, q(\lambda(O_F))}. \tag{2}$$

Let L be a common finite normal splitting field of T, T', T''. We denote $\mathrm{Gal}(L/F)$ by G. First, we provide following elementary lemma.

Lemma 2. *Let $W = X \times Y$ be an abelian group and A be a subgroup of finite index. Then we have the equality*

$$[W\colon A] = [X : AY/Y]\ [Y\colon A \cap Y],$$

where W/Y and $\{1\} \times Y$ are identified with X and Y.

Consider the following short exact sequence of G-modules induced from (1)

$$0 \longrightarrow T'(O_L) \longrightarrow T(O_L) \longrightarrow T''(O_L) \longrightarrow 0.$$

From the long exact sequence derived from this sequence, we have

$$0 \longrightarrow T'(O_F) \xrightarrow{\alpha(O_F)} T(O_F) \xrightarrow{\mu(O_F)} T''(O_F) \longrightarrow H^1(G, T'(O_L)) \longrightarrow H^1(G, T(O_L)) \longrightarrow \cdots.$$

The map $\beta(O_F) \times \mu(O_F)\colon T(O_F) \to T'(O_F) \times T''(O_F)$ shall be written $\lambda(O_F)$. Then, from Lemma 2 and the above long exact sequence, the cokernel of the map $\lambda(O_F)$ satisfies

$$\begin{aligned}
[\mathrm{Cok}\ \lambda(O_F)] &= [T''(O_F)\colon \mu(O_F)(T(O_F))] \times [T'(O_F)\colon \beta(O_F)(T(O_F) \cap \mathrm{Ker}\ \mu(O_F))] \\
&= [\mathrm{Ker}\ (H^1(G, T'(O_L)) \longrightarrow H^1(G, T(O_L)))] \times [T'(O_F)\colon \gamma(O_F)(T'(O_F))] \\
&= [\mathrm{Ker}\ (H^1(G, T'(O_L)) \longrightarrow H^1(G, T(O_L)))]\ [\mathrm{Cok}\ \gamma(O_F)].
\end{aligned}$$

On the other hand, the kernel of the map satisfies

$$[\mathrm{Ker}\ \lambda(O_F)] = [\mathrm{Ker}\ \beta(O_F) \cap \mathrm{Ker}\ \mu(O_F)]$$
$$= [\mathrm{Ker}\ \beta(O_F) \cap \alpha(O_F)(T'(O_F))] = [\mathrm{Ker}\ \gamma(O_F)].$$

Hence we have

$$q(\lambda(O_F)) = q(\gamma(O_F))\ [\mathrm{Ker}\ (H^1(G, T'(O_L)) \longrightarrow H^1(G, T(O_L)))].$$

In the same way as above, the following equality holds for all $\mathfrak{p}$

$$q(\lambda(O_\mathfrak{p})) = q(\gamma(O_\mathfrak{p}))\ [\mathrm{Ker}\ (H^1(G_\mathfrak{P}, T'(O_\mathfrak{P})) \longrightarrow H^1(G_\mathfrak{P}, T(O_\mathfrak{P})))],$$

where $\mathfrak{P}$ is an extension of $\mathfrak{p}$ to L and $G_\mathfrak{P}$ is the decomposition group of $\mathfrak{P}$. Therefore, from the formula (2), we have

$$\frac{h(T)}{h(T')\,h(T'')} = \frac{\tau(T)}{\tau(T')\,\tau(T'')} \times \frac{\prod_\mathfrak{p} q(\gamma(O_\mathfrak{p}))}{q(\hat{\lambda}(F))\,q(\gamma(O_F))}$$
$$\times \frac{\prod_\mathfrak{p} [\mathrm{Ker}\ (H^1(G_\mathfrak{P}, T'(O_\mathfrak{P})) \longrightarrow H^1(G_\mathfrak{P}, T(O_\mathfrak{P})))]}{[\mathrm{Ker}\ (H^1(G, T'(O_L)) \longrightarrow H^1(G, T(O_L)))]},$$

Finally, by virtue of the fact $\dfrac{\prod_\mathfrak{p} q(\gamma(O_\mathfrak{p}))}{q(\gamma(O_F))} = q(\hat{\gamma}(F))$, we have the following theorem.

Theorem 1. *With the notations as above, we have the following class number formula*

$$\frac{h(T)}{h(T')\,h(T'')} = \frac{\tau(T)}{\tau(T')\,\tau(T'')} \times \frac{q(\hat{\gamma}(F))}{q(\hat{\lambda}(F))}$$
$$\times \frac{[\mathrm{Ker}\ (H^1(G, T'(U_L)) \longrightarrow H^1(G, T(U_L)))]}{[\mathrm{Ker}\ (H^1(G, T'(O_L)) \longrightarrow H^1(G, T(O_L)))]}$$
$$= \frac{\tau(T)\,q(\hat{\gamma}(F)) \prod_\mathfrak{p} [T''(O_\mathfrak{p})\colon \mu(O_\mathfrak{p})(T(O_\mathfrak{p}))]}{\tau(T')\,\tau(T'')\,q(\hat{\lambda}(F))\,[T''(O_F)\colon \mu(O_F)(T(O_F))]}.$$

Here U_L denotes the unit group of the idele group $L_A^\times$.

Let K be a finite extension of F and $R_{K/F}$ is the Weil functor restricting the field of definition from K to F. Consider the following special exact sequence of algebraic tori defined over F

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^{(1)}_{K/F}(G_m) & \longrightarrow & R_{K/F}(G_m) & \xrightarrow{N} & G_m & \longrightarrow & 0, \\ & & \| & & \| & & \| & & \\ & & T' & & T & & T'' & & \end{array}$$

where N is the norm map for K/F and $R^{(1)}_{K/F}(G_m) = \mathrm{Ker}\ N$. The invariant $E(K/F)$ is defined by putting

$$E(K/F) = \frac{h(T)}{h(T')\,h(T'')} = \frac{h_K}{h_F\,h_{K/F}},$$

where $h_{K/F}$ denotes the class number $h(T')$. For this case, F-morphism $\beta\colon T \to T'$ is defined by $\beta(x) = x^m(N\,x)^{-1}$ $(m = [K\colon F])$. From the fact that $T' = R^{(1)}_{K/F}(G_m)$ is an anisotropic torus, the elements $q(\hat{\gamma}(F))$ and $q(\hat{\lambda}(F))$ in Theorem 1 are both equal to 1. The Tamagawa numbers $\tau(T) = \tau(T'') = 1$ and $\tau(T') = [K_0\colon F]/[F^\times \cap N_{K/F}K_A^\times\colon N_{K/F}K^\times]$, where K_0 is the maximal abelian extension of F contained in K. Furthermore, we get

$$[T''(O_F)\colon N(O_F)(T(O_F))] = [O_F^\times\colon N_{K/F}O_K^\times],$$

$$\prod_{\mathfrak{p}}[T''(O_\mathfrak{p})\colon N(O_\mathfrak{p})(T(O_\mathfrak{P}))] = \prod_{\mathfrak{p}}[O_\mathfrak{p}^\times\colon \prod_{\mathfrak{P}|\mathfrak{p}} N_{K_\mathfrak{P}/F_\mathfrak{p}}O_\mathfrak{P}^\times]$$
$$= [U_F\colon N_{K/F}U_K].$$

Combining these, we get

$$E(K/F) = \frac{[F^\times \cap N_{K/F}K_A^\times\colon N_{K/F}K^\times]\,[U_F\colon N_{K/F}U_K]}{[K_0\colon F]\,[O_K^\times\colon N_{K/F}O_K^\times]}.$$

L denotes the Galois closure of K/F and G, H denote the Galois groups $\mathrm{Gal}(L/F)$, $\mathrm{Gal}(L/K)$. Then L is a common Galois splitting field of T, T', T''. We denote I. T. Adamson's non-normal cohomology group $H^0([G\colon H], O_L^\times)$ by $H^0(K/F, O_K^\times)$. From [1], Theorem 4.5, we have $H^0(K/F, O_K^\times) \cong O_F^\times/N_{K/F}O_K^\times$. Finally, using these non-normal cohomology groups, we get the following interpretation of $E(K/F)$

$$E(K/F) = \frac{[\mathrm{Ker}\,(H^0(K/F, K^\times) \longrightarrow H^0(K/F, K_A^\times))]\,[H^0(K/F, U_K)]}{[K_0\colon F]\,[H^0(K/F, O_K^\times)]}.$$

Theorem 2. *For any finite extension K/F, we have*

$$E(K/F) = \frac{[\mathrm{Ker}\,(H^0(K/F, K^\times) \longrightarrow H^0(K/F, K_A^\times))]\,[H^0(K/F, U_K)]}{[K_0\colon F]\,[H^0(K/F, O_K^\times)]}$$
$$= \frac{[F^\times \cap N_{K/F}K_A^\times\colon N_{K/F}K^\times]\,[U_F\colon N_{K/F}U_K]}{[K_0\colon F]\,[O_F^\times\colon N_{K/F}O_K^\times]}.$$

When K/F is normal, I. T. Adamson's non-normal cohomology group coincides with usual Tate cohomology group. Hence we have the following class number formula ([9], Theorem).

Corollary 2. *For a normal extension K/F, we have*

$$E(K/F) = \frac{[\mathrm{Ker}\,(H^0(G, K^\times) \longrightarrow H^0(G, K_A^\times))]\,[H^0(G, U_K)]}{[K_0\colon F]\,[H^0(G, O_K^\times)]},$$

where $G = \mathrm{Gal}(K/F)$ *and* $[H^0(G, U_K)] = \prod_{\mathfrak{p}} e_{\mathfrak{p}}^0$. $e_{\mathfrak{p}}^0$ *is the ramification exponent of the maximal abelian subextension over* $F_{\mathfrak{p}}$ *which is contained in* $K_{\mathfrak{P}}$ ($\mathfrak{P}$ *is an extension of* $\mathfrak{p}$ *to* K).

Now, consider the following exact sequence of algebraic tori

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_m & \xrightarrow{\alpha'} & R_{K/F}(G_m) & \xrightarrow{\mu'} & R_{K/F}(G_m)/G_m & \longrightarrow & 0, \\ & & \| & & \| & & \| & & \\ & & T' & & T & & T'' & & \end{array}$$

where K is a finite extension of F of degree m and $\mu'(x) = x \bmod G_m (x \in R_{K/F}(G_m))$. Let β' be the norm map from T to T'. Then there exist F-isogenies $\lambda' = \beta' \times \mu' \colon T \to T' \times T''$ and $\gamma' = \beta' \cdot \alpha' \colon T' \to T'$, where γ' is the map $\gamma'(x) = x^m (x \in T' = G_m)$. L denotes the Galois closure of K over F. Then L is a common splitting field of T, T', T''. We denote the Galois group $\mathrm{Gal}(L/F)$ by G and $\mathrm{Gal}(L/K)$ by H. Let $G = \bigcup_{i=1}^{m} \sigma_i H$ be the right-coset decomposition of G with respect to H. Then the character modules of T, T', T'' are $\hat{T} \cong \mathbf{Z}[G/H] = \mathbf{Z}\langle \sigma_i H | 1 \leqq i \leqq n \rangle \cong \mathrm{Ind}_H^G \mathbf{Z}$, $\hat{T}' \cong \mathbf{Z}$ and $\hat{T}'' \cong I[G/H] = \mathbf{Z}\langle \sigma H - H | \sigma \in G \rangle$, respectively. We denote the integral dual of $I[G/H]$ by $J[G/H]$. $h'_{K/F}$ denotes the class number of the torus $R_{K/F}(G_m)/G_m$. Then the invariant $E'(K/F)$ is defined by $E'(K/F) = \dfrac{h_k}{h_F h'_{K/F}}$. From Theorem 1, we have

$$E'(K/F) = \frac{\tau(T)\, q(\hat{\gamma}'(F))\, [\mathrm{Ker}(H^1(G, U_L) \longrightarrow H^1(H, U_L))]}{\tau(T')\, \tau(T'')\, q(\hat{\lambda}'(F))\, [\mathrm{Ker}(H^1(G, O_L^\times) \longrightarrow H^1(H, O_L^\times))]},$$

where U_L is the unit group of the idele group $L_A^\times$ and $O_L^\times = L^\times \cap U_L$ is the global unit group of L. We shall calculate the Tamagawa numbers, q-symbols,

$$(\mathrm{Ker}\,(H^1(G, U_L) \longrightarrow H^1(G, U_L))] \quad \text{and} \quad [\mathrm{Ker}(H^1(G, O_L^\times) \longrightarrow H^1(H, O_L^\times))].$$

First, one sees the Tamagawa numbers $\tau(T) = \tau(T') = 1$ and

$$\tau(T'') = \frac{[H^1(G, \hat{T}'')]}{[\mathrm{Ker}\,(H^1(G, T''(L)) \longrightarrow H^1(G, T''(L_A)))]} \qquad \text{(cf. [8])}.$$

Consider the following exact sequences of G-modules

$$0 \longrightarrow I[G/H] \longrightarrow \mathbf{Z}[G/H] \longrightarrow \mathbf{Z} \longrightarrow 0,$$
$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}[G/H] \longrightarrow J[G/H] \longrightarrow 0,$$
$$0 \longrightarrow T'(L) \longrightarrow T(L) \longrightarrow T''(L) \longrightarrow 0,$$
$$0 \longrightarrow T'(L_A) \longrightarrow T(L_A) \longrightarrow T''(L_A) \longrightarrow 0.$$

From the fact that $\hat{T}'' \cong I[G/H]$, we have $[H^1(G, \hat{T}'')] = [H^1(G, I[G/H])] = [\mathrm{Cok}\,(H^0(H, \mathbf{Z}) \to H^0(G, \mathbf{Z}))] = [G: H] = [K: F] = m$. Since $T'(L) \cong L^\times$ and $T'(L_A) \cong L_A^\times$ and $H^2(G, L^\times) \to H^2(G, L_A^\times)$ is injective, we have $H^2(G, T'(L)) \to H^2(G, T'(L_A))$ is injective. Since $T(L) \cong \mathbf{Z}[G/H] \otimes L^\times$ and $T(L_A) \cong \mathbf{Z}[G/H] \otimes L_A^\times$, we have $H^1(G, T(L)) \cong H^1(G, \mathbf{Z}[G/H] \otimes L^\times) \cong H^1(H, L^\times) = \{0\}$ and $H^1(G, T(L_A)) \cong H^1(G, \mathbf{Z}[G/H] \otimes L_A^\times) \cong H^1(H, L_A^\times) = \{0\}$. Therefore we have the following commutative diagram with exact rows and colums

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ & & H^1(G, T''(L)) & \longrightarrow & H^1(G, T''(L_A)) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^2(G, T'(L)) & \longrightarrow & H^2(G, T'(L_A)). \end{array}$$

By diagram chasing, we have $H^1(G, T''(L)) \to H^1(G, T''(L_A))$ is injective. Hence the Tamagawa number $\tau(T'')$ equals to m. Since T'' is an anisotropic torus, we have $q(\hat{\lambda}(F)) = [\mathbf{Z}: \mathbf{Z}] = 1$ and $q(\hat{\gamma}(F)) = [\mathbf{Z}: m\mathbf{Z}] = m$. Now, we shall recall the following lemma on non-normal cohomology groups.

Lemma 3 (cf. [1]). *Let G be a finite group and A be a G-module and H be a subgroup of G. Then the following sequence is exact*

$$0 \longrightarrow H^1([G: H], A) \longrightarrow H^1(G, A) \longrightarrow H^1(H, A),$$

where $H^1([G: H], A)$ is I. T. Adamson's non-normal cohomology group.

We denote $H^1([G: H], U_L)$ and $H^1([G: H], O_L^\times)$ by $H^1(K/F, U_K)$ and $H^1(K/F, O_K^\times)$, respectively. Then, from Lemma 3, we have $\mathrm{Ker}\,(H^1(G, U_L) \to H^1(H, U_L)) \cong H^1(K/F, U_K)$ and $\mathrm{Ker}\,(H^1(G, O_L^\times) \to H^1(H, O_L^\times)) \cong H^1(K/F, O_K^\times)$. Hence we have the formula

$$E'(K/F) = \frac{[H^1(K/F, U_K)]}{[H^1(K/F, O_K^\times)]}.$$

In the following, we shall calculate the number $[H^1(K/F, U_K)]$. Let $\mathfrak{P}$ be an extension of $\mathfrak{p}$ to L. Then we have

$$\begin{aligned} H^1(K/F, U_K) &\cong \mathrm{Ker}\,(H^1(G, U_L) \longrightarrow H^1(H, U_L)) \\ &\cong \sum_{\mathfrak{P}} \mathrm{Ker}\,(H^1(G, \mathrm{Ind}_{G_\mathfrak{P}}^G O_\mathfrak{P}^\times) \longrightarrow H^1(H, \mathrm{Ind}_{G_\mathfrak{P}}^G O_\mathfrak{P}^\times)), \end{aligned}$$

where $G_\mathfrak{P}$ is the decomposition group of $\mathfrak{P}$. Let $G = \bigcup_{j=1}^{r} H\tau_j G_\mathfrak{P}$ be a double coset decomposition of G. We denote the extensions of $\mathfrak{p}$ to K by $\mathfrak{p}_1(K)$, $\mathfrak{p}_2(K), \ldots, \mathfrak{p}_r(K)$, where $\mathfrak{p}_j(K) = \mathfrak{p}_1(K)^{\tau_j}$. The ramification exponents of $\mathfrak{P}/\mathfrak{p}$ and $\mathfrak{p}_j(K)/\mathfrak{p}$ shall be written $e(\mathfrak{P}|\mathfrak{p})$ and $e(\mathfrak{p}_j(K)|\mathfrak{p})$, respectively. If $\mathfrak{p}$ is archimedean, $\mathfrak{P}/\mathfrak{p}$ ramifies if and only if $L_\mathfrak{P} = \mathbf{C}$ and $F_\mathfrak{p} = \mathbf{R}$ and the ramification exponent

$e(\mathfrak{P}/\mathfrak{p})=[\mathbf{C}:\mathbf{R}]=2$. Let $\mathfrak{P}'$ be an extension of $\mathfrak{p}_j(K)$ to L. Then $\mathfrak{P}'$ is conjugate to $\mathfrak{P}$ and we obtain the equality $e(\mathfrak{P}'|\mathfrak{p})=e(\mathfrak{P}'|\mathfrak{p}_j(K))\,e(\mathfrak{p}_j(K)|\mathfrak{p})$. Since L/F is normal, we have $e(\mathfrak{P}'|\mathfrak{p})=e(\mathfrak{P}|\mathfrak{p})$. Hence, we may write $e(\mathfrak{P}'|\mathfrak{p}_j(K))$ by $e(\mathfrak{P}|\mathfrak{p}_j(K))$. There exists a commutative diagram for every place $\mathfrak{P}$

$$\begin{array}{ccc}
H^1(G, \mathrm{Ind}_{G_{\mathfrak{P}}}^{G} O_{\mathfrak{P}}^{\times}) & \longrightarrow & H^1(H, \mathrm{Ind}_{G_{\mathfrak{P}}}^{G} O_{\mathfrak{P}}^{\times}) \\
\downarrow\wr & & \downarrow\wr \\
H^1(G_{\mathfrak{P}}, O_{\mathfrak{P}}^{\times}) & \longrightarrow & \sum_{j=1}^{r} H^1(\tau_j^{-1} H\tau_j \cap G_{\mathfrak{P}}, O_{\mathfrak{P}}^{\times}) \\
\downarrow\wr & & \downarrow\wr \\
\mathbf{Z}/e(\mathfrak{P}|\mathfrak{p})\mathbf{Z} & \longrightarrow & \sum_{j=1}^{r} \mathbf{Z}/e(\mathfrak{P}|\mathfrak{p}_j(K))\mathbf{Z}.
\end{array}$$

From induction, one can easily show the following elementary lemma.

Lemma 4. *Let $e, a_1, a_2,\dots, a_r, b_1, b_2,\dots, b_r$ be the natural numbers such that $a_1\cdot b_1=a_2\cdot b_2=\cdots=a_r\cdot b_r=e$. We denote the greatest common divisor of $a_1, a_2,\dots, a_r$ by d and the least common multiple of $b_1, b_2,\dots, b_r$ by g. Then we have $d\cdot g=e$.*

Using this lemma, we have

$$\begin{aligned}
&\mathrm{Ker}\,(\mathbf{Z}/e(\mathfrak{P}|\mathfrak{p})\mathbf{Z} \longrightarrow \sum_{j=1}^{r} \mathbf{Z}/e(\mathfrak{P}|\mathfrak{p}_j(K))\mathbf{Z}) \\
&\cong g\mathbf{Z}/e(\mathfrak{P}|\mathfrak{p})\mathbf{Z} \\
&\cong \mathbf{Z}/e_{\mathfrak{p}}(K)\mathbf{Z}.
\end{aligned}$$

Here g denotes the L.C.M. of $e(\mathfrak{P}|\mathfrak{p}_j(K))$ and $e_{\mathfrak{p}}(K)$ denotes the G.C.D. of $e(\mathfrak{P}_j(K)|\mathfrak{p})$. Hence, we have obtained an isomorphism

$$H^1(K/F, U_K) \cong \sum_{\mathfrak{p}} \mathbf{Z}/e_{\mathfrak{p}}(K)\mathbf{Z},$$

where $\mathfrak{p}$ runs all the ramified places of K/F. Hence we have $[H^1(K/F, U_K)] = \Pi\, e_{\mathfrak{p}}(K)$. Combining these, we have the following theorem.

Theorem 3. *With the notations as above, we have*

$$E'(K/F)=\frac{[H^1(K/F, U_K)]}{[H^1(K/F, O_K^{\times})]}=\frac{\prod_{\mathfrak{p}} e_{\mathfrak{p}}(k)}{[H^1(K/F, O_K^{\times})]}.$$

When K/F is normal, we have the following corollary.

Corollary 3. *When K/F is a finite normal extension, we have*

$$E'(K/F) = \frac{[H^1(G, U_K)]}{[H^1(G, O_K^\times)]} = \frac{\prod_{\mathfrak{p}} e_{\mathfrak{p}}}{[H^1(G, O_K^\times)]},$$

where $\mathfrak{p}$ *runs all the places of* F *and* $e_{\mathfrak{p}}$ *is the ramification exponent of* $\mathfrak{P}$ *over* $\mathfrak{p}$($\mathfrak{P}$ *is an extension of* $\mathfrak{p}$ *to* K).

Remark. We want to take this opportunity to make the following corrections to our paper ([6], Remark 2). In Remark 2, we have written "$[H^1(K/k, U_K)] = \prod_{\mathfrak{p}} [H^1(K_{\mathfrak{P}}/k_{\mathfrak{p}}, O_{\mathfrak{P}}^\times)] = \prod_{\mathfrak{p}} e_{\mathfrak{p}}$, where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{P}$." The correct form of this remark is above Theorem 3. Hence, for "$e_{\mathfrak{p}}$" read "$e_{\mathfrak{p}}(K)$" and for "the ramification index of $\mathfrak{P}$" read "the G.C.D. of the ramification indices of $e(\mathfrak{p}_j(K)|\mathfrak{p})$" and suppress "$\prod_{\mathfrak{p}} [H^1(K_{\mathfrak{P}}/k_{\mathfrak{p}}, O_{\mathfrak{P}}^\times)$"

§2. First, we shall provide elementary tools on Galois modules and Galois cohomology groups. Let G be a finite group and H be a subgroup of index m. Let $G = \bigcup_{i=1}^{m} \sigma_i H$ be the right-coset decomposition of G with respect to H. $J[G/H]$ the integral dual of $I[G/H]$ is the left G-module $\mathbf{Z}[G/H]/\mathbf{Z} \cong \mathbf{Z}\langle \overline{\sigma H} \mid \sigma \in G$ and $\sum_{i=1}^{m} \overline{\sigma_i H} = 0\rangle$. As usual, $J[G/\{1\}]$ and $I[G/\{1\}]$ shall be written $J[G]$ and $I[G]$, respectively. For any G-module A, we have the following lemma.

Lemma 5. *With the notations as above, we have*

$$(I[G/H] \otimes A)^G \cong N_{G/H}^{-1}(0) \cap A^H \text{ and}$$
$$H^0(G, I[G/H] \otimes A) \cong N_{G/H}^{-1}(0) \cap A^H / N_H(D_G A),$$

where $A^H = (a \in A \mid \sigma(a) = a$ *for every* $\sigma \in H\}$ *and* $D_G A = \langle \sigma(a) - a \mid \sigma \in G, a \in A\rangle$.

Proof. Consider the following exact sequence of G-modules

$$0 \longrightarrow A \otimes I[G/H] \longrightarrow A \otimes \mathbf{Z}[G/H] \longrightarrow A \longrightarrow 0.$$

From this exact sequence, we have

$0 \to (A \otimes I(G/H])^G \to (A \otimes \mathbf{Z}[G/H])^G \to A^G \to \cdots$. Since $(A \otimes \mathbf{Z}[G/H])^G = \sum_{i=1}^{m} \sigma_i(A^H \otimes H) \cong A^H$, we obtain $(A \otimes I[G/H])^G \cong \operatorname{Ker}(N_{G/H} : A^H \to A^G) = N_{G/H}^{-1}(0) \cap A^H$. On the other hand, from $A \otimes I[G/H] = \langle a \otimes (\sigma H - H) | a \in A, \sigma \in G\rangle$, we have

$$\langle N_G(a \otimes (\sigma H - H))\rangle = \langle \sum_{\tau \in G} (\tau(a \otimes \sigma H) - \tau(a \otimes H))\rangle$$

$$= \langle \sum_{\tau \in G} \tau((\sigma^{-1} - 1) a \otimes H) \rangle$$

$$= \langle \sum_{i=1}^{m} \sigma_i (N_H((\sigma^{-1} - 1) a \otimes H)) \rangle \cong N_H(D_G A) \subset A^H.$$

Therefore we have

$$H^0(G, A \otimes I[G/H]) \cong (A \otimes I[G/H])^G / N_G(A \otimes I[G/H])$$
$$\cong N_{G/H}^{-1}(0) \cap A^H / N_H(D_G A).$$

In the following, we shall restrict ourselves to the case when G is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$. Let F be an algebraic number field and K be a biquadratic extension of F. We denote the Galois group $\mathrm{Gal}(K/F) = G$ by $\langle \sigma \rangle \times \langle \tau \rangle$. Here $\sigma^2 = \tau^2 = 1$ and $\sigma\tau = \tau\sigma$. Let $K(\sigma)$, $K(\tau)$ and $K(\sigma\tau)$ be the intermediate fields of K/F corresponding to the subgroups $\langle \sigma \rangle$, $\langle \tau \rangle$ and $\langle \sigma\tau \rangle$. Then the character modules of $R_{K/F}^{(1)}(G_m)$ and $R_{K(\sigma\tau)/F}^{(1)}(G_m)$ are isomorphic to $J[G]$ and $J[G/\langle \sigma\tau \rangle]$. There exist isomorphisms

$$J[G] = \mathbf{Z}\langle \bar{\sigma}, \bar{\tau}, \overline{\sigma\tau}, \bar{1} \mid \bar{1} + \bar{\sigma} + \bar{\tau} + \overline{\sigma\tau} = 0 \rangle,$$
$$J[G/\langle \sigma\tau \rangle] \cong \mathbf{Z}\langle \overline{\langle \sigma\tau \rangle} \mid \sigma\overline{\langle \sigma\tau \rangle} + \overline{\langle \sigma\tau \rangle} = 0 \rangle$$
$$\cong \mathbf{Z}\langle \bar{1} + \overline{\sigma\tau} \rangle \subset J[G].$$

We shall construct the following exact sequence of G-modules

$$(3) \qquad 0 \longrightarrow J[G/\langle \sigma \rangle] \oplus J[G/\langle \tau \rangle] \xrightarrow{\hat{\mu}} J[G] \xrightarrow{\hat{\alpha}} J[G/\langle \sigma\tau \rangle] \longrightarrow 0.$$

Here $\hat{\mu}(\overline{\langle \sigma \rangle}) = \bar{1} + \bar{\sigma}$, $\hat{\mu}(\overline{\langle \tau \rangle}) = \bar{1} + \bar{\tau}$ and $\hat{\alpha}(\bar{\rho}) = \overline{\rho\langle \sigma\tau \rangle}$ $(\rho \in G)$. The exactness of (3) follows if $\mathrm{Ker}\ \hat{\alpha} = \mathrm{Im}\ \hat{\mu}$ and μ is injective, because of the fact that $\hat{\alpha}$ is a natural surjective homomorphism of G-modules. From the definition of α, $\mathrm{Ker}\ \hat{\alpha} = \mathbf{Z}\langle a\bar{\sigma} + b\bar{\tau} + c\,\overline{\sigma\tau} \mid c = a + b \rangle = \mathbf{Z}\langle \bar{\sigma} + \overline{\sigma\tau},\ \bar{\tau} + \overline{\sigma\tau} \rangle = \mathbf{Z}\langle \bar{1} + \bar{\sigma},\ \bar{1} + \bar{\tau} \rangle$. On the other hand, we have

$\hat{\mu}$: $J[G/\langle \sigma \rangle] \oplus J[G/\langle \tau \rangle] \cong \mathbf{Z}\langle \bar{1} + \bar{\sigma}, \bar{1} + \bar{\tau} \rangle$. Hence (3) is an exact sequence of G-modules. From the integral dual of (3), we have another exact sequence

$$0 \longrightarrow I[G/\langle \sigma\tau \rangle] \xrightarrow{\hat{\alpha}_0} I[G] \xrightarrow{\hat{\mu}_0} I[G/\langle \sigma \rangle] \oplus I[G/\langle \tau \rangle] \longrightarrow 0,$$

where $\hat{\alpha}_0(\sigma\langle \sigma\tau \rangle - \langle \sigma\tau \rangle) = \sigma + \tau - 1 - \sigma\tau$ and $\hat{\mu}_0(\rho - 1) = (\rho\langle \sigma \rangle - \langle \sigma \rangle, \rho\langle \tau \rangle - \langle \tau \rangle)$ for every $\rho \in G$. Dualizing (3), we obtain the following exact sequence of algebraic tori defined over F

$$(4) \qquad 0 \longrightarrow R_{K(\sigma\tau)/F}^{(1)}(G_m) \xrightarrow{\alpha} R_{K/F}^{(1)}(G_m) \xrightarrow{\mu} R_{K(\sigma)/F}^{(1)}(G_m) \times R_{K(\tau)/F}^{(1)}(G_m) \longrightarrow 0,$$

where $\mu(x) = (N_{K/K(\sigma)}x, N_{K/K(\tau)}x)$ for every $x \in R_{K/F}^{(1)}(G_m)$. Appling Theorem 1 to (4), we have

$$\frac{h_{K/F}}{h_{K(\sigma)/F}\,h_{K(\tau)/F}\,h_{K(\sigma\tau)/F}}=\frac{\tau(K/F)}{\tau(K(\sigma)/F)\,\tau(K(\tau)/F)\,\tau(K(\sigma\tau)/F)}$$
$$\times\frac{q(\hat{\gamma}(F))}{q(\hat{\lambda}(F))}\times\frac{[\mathrm{Ker}\,(H^1(G,I[G/\langle\sigma\tau\rangle]\otimes U_K)\to H^1(G,I[G]\otimes U_K))]}{[\mathrm{Ker}\,(H^1(G,I[G/\langle\sigma\tau\rangle]\otimes O_K^\times)\to H^1(G,I[G]\otimes O_K^\times))]},$$

where $\tau(K/F)$ is the Tamagawa number $\tau(R^{(1)}_{K/F}(G_m))$ and $\gamma=\beta\cdot\alpha\colon R^{(1)}_{K/F}(G_m)\to R^{(1)}_{K/F}(G_m)$ and $\lambda=\mu\times\beta$. Here β and γ are the morphisms $\beta=N_{K/K(\sigma\tau)}$ and $\gamma(x)=x^2$ for every $x\in R^{(1)}_{K/F}(G_m)$. First we have $q(\hat{\gamma}(F))=q(\hat{\lambda}(F))=1$ because of the fact that all the tori of (4) are anisotropic. The Tamagawa numbers $\tau(K(\sigma)/F)=\tau(K(\tau)/F)=\tau(K(\sigma\tau)/F)=2$, because Hasse's norm theorem holds for the quadratic extension, and $\tau(K/F)=4/[F^\times\cap N_{K/F}K_A^\times\colon N_{K/F}K^\times]$ from [8]. Let A_K be either U_K or $O_K^\times$. Then, using Lemma 5 and the exact sequence (4), we have

$$\begin{aligned}&\mathrm{Ker}\,(H^1(G,I[G/\langle\sigma\rangle]\otimes A_K)\longrightarrow H^1(G,I[G]\otimes A_K))\\
&\quad\cong\mathrm{Cok}\,((I[G]\otimes A_K)^G\longrightarrow((I[G/\langle\sigma\rangle]\oplus I\{G/\langle\tau\rangle])\otimes A_K)^G)\\
&\quad\cong[(N^{-1}_{K(\sigma)/F}(1)\cap A_{K(\sigma)})\times(N^{-1}_{K(\tau)/F}(1)\cap A_{K(\tau)/F})\colon(N_{K/K(\sigma)}\times N_{K/K(\sigma)})(N^{-1}_{K/F}(1)\cap A_K)]\\
&\quad\cong\mathrm{Cok}\,(H^0(G,I[G]\otimes A_K)\longrightarrow H^0(G,(I[G/\langle\sigma\rangle]\oplus I[G/\langle\tau\rangle])\otimes A_K))\\
&\quad\cong\mathrm{Cok}\,((N^{-1}(1)\cap A_K)/D_G A_K\longrightarrow((N^{-1}_{K(\sigma)/F}(1)\cap A_{K(\sigma)})/N_{K/K(\sigma)}(D_G A_K))\\
&\qquad\times((N^{-1}_{K(\tau)/F}(1)\cap A_{K(\tau)})/N_{K/K(\tau)}(D_G A_K))).\end{aligned}$$

Theorem 4. *With the notations as above, we have*

$$\frac{h_{K/F}}{h_{K(\sigma)/F}\,h_{K(\tau)/F}\,h_{K(\sigma\tau)/F}}=\frac{1}{2[F^\times\cap N_{K/F}K_A^\times\colon N_{K/F}K^\times]}$$
$$\times\frac{[\mathrm{Ker}\,(H^1(G,I[G/\langle\sigma\tau\rangle]\otimes U_K)\longrightarrow H^1(G,I[G]\otimes U_K))]}{[\mathrm{Ker}\,(H^1(G,I[G/\langle\sigma\tau\rangle]\otimes O_K^\times)\longrightarrow H^1(G,I[G]\otimes O_K^\times))]},$$

where the last factor equals to

$$\frac{[(N^{-1}_{K(\sigma)/F}(1)\cap U_{K(\sigma)})\times(N^{-1}_{K(\tau)/F}(1)\cap U_{K(\tau)})\colon N(N^{-1}_{K/F}(1)\cap U_K)]}{[(N^{-1}_{K(\sigma)/F}(1)\cap O^\times_{K(\sigma)})\times(N^{-1}_{K(\tau)/F}(1)\cap O^\times_{K(\tau)})\colon N(N^{-1}_{K/F}(1)\cap O^\times_K)]}.$$

Here N is the map $N=N_{K/K(\sigma)}\times N_{K/K(\tau)}$.

§3. In this section, we shall obtain explicit form of Theorem 4 when $F=\mathbf{Q}$ and $K=\mathbf{Q}(\sqrt{q},\sqrt{-1})$ (q is a prime). We define σ, τ by putting

$$(\sqrt{q})^\sigma=\sqrt{q},\ (\sqrt{-1})^\sigma=-\sqrt{-1}\ \text{and}\ (\sqrt{q})^\tau=-\sqrt{q},\ (\sqrt{-1})^\tau=\sqrt{-1}.$$

For the sake of simplicity, we shall restrict ourselves to the case when $q\equiv 1 \pmod 4$. In the following, we shall calculate the right factors of Theorem 4. First, we shall show

$$[\mathrm{Ker}\,(H^1(G,I[G/\langle\sigma\rangle]\otimes O_K^\times)\longrightarrow H^1(G,I[G]\otimes O_K^\times))]=4.$$

There exist isomorphisms

$$\mathrm{Ker}\,(H^1(G, I[G/\langle\sigma\tau\rangle]\otimes O_K^\times) \longrightarrow H^1(G, I[G]\otimes O_K^\times))$$
$$\cong \mathrm{Cok}\,(H^0(G, I[G]\otimes O_K^\times) \longrightarrow H^0(G, (I[G/\langle\sigma\rangle]\oplus I[G/\langle\tau\rangle])\otimes O_K^\times)).$$

Let ε be the fundamental unit of $K(\sigma)=\mathbf{Q}(\sqrt{q})$. Since $q\equiv 1 \pmod 4$, $N_{K(\sigma)/\mathbf{Q}}\varepsilon = -1$ and the unit group $O_K^\times = \langle\varepsilon\rangle\times\langle\sqrt{-1}\rangle$. $O_\sigma^\times$, $O_\tau^\times$, $O_{\sigma\tau}^\times$ denote the unit groups of $\mathbf{Q}(\sqrt{q})$, $\mathbf{Q}(\sqrt{-1})$, $\mathbf{Q}(\sqrt{-q})$, respectively. Then $O_\sigma^\times=\langle -1\rangle\times\langle\varepsilon\rangle$ and $O_\tau^\times=\langle\sqrt{-1}\rangle$ and $O_{\sigma\tau}^\times=\langle -1\rangle$. From the fact $N_{K/\mathbf{Q}}(O_K^\times)=N_{K(\sigma\tau)/\mathbf{Q}}(N_{K/K(\sigma\tau)}O_K^\times)=\{1\}$ and Lemma 5, one sees

$$H^0(G, I[G]\otimes O_K^\times)\cong N_{K/\mathbf{Q}}^{-1}(1)/D_G O_K^\times = O_K^\times/D_G O_K^\times.$$

From the fact that $\varepsilon^\sigma=\varepsilon$, $(\sqrt{-1})^\sigma=-\sqrt{-1}$, $\varepsilon^\tau=-\varepsilon^{-1}$, $(\sqrt{-1})^\tau=\sqrt{-1}$, we see $D_G O_K^\times=\langle\varepsilon^2\rangle\times\langle -1\rangle$. Hence we have

$$H^0(G, I[G]\otimes O_K^\times)\cong\langle\varepsilon\rangle\times\langle\sqrt{-1}\rangle/\langle\varepsilon^2\rangle\times\langle -1\rangle\cong(\mathbf{Z}/2\mathbf{Z})^2.$$

Next, we have

$$\begin{aligned}H^0(G, I[G/\langle\sigma\rangle]\otimes O_K^\times) &\cong (O_\sigma^\times\cap N_{K(\sigma)/\mathbf{Q}}^{-1}(1))/N_{K/K(\sigma)}(D_G O_K^\times)\\ &=\langle\varepsilon^2\rangle\times\langle -1\rangle/(\langle\varepsilon^2\rangle\times\langle -1\rangle)^{\sigma+1}\\ &=\langle\varepsilon^2\rangle\times\langle -1\rangle/\langle\varepsilon^4\rangle\cong(\mathbf{Z}/2\mathbf{Z})^2.\end{aligned}$$

Finally we have

$$\begin{aligned}H^0(G, I[G/\langle\tau\rangle]\otimes O_K^\times) &\cong (O_\tau^\times\cap N_{K(\tau)/\mathbf{Q}}^{-1}(1))/N_{K/K(\tau)}(D_G O_K^\times)\\ &=\langle\sqrt{-1}\rangle/(\langle\varepsilon^2\rangle\times\langle -1\rangle)^{\tau+1}\\ &=\langle\sqrt{-1}\rangle\cong\mathbf{Z}/4\mathbf{Z}.\end{aligned}$$

Since $(N_{K/K(\sigma)}\varepsilon, N_{K/K(\tau)}\varepsilon)=(\varepsilon^{\sigma+1}, \varepsilon^{\tau+1})=(\varepsilon^2, -1)$ and $(N_{K/(\sigma)}(\sqrt{-1}), N_{K/K(\tau)}(\sqrt{-1}))=(1, -1)$, we have the equality

$$[\mathrm{Cok}\,(H^0(G, I[G]\otimes O_K^\times)\to H^0\,(G, (I[G/\langle\sigma\rangle]\oplus I[G/\langle\tau\rangle])\otimes O_K^\times))]$$
$$=4\times 2\times 2/2\times 2=4.$$

Next, we shall calculate the number

$$[\mathrm{Cok}\,(H^0(G, I[G]\otimes U_K)\longrightarrow H^0(G, (I[G/\langle\sigma\rangle]\oplus I(G/\langle\tau\rangle])\otimes U_K))].$$

For any prime p, $\mathfrak{P}$ denotes an extension of p to K.. $\mathbf{Q}_p(\sqrt{q}, \sqrt{-1})$ the completion of K at $\mathfrak{P}$ shall be written $K_{\mathfrak{P}}$. $O_{\mathfrak{P}}^\times$ denotes the local unit group of $K_{\mathfrak{P}}$. We denote the decomposition group Gal $(K_{\mathfrak{P}}/\mathbf{Q}_p)$ by $G_{\mathfrak{P}}$, which is considered as a subgroup of G. Then, as G-module, U_K is isomorphic to $\sum_p \mathrm{Ind}_{G_{\mathfrak{P}}}^G O_{\mathfrak{P}}^\times$. We denote $\mathrm{Ind}_{G_P}^G O_{\mathfrak{P}}^\times$ by U_p. f_p denotes the number $[\mathrm{Cok}\,(H^0(G, I[G]\otimes U_p)\to H^0(G, (I[G/\langle\sigma\rangle]\oplus I[G/\langle\tau\rangle])\otimes U_p))]$. For the archimedean place, we denote this number by f_∞. Then the number $[\mathrm{Cok}\,(H^0(G, I[G]\otimes U_K)\to H^0(G, (I[G/$

$\langle\sigma\rangle]\oplus I[G/\langle\tau\rangle])\otimes U_K))]$ equals to $\Pi\ f_p\times f_\infty$. When $K_{\mathfrak{P}}/\mathbf{Q}_p$ is unramified, that is, $p\nmid 2q$, U_p is cohomologically trivial. For any subgroup $H\subset G$, $\mathbf{Z}[G/H]\otimes U_p$ is also cohomologically trivial. Hence $I[G/H]\otimes U_p$ is also cohomologically trivial. Therefore we have $f_p=1$ when $p\neq 2$, $p\neq q$, $p\neq\infty$. Hence we have $\prod_p f_p\times f_\infty=f_2\times f_q\times f_\infty$.

(i) Calculation of $f_p(p=q)$.

Since $\left(\frac{-1}{p}\right)=1$, we see $G_{\mathfrak{P}}=\langle\tau\rangle$ and $U_p=(O_{\mathfrak{P}}^\times)^\sigma\times O_{\mathfrak{P}}^\times$. Hence we have

$$H^0(G, I[G]\otimes U_p)\cong H^{-1}(G, U_p)$$
$$=\{(x^\sigma, y)|(xy)^{\tau+1}=1 \text{ and } x, y\in O_{\mathfrak{P}}^\times\}/D_G U_p\cong H^{-1}(G_{\mathfrak{P}}, O_{\mathfrak{P}}^\times)$$

$\cong\mathbf{Z}/2\mathbf{Z}$. Here $D_G U_p=\langle(x^{\sigma\tau-\sigma}, y^{\tau-1}), (x^\sigma, x^{-1})|x, y\in O_{\mathfrak{P}}^\times\rangle$. Next, we have

$$H^0(G, I[G/\langle\sigma\rangle]\otimes U_p)\cong N_{G/\langle\sigma\rangle}^{-1}(1)\cap U_p^{\langle\sigma\rangle}/N_{\langle\sigma\rangle}(D_G U_p),$$

where $U_p^{\langle\sigma\rangle}=\{(x^\sigma, x)|x\in O_{\mathfrak{P}}^\times\}$. On the other hand, we have

$N_{G/\langle\sigma\rangle}^{-1}(1)\cap U_p^{\langle\sigma\rangle}=\{(x^\sigma, x)|x\in O_{\mathfrak{P}}^\times, x^{\tau+1}=1\}$ and

$N_{\langle\sigma\rangle}(D_G U_p)=\{(x^{\sigma\tau-\tau}, x^{\tau-1})|x\in O_{\mathfrak{P}}^\times\}$. Hence we have

$H^0(G, I[G/\langle\sigma\rangle]\otimes U_p)\cong N_{\langle\tau\rangle}^{-1}(1)\cap O_{\mathfrak{P}}^\times/D_{\langle\tau\rangle}O_{\mathfrak{P}}^\times\cong H^{-1}(\langle\tau\rangle, O_{\mathfrak{P}}^\times)\cong\mathbf{Z}/2\mathbf{Z}$.

Next, we have

$$H^0(G, I[G/\langle\tau\rangle]\otimes U_p)\cong N_{G/\langle\tau\rangle}^{-1}(1)\cap U_p^{\langle\tau\rangle}/N_{\langle\tau\rangle}(D_G U_p),$$

where $U_p^{\langle\tau\rangle}=\{(x^\sigma, y)|x, y\in(O_{\mathfrak{P}}^\times)^{\langle\tau\rangle}=\mathbf{Z}_p^\times\}$. On the other hand, we have

$N_{G/\langle\tau\rangle}^{-1}(1)\cap U_p^{\langle\tau\rangle}=\{(x^\sigma, X^{-1})|x\in\mathbf{Z}_p^\times\}$ and

$N_{\langle\tau\rangle}(D_G U_p)=\{(x^{\sigma\tau+\sigma}, x^{-\tau-1})|x\in O_{\mathfrak{P}}^\times\}$. Hence we have

$H^0(G, I[G/\langle\tau\rangle]\otimes U_p)\cong H^0(\langle\tau\rangle, O_{\mathfrak{P}}^\times)\cong\mathbf{Z}_p^\times/N_{\langle\tau\rangle}O_P^\times\cong\mathbf{Z}/2\mathbf{Z}$.

Therefore we have $f_q=[\mathrm{Cok}\ (\mathbf{Z}/2\mathbf{Z}\to(\mathbf{Z}/2\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}))]$. Hence $f_q=2$ or 4. It is easy to show $(x^\sigma, 1)$ mod $D_G\,O_{\mathfrak{P}}^\times\in H^{-1}(G, U_p)$ for any $x\in O_{\mathfrak{P}}^\times$ such that $x^{\tau+1}=1$. On the other hand, for any $x(x\in O_{\mathfrak{P}}^\times$ and $x^{\tau+1}=1)$, we have $N_{K/K(\sigma)}\times N_{K/K(\tau)}(x^\sigma, 1)=(x^\sigma, x)$ mod $N_{\langle\sigma\rangle}(D_G U_p)\times(1, 1)$ mod $N_{\langle\tau\rangle}(D_G U_p)$. Hence

$$\mathrm{Cok}\ (H^0(G, I[G]\otimes U_p)\longrightarrow H^0(G, (I[G/\langle\sigma\rangle]\oplus I[G/\langle\tau\rangle])\otimes U_p))$$

$\cong H^0(G, I[G/\langle\tau\rangle]\otimes U_p)\cong H^0(\langle\tau\rangle, O_{\mathfrak{P}}^\times)\cong\mathbf{Z}/2\mathbf{Z}$. Therefore we have $f_q=2$.

(ii) Calculation of f_2.

First, we treat the case when $q\equiv 1\pmod 8$. Then $\left(\frac{q}{2}\right)=1$, that is, $\mathbf{Q}_2(\sqrt{q})=\mathbf{Q}_2$ and the decomposition group $G_{\mathfrak{P}}$ equals to $\langle\sigma\rangle$. Therefore, in the same way as above, we have

$$\mathrm{Cok}\ (H^0(G, I[G]\otimes U_p)\longrightarrow H^0(G, (I[G/\langle\sigma\rangle]\oplus I[G/\langle\tau\rangle])\otimes U_p))]$$

$\cong H^0(G, I[G/\langle\sigma\rangle]\otimes U_p)\cong H^0(\langle\sigma\rangle, O_{\mathfrak{P}}^\times)\cong \mathbf{Z}/2\mathbf{Z}$. Hence $f_2=2$.

For $q\equiv 5 \pmod 8$, one sees $\left(\frac{q}{2}\right)=-1$, that is, $\mathbf{Q}_2(\sqrt{q})$ is the unramified quadratic extension of $\mathbf{Q}_2$. Since $\mathbf{Q}_2(\sqrt{-1})/\mathbf{Q}_2$ is a ramified quadratic extension, the decomposition group $G_{\mathfrak{P}}$ equals to $\langle\sigma\rangle\times\langle\tau\rangle$. Hence $U_p=O_{\mathfrak{P}}^\times$. Therefore we have

$$\begin{aligned} H^0(G, I[G]\otimes U_p) &\cong H^{-1}(G, U_p)=H^{-1}(G, O_{\mathfrak{P}}^\times)\\ &=\{x\in O_{\mathfrak{P}}^\times \mid x^{\sigma\tau+\sigma+\tau+1}=1\}/D_G O_{\mathfrak{P}}^\times. \end{aligned}$$

Here $D_G O_{\mathfrak{P}}^\times=\langle x^{1-\sigma}, y^{1-\tau}\mid x, y\in O_{\mathfrak{P}}^\times\rangle$.

Next we have

$$H^0(G, I[G/\langle\sigma\rangle]\otimes U_p)=H^0(G, I[G/\langle\sigma\rangle]\otimes O_{\mathfrak{P}}^\times)\cong N_{G/\langle\sigma\rangle}^{-1}(1)\cap U_\sigma/N_{\langle\sigma\rangle}(D_G O_{\mathfrak{P}}^\times)$$

$\cong\{x\in U_\sigma\mid x^{\tau+1}=1\}/\{y^{(\sigma+1)(1-\tau)}\mid y\in O_{\mathfrak{P}}^\times\}$. Here U_σ is the local unit group $(O_{\mathfrak{P}}^\times)^{\langle\sigma\rangle}$. Since $\mathbf{Q}_2(\sqrt{q})/\mathbf{Q}_2$ is an unramified extension, U_σ is a cohomologically trivial $G/\langle\sigma\rangle$-module. Hence we have $H^{-1}(G/\langle\sigma\rangle, U_\sigma)=1$, that is, $N_{G/\langle\sigma\rangle}^{-1}(1)\cap U_\sigma=\{x^{1-\tau}\mid x\in U_\sigma\}$. Therefore we can define the following surjective homomorphism δ

$\delta\colon \mathbf{Z}/2\mathbf{Z}\cong H^0(\langle\sigma\rangle, O_{\mathfrak{P}}^\times)\to H^0(G, I[G/\langle\sigma\rangle]\otimes O_{\mathfrak{P}}^\times)$, where δ is defined by $\delta(x \bmod N_{\langle\sigma\rangle}O_{\mathfrak{P}}^\times)=x^{1-\tau} \bmod N_{\langle\sigma\rangle}(D_G O_{\mathfrak{P}}^\times)$. One can easily show the kernel of δ equals to $\mathbf{Z}_p^\times N_{\langle\sigma\rangle}O_{\mathfrak{P}}^\times/N_{\langle\sigma\rangle}O_{\mathfrak{P}}^\times$. Hence we have

$H^0(G, I[G/\langle\sigma\rangle]\otimes O_{\mathfrak{P}}^\times)\cong U_\sigma/\mathbf{Z}_p^\times N_{\langle\sigma\rangle}O_{\mathfrak{P}}^\times$. Hence we have

$[H^0(G, I[G/\langle\sigma\rangle]\otimes O_{\mathfrak{P}}^\times)]\leqq 2$. Since $H^0(G/\langle\sigma\rangle, U_\sigma)=1$, that is, $\mathbf{Z}_p^\times=N_{\langle\sigma\rangle}O_{\mathfrak{P}}^\times$, we can define the following surjective homomorphism

$$\delta'\colon H^0(G, I[G/\langle\sigma\rangle]\otimes O_{\mathfrak{P}}^\times\cong U_\sigma/\mathbf{Z}_p^\times N_{\langle\sigma\rangle}O_{\mathfrak{P}}^\times\longrightarrow \mathbf{Z}_p^\times/(\mathbf{Z}_p^\times)^2 N_G O_{\mathfrak{P}}^\times.$$

Here δ' is defined by $\delta'(x \bmod \mathbf{Z}_p^\times N_{\langle\sigma\rangle}O_{\mathfrak{P}}^\times)=x^{\tau+1} \bmod (\mathbf{Z}_p^\times)^2 N_G O_{\mathfrak{P}}^\times$. Since $H^0(G, O_{\mathfrak{P}}^\times)=\mathbf{Z}_p^\times/N_G O_{\mathfrak{P}}^\times\cong\mathbf{Z}/2\mathbf{Z}$, we have $N_G O_{\mathfrak{P}}^\times\supset(\mathbf{Z}_p^\times)^2$. Therefore $Z_p^\times/(\mathbf{Z}_p^\times)^2 N_G O_{\mathfrak{P}}^\times\cong\mathbf{Z}_p^\times/N_G O_{\mathfrak{P}}^\times\cong H^0(G, O_{\mathfrak{P}}^\times)\cong\mathbf{Z}/2\mathbf{Z}$. Therefore $[H^0(G, I[G/\langle\sigma\rangle]\otimes O_{\mathfrak{P}}^\times)]\geqq 2$. Combining these, we have $H^0(G, I[G/\langle\sigma\rangle]\otimes O_{\mathfrak{P}}^\times)\cong\mathbf{Z}/2\mathbf{Z}$. Finally we have

$$H^0(G, I[G/\langle\tau\rangle]\otimes U_p)=H^0(G, I[G/\langle\tau\rangle]\otimes O_{\mathfrak{P}}^\times)\cong N_{G/\langle\tau\rangle}^{-1}(1)\cap U_\tau/N_{\langle\tau\rangle}(D_G O_{\mathfrak{P}}^\times)$$

$\cong\{x\in U_\tau\mid x^{\sigma+1}=1\}/\{y^{(\tau+1)(1-\sigma)}\mid y\in O_{\mathfrak{P}}^\times\}$. Here U_τ is the local unit group $(O_{\mathfrak{P}}^\times)^{\langle\tau\rangle}$. Since $H^0(\langle\tau\rangle, O_{\mathfrak{P}}^\times)=1$, we have $N_{\langle\tau\rangle}O_{\mathfrak{P}}^\times=U_\tau$. Therefore we have $H^0(G, I[G/\langle\tau\rangle]\otimes O_{\mathfrak{P}}^\times)\cong\{x\in U_\tau\mid x^{\sigma+1}=1\}/\{y^{1-\sigma}\mid y\in U_\tau\}=H^{-1}(G/\langle\tau\rangle, U_\tau)\cong\mathbf{Z}/2\mathbf{Z}$.

Let α be the following map

$$\begin{aligned}\alpha\colon \{x\in O_{\mathfrak{P}}^\times\mid x^{\sigma\tau+\sigma+\tau+1}=1\}/D_G O_{\mathfrak{P}}^\times &\longrightarrow(\{x\in U_\sigma\mid x^{\tau+1}=1\}/\{y^{(\sigma+1)(1-\tau)}=1\mid y\in O_{\mathfrak{P}}^\times\})\\ &\times(\{x\in U_\tau\mid x^{\sigma+1}=1\}/\{y^{(\tau+1)(1-\sigma)}=1\mid y\in O_{\mathfrak{P}}^\times\}).\end{aligned}$$

Here α is defined by $\alpha(x \bmod D_G O_{\mathfrak{P}}^\times)=(x^{\sigma+1} \bmod N_{\langle\sigma\rangle}(D_G O_{\mathfrak{P}}^\times), x^{\tau+1} \bmod N_{\langle\tau\rangle}(D_G O_{\mathfrak{P}}^\times))$. From Lemma 2 and $H^0(\langle\tau\rangle, O_{\mathfrak{P}}^\times)=1$, we have $[\mathrm{Cok}\,\alpha]=2/a$. Here a is the order of the following group A

$$A = (x^{\sigma+1} | x \in O_{\mathfrak{P}}^{\times} \text{ and } x^{\tau+1} = y^{(\tau+1)(1-\sigma)} \quad \text{for some } y \in O_{\mathfrak{P}}^{\times}\}/N_{\langle\sigma\rangle}(D_G O_{\mathfrak{P}}^{\times}).$$

For any $x \in O_{\mathfrak{P}}^{\times}$ such that $x^{\tau+1} = y^{(1-\sigma)(\tau+1)}$, we have $(x \times y^{\sigma-1})^{\tau+1} = 1$. Since $H^{-1}(\langle\tau\rangle, O_{\mathfrak{P}}^{\times}) = 1$, there exists an element $z \in O_{\mathfrak{P}}^{\times}$ such that $x \times y^{\sigma-1} = z^{1-\tau}$. Hence $x = y^{1-\sigma} \times z^{1-\sigma}$. Therefore $x^{\sigma+1} = z^{(\sigma+1)(1-\tau)} \in N_{\langle\sigma\rangle}(D_G O_{\mathfrak{P}}^{\times})$. Hence $a = 1$. Therefore f_2 is also equals to 2 for $q \equiv 5 \pmod 8$.

(iii) Calculation of f_∞.

From $U_\infty \cong \mathbf{C}^\times \times \mathbf{C}^\times$, we have $H^0(G, I[G] \otimes U_\infty) \cong H^{-1}(G, U_\infty)$ $\cong \{(x^\sigma, y) | x, y \in \mathbf{C}^\times, (xy)^{\tau+1} = 1\}/\langle (x^\sigma, x^{-1}), (1, y^{1-\tau}) | x, y \in \mathbf{C}^\times \rangle$. In the same way as the calculation of f_q, we have $H^0(G, I[G/\langle\sigma\rangle] \otimes U_\infty) \cong N^{-1}(\langle\tau\rangle, \mathbf{C}^\times) \cong \{1\}$ (Hilbert's Theorem 90). On the other hand

$$H^0(G, I[G/\langle\tau\rangle] \otimes U_\infty) \cong N_{G/\langle\tau\rangle}^{-1}(1) \cap U_\infty^{\langle\tau\rangle}/N_{\langle\tau\rangle}(D_G U_\infty)$$
$$\cong \{(x^\sigma, x^{-1}) | x \in \mathbf{R}^\times\}/\{(x^{\sigma\tau+\sigma}, x^{-1-\tau}) | x \in \mathbf{C}^\times\} \cong H^0(\langle\tau\rangle, \mathbf{C}^\times) \cong \mathbf{R}^\times/\mathbf{R}_+^\times \cong \mathbf{Z}/2\mathbf{Z}.$$

Let β be the map

$$\beta\colon H^0(G, I[G] \otimes U_\infty) \longrightarrow H^0(G, I[G/\langle\tau\rangle] \otimes U_\infty)$$

defined by putting

$$\beta((x^\sigma, y) \bmod D_G U_\infty) = (x^{\sigma\tau+\sigma}, y^{\tau+1}) \bmod N_{\langle\tau\rangle}(D_G U_\infty).$$

From the definition, we have $y^{\tau+1} = x^{-1-\tau}$. Hence we obtained $f_\infty = [\operatorname{Cok} \beta] = 2$. Therefore, we have obtained $f_2 = f_q = f_\infty = 2$ for $q \equiv 1 \pmod 4$. Hence, from Theorem 4, we have

$$\frac{h_{K/\mathbf{Q}}}{h_{K(\sigma)/\mathbf{Q}}\, h_{K(\tau)/\mathbf{Q}}\, h_{K(\sigma\tau)/\mathbf{Q}}} = \frac{2 \times 2 \times 2}{8 \times [\mathbf{Q}^\times \cap N_{K/\mathbf{Q}} K_A^\times : N_{K/\mathbf{Q}} K^\times]}$$
$$= \frac{1}{[\mathbf{Q}^\times \cap N_{K/\mathbf{Q}} K_A^\times : N_{K/\mathbf{Q}} K^\times]}.$$

It is known that Scholz's number knot group $\mathbf{Q}^\times \cap N_{K/\mathbf{Q}} K_A^\times / N_{K/\mathbf{Q}} K^\times$ is isomorphic to $\operatorname{Ker}(H^3(G, \mathbf{Z}) \to \sum_p H^3(G_{\mathfrak{P}}, \mathbf{Z}))$, where $G_{\mathfrak{P}}$ is the decomposition group for every prime p. From Lyndon's formula, we have $H^3(G, \mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$ for this case. Hence we have

$$\mathbf{Q}^\times \cap N_{K/\mathbf{Q}} K_A^\times / N_{K/\mathbf{Q}} K^\times \cong \mathbf{Z}/2\mathbf{Z} \qquad (q \equiv 1 \pmod 8)),$$
$$\mathbf{Q}^\times \cap N_{K/\mathbf{Q}} K_A^\times / N_{K/\mathbf{Q}} K^\times \cong \{1\} \qquad (q \equiv 5 \pmod 8)).$$

Hence we have obtained the following formula

$$(4) \qquad \frac{h_{K/\mathbf{Q}}}{h_{K(\sigma)/\mathbf{Q}}\, h_{K(\tau)/\mathbf{Q}}\, h_{K(\sigma\tau)/\mathbf{Q}}} = \begin{cases} 1/2 & (q \equiv 1 \pmod 8)) \\ 1 & (q \equiv 5 \pmod 8)) \end{cases}.$$

From Corollary 2, we have $E(K(\sigma)/\mathbf{Q}) = 1$, $E(K(\tau)/\mathbf{Q}) = 2$, $E(K(\sigma\tau)/\mathbf{Q}) = 1$.

From the above calculation on the order of Scholz's knot group, we have

$$E(K/\mathbf{Q})=\begin{cases}2 & (q\equiv 1 \ (\mathrm{mod}\ 8))\\ 1 & (q\equiv 5 \ (\mathrm{mod}\ 8))\end{cases}.$$

Let h_σ, h_τ, $h_{\sigma\tau}$ be the class numbers of the quadratic fields $\mathbf{Q}(\sqrt{q})$, $\mathbf{Q}(\sqrt{-1})$, $\mathbf{Q}(\sqrt{-q})$, respectively. There exists an equation

$$\frac{h_K}{h_\sigma\, h_\tau\, h_{\sigma\tau}}=\frac{E(K/\mathbf{Q})}{E(K(\sigma)/\mathbf{Q})\; E(K(\tau)/\mathbf{Q})\; E(K(\sigma\tau)/\mathbf{Q})}\times\frac{h_{K/\mathbf{Q}}}{h_{K(\sigma)/\mathbf{Q}}\; h_{K(\tau)/\mathbf{Q}}\; h_{K(\sigma\tau)/\mathbf{Q}}}.$$

Combining these and the fact $h_\tau=1$, we have $\dfrac{h_K}{h_\sigma\, h_{\sigma\tau}}=\dfrac{1}{2}$.

Finally, we have obtained the following Dirichlet's class number formula.

Corollary 4. *With the notations as above, we have*

$$h_K=\frac{h_\sigma\, h_{\sigma\tau}}{2} \qquad (q\equiv 1\ (\mathrm{mod}\ 4)).$$

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