J. Math. Kyoto Univ. (JMKYAZ)
39-4 (1999) 607–618

On the associated graded module of an ideal generated by an unconditioned strong d-sequence

By

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0. Introduction

Throughout this paper, A is a commutative ring with non-zero identity, $x_1, \ldots, x_s$ is a sequence of elements of A of length $s > 0$, $\mathfrak{a}$ is an ideal of A and M is an A-module. We use $\mathbf{N}$ (respectively $\mathbf{N}_0$) to denote the set of positive (respectively non-negative) integers. For each i $(1 \leq i \leq s)$, let $\mathfrak{q}_i = (x_1, \ldots, x_i)$, $\mathfrak{q} = (x_1, \ldots, x_s)$ and $\mathfrak{q}_0 = (0)$. If there is no confusion, the associated graded ring $G_{\mathfrak{q}}(A) = \bigoplus_{n \geq 0} \mathfrak{q}^n/\mathfrak{q}^{n+1}$ and the associated graded module $G_{\mathfrak{q}}(M) = \bigoplus_{n \geq 0} \mathfrak{q}^n M/\mathfrak{q}^{n+1} M$ are denoted by G and $G(M)$ respectively. We put $h_i = x_i \bmod \mathfrak{q}^2$ $(1 \leq i \leq s)$, the initial forms of x_i's in G.

The concept of a d-sequence is given by Huneke (see [5]) and it plays an important role in the theory of Buchsbaum modules and in the theory of Blow up algebra, e.g. Ress Algebra. The sequence $x_1, \ldots, x_s$ of elements of A is called a d-sequence on M if, for each $i = 0, 1, \ldots, s-1$, the equality

$$\left(\sum_{j=1}^{i} Ax_j\right) M :_M x_{i+1}x_k = \left(\sum_{j=1}^{i} Ax_j\right) M :_M x_k$$

holds for all $k \geq i+1$ (this is actually a slight weakening of Huneke's definition); it is an unconditioned strong d-sequence (u.s.d-sequence) on M if $x_1^{\alpha_1}, \ldots, x_s^{\alpha_s}$ is a d-sequence in any order for all positive integers $\alpha_1, \ldots, \alpha_s$.

It is well known that if A is local, M is finitely generated and $x_1, \ldots, x_s$ is a system of parameters for M, then $x_1, \ldots, x_s$ is an u.s.d-sequence on M if and only if the natural homomorphism $H_i(\mathfrak{q}, M) \to H_{\mathfrak{q}}^i(M)$ is surjective for all $i < s$. Although these natural homomorphisms do provide a satisfactory characterization of u.s.d-sequence, they have the disadvantage that their underlying ring is local and the ideal $\mathfrak{q}$ is a parameter ideal of M.

In [7], for a sequence $x = x_1, \ldots, x_s$ of elements of A, we established the canonical homomorphisms $\overline{\Psi}^{\bullet}_{x,M}$ between the homology modules of the Koszul complex $K_{\bullet}(x, M)$ and the homology modules of a complex $C(\mathscr{A}(x), M)$ of A-modules which involves modules of generalized fractions derived from M and the sequence x. Then we showed that these canonical homomorphisms do provide

Communicated by Prof. K. Ueno, July 6, 1998

useful criteria for u.s.d-sequences without any restriction on A and M. The purpose of this paper is to show that our criteria for u.s.d-sequences is good help when we treat the u.s.d-sequences in relation with associated graded modules. Indeed we shall prove, among other things, the following two theorems.

Theorem A. *If $x_1, \dots, x_s$ is an u.s.d-sequence on M, then it is an unconditioned $\mathfrak{q}$-filter regular sequence on M and the sequence $h_1, \dots, h_s$ constitute an u.s.d-sequence on $G_{\mathfrak{q}}(M)$. Moreover if A is Noetherian and M is finitely generated, the converse is also true.*

The proof of Theorem A is divided in two parts. The proof of the first part of the theorem is given in 2.3, while the second part of the theorem is a consequence of 2.4. It is shown, in 2.5, that the result [3, 2.12] of Goto and Yamagishi can be deduced from Theorem A.

Theorem B. *For an ideal $\mathfrak{a}$ of a Noetherian ring A, a finitely generated A-module M and a positive integer s, the following statements are equivalent:*

(i) *$H^j_{\mathfrak{a}}(M)$ is finitely generated for all $j < s$,*

(ii) *There is an $\mathfrak{a}$-filter regular sequence $x_1, \dots, x_s$ on M such that $h_1, \dots, h_s$ is an unconditioned I-filter regular sequence on $G_{\mathfrak{q}}(M)$ and $H^j_I(G_{\mathfrak{q}}(M))$ is finitely generated G-module for all $j < s$, where $I = \sum_{i=1}^s h_i G_{\mathfrak{q}}(A)$.*

1. Notations and preparatory results

We say that a sequence $x_1, \dots, x_s$ of elements of A is an $\mathfrak{a}$-filter regular sequence on M if $x_1, \dots, x_s \in \mathfrak{a}$ and

$$\operatorname{Supp}\left(\left(\left(\sum_{j=1}^{i-1} Ax_j\right)M :_M x_i\right) \Big/ \left(\sum_{j=1}^{i-1} Ax_j\right)M\right) \subseteq V(\mathfrak{a})$$

for all $i = 1, \dots, s$, where $V(\mathfrak{a})$ denotes the set of prime ideals containing $\mathfrak{a}$. When such property holds in any order, we will say that the sequence $x_1, \dots, x_s$ form an unconditioned $\mathfrak{a}$-filter regular sequence on M. The concept of an $\mathfrak{a}$-filter regular sequence on M is a generalization of the one of a filter regular sequence which has been studied in [9], [12], [13] and has led to some interesting results. Note that both concepts coincide if A is local, M is finitely generated, and $\mathfrak{a}$ is the maximal ideal of A. Also note that $x_1, \dots, x_s$ is a poor M-sequence [15, §2] if and only if it is an A-filter regular sequence on M. D-sequences are closely related to filter regular sequences. It is easy to see that if $x_1, \dots, x_s$ is a d-sequence on M, then it is an $\sum_{i=1}^s Ax_i$-filter regular sequence on M. For the converse, we have the following

1.1. Remarks. Consider the special case in which A is Noetherian and M is finitely generated.

(i) By slight modification in the arguments of [13, 2.1], one can show that if $x_1, \dots, x_s$ is an $\mathfrak{a}$-filter regular sequence on M, then, for each $k > 0$, there exists an ascending sequence of integers $k \leq r_1 \leq \cdots \leq r_s$ such that $x_1^{r_1}, \ldots, x_s^{r_s}$ is a d-sequence on M.

(ii) [3, 6.12] Let A be local with maximal ideal $\mathfrak{m}$ and let $x_1, \ldots, x_s$ be an unconditioned $\mathfrak{m}$-filter regular sequence on M. Then the following conditions are equivalent:

(a) $x_1, \ldots, x_s$ form an u.s.d-sequence on M;

(b) $x_{j+1} H_{\mathfrak{m}}^i(M/\mathfrak{q}_j M) = 0$ for every $0 \leq i + j < s$.

Now we recall some facts about d-sequences which are needed for the proof of the main results in this paper. The reader is referred to [4, 5.1.1] and [3, 1.3, 1.6 and 1.9(2)] for their proofs.

1.2. Proposition. (i) *$x_1, \ldots, x_s$ form a d-sequence on M if and only if the equality*

$$[\mathfrak{q}_{i-1} M :_M x_i] \cap \mathfrak{q} M = \mathfrak{q}_{i-1} M$$

holds for all $1 \leq i \leq s$.

(ii) *if $x_1, \ldots, x_s$ form a d-sequence on M, then the equalities*

$$\mathfrak{q}_{i-1} M \cap \mathfrak{q}^n M = \mathfrak{q}_i \mathfrak{q}^{n-1} M \qquad \text{and} \qquad x_1^m M \cap \mathfrak{q}^n M = x_1^m \mathfrak{q}^{n-m} M$$

hold for every $1 \leq i \leq s$, $m > 0$ and $n \in \mathbf{Z}$.

(iii) *$h_1, \ldots, h_s$ form a d-sequence on $G_{\mathfrak{q}}(M)$ if and only if the equality*

$$[\mathfrak{q}_{i-1} \mathfrak{q}^n M + \mathfrak{q}^{n+2} M :_M x_i] \cap \mathfrak{q}^n M = \mathfrak{q}_{i-1} \mathfrak{q}^{n-1} M + \mathfrak{q}^{n+1} M$$

holds for all $1 \leq i \leq s$ and $n > 0$.

(iv) *If $x_1, \ldots, x_s$ form a d-sequence on M, then the equality*

$$[\mathfrak{q}_i M :_M x_{i+1}] \cap \mathfrak{q}^n M = \mathfrak{q}_i \mathfrak{q}^{n-1} M$$

holds for every $0 \leq i \leq s$ and $n > 0$, where $x_{s+1} = 1$.

For a system of elements $x = x_1, \ldots, x_s$ of A, let $K_\bullet(x, M)$ and $H_*(x, M)$ denote the Koszul complex generated by x over M and the homology module of the Koszul complex, respectively. When discussing the Koszul complex, we shall use the notation of [8]. In particular, we shall abbreviate $K_p(x, M)$ to $K_p(M)$ when no confusion is possible. Also, in this paper, we shall use the concept of a modules of generalized fractions introduced in [11]. The notations and terminology concerning triangular subset of A^n (for $n \in \mathbf{N}$) and modules of generalized fractions will be the same as that used in [7, §2]. In particular, $C(\mathscr{A}(x), M)$ denotes the associated complex of modules of generalized fractions derived from x and M.

In [7, §2], we established the homomorphism $\overline{\Psi}_{x,M}^p$ between the Koszul homology module $H_{s-p}(x, M)$ and the p-th homology module of the complex $C(\mathscr{A}(x), M)$. Let us recall briefly the construction of these morphisms and review the main result of [7, §2] which play a significant role in the proof of the main results of this paper.

Write the associated complex $C(\mathscr{A}(x), M)$ as

$$0 \xrightarrow{e_{x,M}^{-1}} M \xrightarrow{e_{x,M}^{0}} U(x)_1^{-1}M \xrightarrow{e_{x,M}^{1}} \cdots \longrightarrow U(x)_i^{-i}M \xrightarrow{e_{x,M}^{i}} U(x)_{i+1}^{-i-1}M \longrightarrow \cdots.$$

For each integer p with $0 \leq p \leq s$, we define

$$\Psi_{x,M}^{p} : K_{s-p}(M) \longrightarrow U(x)_p^{-p}M$$

as follows. $\Psi_{x,M}^{0}$ is the identity map, $\Psi_{x,M}^{s}(b) = \dfrac{b}{(x_1, \ldots, x_s)}$ for all $b \in M$ and, for each $1 \leq p \leq s-1$, $\Psi_{x,M}^{p}$ is defined by the rule

$$\Psi_{x,M}^{p}(be_{i_1 \cdots i_{s-p}}) = \begin{cases} \dfrac{b}{(x_1, \ldots, x_p)} & \text{if } (i_1, \ldots, i_{s-p}) = (p+1, \ldots, s) \\ 0 & \text{otherwise} \end{cases}$$

for all $b \in M$. It is easily seen that, for all $0 \leq p \leq s$, $\Psi_{x,M}^{p}$ is an A-homomorphism and that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_s(M) & \longrightarrow & K_{s-1}(M) & \longrightarrow \cdots \longrightarrow & K_1(M) & \longrightarrow & K_0(M) \\ & & \downarrow{\scriptstyle \Psi_{x,M}^{0}} & & \downarrow{\scriptstyle \Psi_{x,M}^{1}} & & \downarrow{\scriptstyle \Psi_{x,M}^{s-1}} & & \downarrow{\scriptstyle \Psi_{x,M}^{s}} \\ 0 & \xrightarrow{e_{x,M}^{-1}} & M & \xrightarrow{e_{x,M}^{0}} & U(x)_1^{-1}M & \longrightarrow \cdots \longrightarrow & U(x)_{s-1}^{-s+1}M & \xrightarrow{e_{x,M}^{s-1}} & U(x)_s^{-s}M \end{array}$$

is commutative. Therefore, for all $0 \leq p \leq s-1$, $\Psi_{x,M}^{p}$ induces an A-homomorphism $H_{s-p}(x, M) \rightarrow \dfrac{\ker e_{x,M}^{p}}{\operatorname{im} e_{x,M}^{p-1}}$ which is denoted by $\overline{\Psi}_{x,M}^{p}$.

1.3. Theorem. [7, 2.4]. *The following conditions are equivalent:*

(i) $x_1, \ldots, x_s$ *is an u.s.d-sequence on* M*;*

(ii) *For any permutation* σ *of the set* $\{1, \ldots, s\}$, *the canonical homomorphism*

$$\overline{\Psi}_{\sigma(x),M}^{p} : H_{s-p}(\sigma(x), M) \rightarrow \frac{\ker e_{\sigma(x),M}^{p}}{\operatorname{im} e_{\sigma(x),M}^{p-1}}$$

is surjective for all p *with* $0 \leq p \leq s-1$, *where* $\sigma(x) := x_{\sigma(1)}, \ldots, x_{\sigma(s)}$.

For an ideal $\mathfrak{b}$ of A and $b \in A$, we shall denote the submodule

$$\{m \in M : b^r m \in \mathfrak{b}M \text{ for some } r \in \mathbf{N}_0\}$$

of M by $\mathfrak{b}M :_M \langle b \rangle$. Assume that $x_1, \ldots, x_s$ form an unconditioned $\mathfrak{a}$-filter regular sequence on M and that x_s is a non-zero-divisor on M. Then, by using the fact that $(\sum_{j=1}^{i-1} Ax_j^{\alpha_j})M :_M \langle x_i \rangle = (\sum_{j=1}^{i-1} Ax_j^{\alpha_j})M :_M \langle x_s \rangle$ for all $1 \leq i \leq s$ and $\alpha_1, \ldots, \alpha_{i-1} \in \mathbf{N}$, we may apply the same arguments as in the proof [7, 2.3] to obtain, for each $0 \leq i \leq s$, the exact sequence

$$0 \longrightarrow U(x)_i^{-i}M \xrightarrow{x_s} U(x)_i^{-i}M \longrightarrow U(x)_i^{-i}(M/x_sM) \longrightarrow 0,$$

where $U(x)_i^{-i}M \longrightarrow U(x)_i^{-i}(M/x_sM)$ is the natural homomorphism. Put $\overline{M} = M/x_sM$. Then the above exact sequence induces the exact sequence of complexes

$$0 \longrightarrow C(\mathscr{A}(x), M) \xrightarrow{x_s} C(\mathscr{A}(x), M) \longrightarrow C(\mathscr{A}(x), \overline{M}) \longrightarrow 0$$

which, in turn, yields the exact complex

$$\cdots \longrightarrow H^i(C(\mathscr{A}(x), M)) \xrightarrow{x_s} H^i(C(\mathscr{A}(x), M))$$
$$\longrightarrow H^i(C(\mathscr{A}(x), \overline{M})) \xrightarrow{\Delta_i} H^{i+1}(C(\mathscr{A}(x), M)) \longrightarrow \cdots. \qquad (*)$$

Throughout the paper, we shall appeal to such exact complexes without further comments.

1.4. Remark. In this note we shall employ the notion of graded modules. For an integer n and a graded module X, we define $X(n)$ as the module X whose grading is given by $[X(n)]_m = X_{n+m}$. Also it should be observed that, if X is a graded module over a graded commutative ring R (with identity) and U is a triangular subset of R^n $(n \in \mathbf{N})$ composed of homogeneous elements, then $U^{-n}X$ has graded structure as R-module which is such that, for a homogeneous element $x \in X$ and $(u_1, \dots, u_n) \in U$, the degree of the fraction $\dfrac{x}{(u_1, \dots, u_n)}$ is $\deg x - \sum_{i=1}^n \deg u_i$. Hence, for a chain of graded triangular subsets $\mathscr{U}$ on R, every homology module of the complex $C(\mathscr{U}, X)$ has graded structure as R-module (see [1]). When discussing such complexes, we shall use the above mentioned grading.

2. Proof of the main results

It was shown in [12, Appendix 2(i)] that whenever A is local (Noetherian) with maximal ideal $\mathfrak{m}$, M is finitely generated and s is a positive integer, then there exists an $\mathfrak{m}$-filter regular sequence on M of length s. The following proposition establishes a similar result for unconditioned filter regular sequences.

2.1. Proposition. *Suppose that A is Noetherian and that M is finitely generated. If $x_1, \dots, x_r$ is an unconditioned $\mathfrak{a}$-filter regular sequence on M, then there exists an element $x_{r+1} \in \mathfrak{a}$ such that $x_1, \dots, x_r, x_{r+1}$ is an unconditioned $\mathfrak{a}$-filter regular sequence on M.*

Proof. If $r = 0$, then choose $x_1 \in \mathfrak{a} \backslash \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M) \backslash V(\mathfrak{a})} \mathfrak{p}$ arbitrary. So suppose that $r \geq 1$. Set

$$S := \left\{\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}\left(M \middle/ \left(\sum_{i \in I} Ax_i\right) M\right), I \subseteq \{1, \dots, r\}\right\}$$

and let $x_{r+1} \in \mathfrak{a} \backslash \bigcup_{\mathfrak{p} \in S \backslash V(\mathfrak{a})} \mathfrak{p}$. Let $y_1, \dots, y_{r+1}$ be any permutation of $x_1, \dots, x_{r+1}$ and suppose that $y_l = x_{r+1}$ for some $1 \leq l \leq r+1$. To complete the proof, it is now sufficient to show that, for each $i = 1, \ldots, r+1$, $y_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M/(\sum_{j=1}^{i-1} Ay_j)M) \setminus V(\mathfrak{a})$. To do this assume contrary. Then there exist an integer i, with $l+1 \le i \le r+1$, and $\mathfrak{p} \in \mathrm{Ass}(M/(\sum_{j=1}^{i-1} Ay_j)M) \setminus V(\mathfrak{a})$ such that $y_i \in \mathfrak{p}$. It is easy to see that $y_1, \ldots, \check{y}_l, \ldots, y_i, y_l$ is an $\mathfrak{a}$-filter regular sequence on M, where the character with $\check{}$ means that it is deleted. Now, by slight modification in the arguments of [9, 2.2], one can show that $\frac{y_1}{1}, \ldots, \frac{\check{y}_l}{1}, \ldots, \frac{y_i}{1}, \frac{y_l}{1} \in \mathfrak{p}A_{\mathfrak{p}}$ is an $M_{\mathfrak{p}}$-sequence. Hence, by [8, p. 127], $\frac{y_1}{1}, \ldots, \frac{y_l}{1}, \ldots, \frac{y_i}{1}$ forms an $M_{\mathfrak{p}}$-sequence too. Therefore $\mathfrak{p}A_{\mathfrak{p}} \notin \mathrm{Ass}(M_{\mathfrak{p}}/\sum_{j=1}^{i-1} y_j M_{\mathfrak{p}})$, which is impossible by the choice of $\mathfrak{p}$.

2.2. Lemma. *Let $x_1, \ldots, x_s$ be an u.s.d-sequence on M. Then, for all $\alpha \in \mathbf{N}$,*

$$0 :_{G_{\mathfrak{q}}(M)} h_1^{\alpha} = (0 :_M x_1)(0).$$

Proof. Let $g \in 0 :_{G(M)} h_1^{\alpha}$ be a homogeneous element of degree $n(\ge 0)$. Choose an element y of $\mathfrak{q}^n M$ such that $g = y \bmod \mathfrak{q}^{n+1}M$ in $[G(M)]_n$. Then $x_1^{\alpha} y \in x_1^{\alpha} M \cap \mathfrak{q}^{n+\alpha+1}M$. Hence, by 1.2(ii), $x_1^{\alpha} y \in x_1^{\alpha}\mathfrak{q}^{n+1}M$. Therefore, it is the case that $y = u \bmod \mathfrak{q}M$ for some $u \in 0 :_M x_1^{\alpha}$ if $n = 0$, but $y \in \mathfrak{q}^{n+1}M$ if $n > 0$. Hence the inclusion $\subseteq$ holds. As the opposite inclusion is trivially true, the result follows.

Next, we show that the result [3, 2.10] of Goto and Yamagishi is quickely derived from our criteria 1.3 for u.s.d-sequences.

2.3. Theorem. *Let $x_1, \ldots, x_s$ be an u.s.d-sequence on M. Then $h_1, \ldots, h_s$ form an u.s.d-sequence on $G_{\mathfrak{q}}(M)$.*

Proof. Let $I = \sum_{i=1}^{s} h_i G$. It follows from 1.2 (ii) (iii) (iv) that every permutation of $h_1, \ldots, h_s$ is a d-sequence on $G(M)$. Hence, in particular,

$$0 :_{G(M)} h_i = 0 :_{G(M)} I \tag{1}$$

for all $1 \le i \le s$. Let $h = h_1, \ldots, h_s$. In order to prove the result, it suffices, in view of 1.3, to show that $\overline{\Psi}^p_{h, G(M)}$ is surjective for all integer p with $0 \le p \le s-1$. We prove this by induction on p. By (1) it is clear that the canonical homomorphism $\overline{\Psi}^0_{h, G(M)} : H_s(I, G(M)) \longrightarrow \dfrac{\ker e^0_{h, G(M)}}{\mathrm{im}\, e^{-1}_{h, G(M)}}$ is surjective. Let p be an integer with $1 \le p \le s-1$ and suppose that the result has been proved for $p-1$. Set $\tilde{G} := G(M)/(0 :_{G(M)} h_s)$. In view of (1), it is easy to see that $U(h)_p^{-p}(0 :_{G(M)} h_s) = 0$ for all $p \ge 1$. Therefore the exact sequence

$$0 \longrightarrow (0 :_{G(M)} h_s) \longrightarrow G(M) \longrightarrow \tilde{G} \longrightarrow 0$$

yields an exact complex similar to $(*)$ which in turn implies that $\dfrac{\ker e^{p}_{h,G(M)}}{\operatorname{im} e^{p-1}_{h,G(M)}} \cong \dfrac{\ker e^{p}_{h,\tilde{G}}}{\operatorname{im} e^{p-1}_{h,\tilde{G}}}$ for all $p \geq 1$. On the other hand, it follows from (1) that, the Koszul homology module $H_{s-p}(h, 0 :_{G(M)} h_s)$ is a direct sum of copies of $0 :_{G(M)} h_s$ for all $p = 1, \ldots, s$. Now, using the elementrary fact on the Koszul complex together with 1.2(i), we may deduce that the map $H_{s-p}(h, 0 :_{G(M)} h_s) \longrightarrow H_{s-p}(h, G(M))$ is injective for all $p = 1, \ldots, s$. Therefore, for all $p = 1, \ldots, s-1$, we obtain the commutative diagram

$$\begin{array}{ccccc}
H_{s-p}(h, G(M)) & \longrightarrow & H_{s-p}(h, \tilde{G}) & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \overline{\Psi}^{p}_{h,G(M)}} & & \Big\downarrow{\scriptstyle \overline{\Psi}^{p}_{h,\tilde{G}}} & & \\
\dfrac{\ker e^{p}_{h,G(M)}}{\operatorname{im} e^{p-1}_{h,G(M)}} & \longrightarrow & \dfrac{\ker e^{p}_{h,\tilde{G}}}{\operatorname{im} e^{p-1}_{h,\tilde{G}}} & &
\end{array}$$

in which the upper row is exact and the lower row is the natural isomorphism. Hence we may assume, without loss of generality, that h_s is a non-zero-divisor on $G(M)$. Put $A' = A/x_s^2 A$, $M' = M/x_s^2 M$, $\mathfrak{q}' = \mathfrak{q}A'$ and $G(M') = G_{\mathfrak{q}'}(M')$. Then, using the exact sequence

$$0 \longrightarrow G(M)(-2) \xrightarrow{h_s^2} G(M) \longrightarrow G(M') \longrightarrow 0,$$

we obtain, for all integer p, the commutative diagram

$$\begin{array}{ccccc}
H_{s-(p-1)}(h, G(M')) & \longrightarrow & H_{s-p}(h, G(M)(-2)) & \xrightarrow{h_s^2} & H_{s-p}(h, G(M)) \\
\Big\downarrow{\scriptstyle \overline{\Psi}^{p-1}_{h,G(M')}} & & \Big\downarrow{\scriptstyle \overline{\Psi}^{p}_{h,G(M)(-2)}} & & \Big\downarrow{\scriptstyle \overline{\Psi}^{p}_{h,G(M)}} \\
\dfrac{\ker e^{p-1}_{h,G(M')}}{\operatorname{im} e^{p-2}_{h,G(M')}} & \longrightarrow & \dfrac{\ker e^{p}_{h,G(M)(-2)}}{\operatorname{im} e^{p-1}_{h,G(M)(-2)}} & \xrightarrow{h_s^2} & \dfrac{\ker e^{p}_{h,G(M)}}{\operatorname{im} e^{p-1}_{h,G(M)}}
\end{array}$$

in which the rows are exact and, by inductive hypothesis, the map $\overline{\Psi}^{p-1}_{h,G(M')}$ is surjective. Therefore in order to complete the inductive step it is enough to show that $h_s^2 \left(\dfrac{\ker e^{p}_{h,G(M)(-2)}}{\operatorname{im} e^{p-1}_{h,G(M)(-2)}} \right) = 0$. Now, let $Y \in \dfrac{\ker e^{p}_{h,G(M)(-2)}}{\operatorname{im} e^{p-1}_{h,G(M)(-2)}} = 0$. Then, by employing a method of proof which is similar to that used in [14, 2.3(ii)], there exists $t \in \mathbf{N}$ such that $h_s^t Y = 0$. If $t \geq 2$, then, using the above diagram, there exists $Z \in H_{s-p}(h, G(M)(-2))$ such that $\overline{\Psi}^{p}_{h,G(M)(-2)}(Z) = h_s^{t-2} Y$; which implies that $h_s^{t-1} Y = 0$, since $h_s Z = 0$. Now, one can repeat the same arguments to achieve that $h_s^2 Y = 0$ as required.

By the example (1) of [3, 1.12] we know that x_i's do not necessarily form an u.s.d-sequence on M even though the h_i's form an u.s.d-sequence on $G(M)$. In the following theorem we discuss about this fact.

2.4. Theorem. *Suppose that A is Noetherian and that M is finitely generated. Let $x_1, \ldots, x_s$ be an unconditioned $\mathfrak{q}$-filter regular sequence on M such that $h_1^t, \ldots, h_s^t$ forms an u.s.d-sequence on $G_{\mathfrak{q}}(M)$ for some $t \in \mathbf{N}$. Then $x_1^l, \ldots, x_s^l$ forms an u.s.d-sequence on M for $l = st - s + 1$.*

Proof. Let $l = st - s + 1$ and let $x^l = x_1^l, \ldots, x_s^l$. In view of 1.3, we have to show that $\overline{\Psi}_{x^l, M}^p$ is surjective for all integer p with $0 \leq p \leq s - 1$. To do this, first we claim that

$$(0 :_M x_i^l) \cap \left(\sum_{j=1}^{s} A x_j^l \right) M = 0 \qquad \text{for every } 1 \leq i \leq s. \tag{2}$$

Let $r \in (0 :_M x_i^l) \cap (\sum_{j=1}^s A x_j^l) M$ for some i with $1 \leq i \leq s$. Let g be a homogenous element of degree l of $G_{\mathfrak{q}}(M)$ such that $g = r \bmod \mathfrak{q}^{l+1} M$ in $[G_{\mathfrak{q}}(M)]_l$. So $g \in 0 :_{G_{\mathfrak{q}}(M)} h_i^l$. As h_i^t is a d-sequence on $G_{\mathfrak{q}}(M)$ and $l \geq t$ we have that $g \in 0 :_{G_{\mathfrak{q}}(M)} h_i^t$. Also it is easy to see that $g \in (\sum_{j=1}^s G_{\mathfrak{q}}(A) h_j^t) G_q(M)$. Hence, by 1.2 (i), $g = 0$; i.e. $r \in \mathfrak{q}^{l+1} M$. Now, one can repeat the same arguments to achieve that $r \in \mathfrak{q}^{\beta} M$ for all $\beta \geq l$. On the other hand, by 1.1 (i), there exist $n_1, \ldots, n_s \in \mathbf{N}$ such that $x_1^{n_1}, \ldots, x_s^{n_s}$ is a d-sequence on M. Therefore $r \in (\sum_{j=1}^s A x_j^{n_j}) M$; hence, by 1.2 (i), we have $r = 0$ and the claim follows.

Now, let $1 \leq i \leq s$ and let $r \in 0 :_M x_i^{\alpha l}$ for some integer α with $\alpha \geq 2$. Then, by (2), $r \in 0 :_M x_i^l$. Therefore $0 :_M x_i^{\alpha l} = 0 :_M x_i^l$. Hence, using the assumption that $x_1, \ldots, x_s$ form an unconditioned $\mathfrak{q}$-filter regular sequence on M, we have

$$0 :_M \left(\sum_{j=1}^{s} A x_j^l \right) = 0 :_M x_i^{\alpha l} \qquad \text{for all } i = 1, \ldots, s \qquad \text{and } \alpha \in \mathbf{N}. \tag{3}$$

Thus the canonical homomorphism

$$\overline{\Psi}_{x^l, M}^0 : H_s(x^l, M) \longrightarrow \frac{\ker e_{x^l, M}^0}{\operatorname{im} e_{x^l, M}^{-1}}$$

is surjective. Next, consider the exact sequence

$$0 \longrightarrow (0 :_M x_s^l) \longrightarrow M \longrightarrow (M/(0 :_M x_s^l)) \longrightarrow 0$$

to deduce the long exact sequence

$$\begin{aligned} \cdots \longrightarrow H_p(x^l, 0 :_M x_s^l) &\longrightarrow H_p(x^l, M) \\ &\longrightarrow H_p(x^l, M/(0 :_M x_s^l)) \longrightarrow H_{p-1}(x^l, 0 :_M x_s^l) \longrightarrow \cdots. \end{aligned}$$

It follows, in view of (3), that $H_p(x^l, 0 :_M x_s^l)$ is a direct sum of some copies of $0 :_M x_s^l$ for all $p = 0, 1, \ldots, s$. Therefore, using (2), it is easy to see that the map

$$H_p(x^l, 0 :_M x_s^l) \longrightarrow H_p(x^l, M)$$

is injective. So, for all $1 \leq p \leq s-1$, we have the commutative diagram

$$\begin{array}{ccccc} H_{s-p}(x^l, M) & \longrightarrow & H_{s-p}(x^l, M/(0:_M x_s^l)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \dfrac{\ker e^p_{x,M}}{\operatorname{im} e^{p-1}_{x,M}} & \xrightarrow{\cong} & \dfrac{\ker e^p_{x,M/(0:_M x_s^l)}}{\operatorname{im} e^{p-1}_{x,M/(0:_M x_s^l)}} & & \end{array}$$

in which the upper row is exact and the lower row is the natural isomorphism. Therefore we may assume, without loss of generality, that x_s is non-zero-divisor on M. Now, by the same arguments as in the proof of 2.3 we can complete the proof.

As we mentioned in the introduction, Theorem A is an immediate consequence of 2.3 and 2.4. Let us now indicate how the result [3, 2.12] of Goto and Yamagishi can be deduced from Theorems 2.3 and 2.4.

2.5. Consequence. Consider the special case in which A is Noetherian, M is finitely generated and $x_1, \dots, x_s$ is contained in the Jacobson radical of A. Then, using 1.2 (iii), it is straightforward to see that $x_1, \dots, x_s$ forms an unconditioned $\mathfrak{q}$-filter regular sequence on M if $h_1, \dots, h_s$ is an u.s.d-sequence on $G_{\mathfrak{q}}(M)$. Hence, in view of 2.3 and 2.4, the following conditions are equivalent:

(i) $x_1, \dots, x_s$ is an u.s.d-sequence on M;
(ii) $h_1, \dots, h_s$ is an u.s.d-sequence on $G_{\mathfrak{q}}(M)$.

2.6. Remark. Suppose that A is Noetherian and M is finitely generated. Then the existance of u.s.d-sequence on M in $\mathfrak{a}$ are closely related to the finiteness properties of $H^i_{\mathfrak{a}}(M)$. In fact if $x_1, \dots, x_n$ is an u.s.d-sequence on M, then, in view of [14, 2.4], [7, 2.4] and [2, Lemma 3], it is easy to see that $H^i_{(x_1,\dots,x_n)}(M)$ is finite for all $0 \leq i \leq n-1$. If, in addition, $x_1, \dots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M then, by [7, 1.3(ii)], $H^i_{\mathfrak{a}}(M)$ is finite for all $0 \leq i \leq n-1$.

Proof of Theorem B. (i) $\Rightarrow$ (ii) By [6, Theorem], there exists $k \in \mathbf{N}$ such that every $\mathfrak{a}$-filter regular sequence on M of length s is an $\mathfrak{a}^k$-weak M-sequence. Now suppose that $x_1, \dots, x_s$ is an unconditioned $\mathfrak{a}$-filter regular sequence on M in $\mathfrak{a}^k$. (Note that the existence of such a sequence is guaranteed by 2.1.) Then $x_1, \dots, x_s$ is an u.s.d-sequence on M. Hence, by 2.3, $h_1, \dots, h_s$ is an u.s.d-sequence on $G(M)$. Thus, for $0 \leq i \leq s-1$, $H^i_I(G(M))$ is finitely generated, as required.

(ii) $\Rightarrow$ (i) First of all, using [6, Theorem], we may deduce that $h_1^{\alpha}, \dots, h_s^{\alpha}$ is an u.s.d-sequence on $G(M)$ for some $\alpha \in \mathbf{N}$. Hence, by 2.4, $x_1^{\beta}, \dots, x_s^{\beta}$ is an u.s.d-sequence on M for some $\beta \in \mathbf{N}$. Moreover, by our assumption, $x_1, \dots, x_s$ is an $\mathfrak{a}$-filter regular sequence on M. Therefore, by 2.6, $H^j_{\mathfrak{a}}(M)$ is finitely generated for all $0 \leq j \leq s-1$.

2.7. Corollary. [10, 4.2]. *Suppose that A is local with maximal ideal $\mathfrak{m}$ and that M is finitely generated of dimension s (>0). Then the following conditions are equivalent:*

(i) *$H^i_{\mathfrak{m}}(M)$ is finitely generated for all $0 \le i \le s-1$;*

(ii) *There is a system of parameters $x_1, \ldots, x_s$ for M such that $H^i_{\mathfrak{m}^*}(G_{\mathfrak{q}}(M))$ is finitely generated for $0 \le i \le s-1$, where $\mathfrak{q} = \sum_{i=1}^n Ax_i$ and $\mathfrak{m}^*$ is the unique graded maximal ideal of $G_{\mathfrak{q}}(A)$.*

Proof. In view of Theorem B ((i) $\Rightarrow$ (ii)) it is enough to prove the implication (ii) $\Rightarrow$ (i). To do this, note that, by the assumption, $h_1, \ldots, h_s$ is a system of parameters for $G_{\mathfrak{q}}(M)$ and that, since $H^i_{\mathfrak{m}^*}(G_{\mathfrak{q}}(M))$ is finitely generated for all $i = 0, 1, \ldots, s-1$, there exists $t \in \mathbf{N}$ such that $h_1^t, \ldots, h_s^t$ is an u.s.d-sequence on $G_{\mathfrak{q}}(M)$. Let $y_1, \ldots, y_s$ be any permutation of $x_1, \ldots, x_s$. Then, by applying 1.2, it is easy to check that the equality

$$\left[\sum_{j=1}^{i-1} y_j^t \mathfrak{q}^n M + \mathfrak{q}^{n+t+1} M :_M y_i^t\right] \cap \mathfrak{q}^n M = \sum_{j=1}^{i-1} y_j^t \mathfrak{q}^{n-t} M + \mathfrak{q}^{n+1} M \tag{4}$$

holds for all $1 \le i \le s$ and $n \ge st - s - 1$. Since the sequence $x_1, \ldots, x_s$ is contained in the Jacobson radical of A, we can deduce from (4) that $x_1, \ldots, x_s$ is an unconditioned $\mathfrak{q}$-filter regular sequence on M. Now the assertion follows from the implication (ii) $\Rightarrow$ (i) of Theorem B.

The following theorem clarify the structure of the homology modules of the complex $C(\mathscr{A}(h), G(M))$ of $G(A)$-modules which involves modules of generalized fractions derived from $G(M)$ and the u.s.d-sequence $h := h_1, \ldots, h_s$ on $G(M)$. It follows from this theorem in conjunction with [14, 2.4] that if A is Noetherian, then i-th local cohomology module $H^i_{\mathfrak{q}}(M)(i)$ and $H^i_Q(G(M))$, where $Q = \sum_{i=1}^d h_i G$, are isomorphic. Thus, under Noetherian hypothesis on A, the next theorem provide an alternative proof of [3, 4.2].

2.8. Theorem. *Let $x_1, \ldots, x_s$ be an u.s.d-sequence on M. Then*

$$\frac{\ker e^i_{h, G(M)}}{\operatorname{im} e^{i-1}_{h, G(M)}} \cong \frac{\ker e^i_{x, M}}{\operatorname{im} e^{i-1}_{x, M}}(i)$$

for all $i = 0, 1, \ldots, s-1$.

Proof. We prove this by induction on s. If $s = 1$, by 2.2, we have noting to do any more. So, suppose, inductively, that $s > 1$ and that the result has been proved for smaller values of s. In order to prove the assertion for s we use induction on i. By 2.2, it is trivial in case $i = 0$, i.e. $\dfrac{\ker e^0_{h, G(M)}}{\operatorname{im} e^{-1}_{h, G(M)}} \cong \dfrac{\ker e^0_{x, M}}{\operatorname{im} e^{-1}_{x, M}}(0)$. Now, suppose that $1 \le i \le s-1$ and that the result holds for smaller values of i. Put $\overline{M} = M/(0 :_M x_s)$ and $\bar{G} = G_{\mathfrak{q}}(\overline{M})$. Consider the exact sequences

$$0 \longrightarrow (0 :_M x_s) \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0$$

and

$$0 \longrightarrow (0 :_{G(M)} h_s) \longrightarrow G(M) \longrightarrow \bar{G} \longrightarrow 0$$

and apply 2.3 to obtain

$$\frac{\ker e^{i}_{x,M}}{\operatorname{im} e^{i-1}_{x,M}} \cong \frac{\ker e^{i}_{x,\bar{M}}}{\operatorname{im} e^{i-1}_{x,\bar{M}}} \quad \text{and} \quad \frac{\ker e^{i}_{h,G(M)}}{\operatorname{im} e^{i-1}_{h,G(M)}} \cong \frac{\ker e^{i}_{h,\bar{G}}}{\operatorname{im} e^{i-1}_{h,\bar{G}}}$$

for all $i = 0, 1, \ldots, s-1$. Thus, without loss of generality, we may assume that x_s (respectively h_s) is a non-zero-divisor on M (respectively $G(M)$). Let $A' = A/x_sA$, $\mathfrak{q}' = \mathfrak{q}A'$, $M' = M/x_sM$ and $G(M') = G_{\mathfrak{q}'}(M')$. Consider the exact sequences

$$0 \longrightarrow G(M)(-1) \xrightarrow{h_s} G(M) \longrightarrow G(M') \longrightarrow 0 \tag{5}$$

and

$$0 \longrightarrow M \xrightarrow{x_s} M \longrightarrow M' \longrightarrow 0. \tag{6}$$

Since, by 2.3, $h_1, \ldots, h_s$ is an u.s.d-sequence on $G(M)$, we have that $h_s \dfrac{\ker e^{i}_{h,G(M)(-1)}}{\operatorname{im} e^{i-1}_{h,G(M)(-1)}} = 0$ for all $i = 0, 1, \ldots, s-1$. Now, from (5), we obtain the induced exact sequence

$$0 \longrightarrow \frac{\ker e^{i-1}_{h,G(M)}}{\operatorname{im} e^{i-2}_{h,G(M)}} \longrightarrow \frac{\ker e^{i-1}_{h,G(M')}}{\operatorname{im} e^{i-2}_{h,G(M')}} \longrightarrow \frac{\ker e^{i}_{h,G(M)(-1)}}{\operatorname{im} e^{i-1}_{h,G(M)(-1)}} \longrightarrow 0$$

which in turn yields, by applying inductive hypothesis on the module $G(M')$, $\left[\dfrac{\ker e^{i}_{h,G(M)(-1)}}{\operatorname{im} e^{i-1}_{h,G(M)(-1)}}\right]_n = 0$ for all $n \neq -i+1$. Similarly, from (6), we obtain the exact sequence

$$0 \longrightarrow \frac{\ker e^{i-1}_{x,M}}{\operatorname{im} e^{i-2}_{x,M}} \longrightarrow \frac{\ker e^{i-1}_{x,M'}}{\operatorname{im} e^{i-2}_{x,M'}} \longrightarrow \frac{\ker e^{i}_{x,M}}{\operatorname{im} e^{i-1}_{x,M}} \longrightarrow 0.$$

Now, using inductive hypothesis, we may obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & \left[\dfrac{\ker e^{i-1}_{h,G(M)}}{\operatorname{im} e^{i-2}_{h,G(M)}}\right]_{-i+1} \arrow[r] & \left[\dfrac{\ker e^{i-1}_{h,G(M')}}{\operatorname{im} e^{i-2}_{h,G(M')}}\right]_{-i+1} \arrow[r] & \left[\dfrac{\ker e^{i}_{h,G(M)(-1)}}{\operatorname{im} e^{i-1}_{h,G(M)(-1)}}\right]_{-i+1} \arrow[r] & 0 \\
0 \arrow[r] & \dfrac{\ker e^{i-1}_{x,M}}{\operatorname{im} e^{i-2}_{x,M}} \arrow[r] \arrow[u, "\varphi"] & \dfrac{\ker e^{i-1}_{x,M'}}{\operatorname{im} e^{i-2}_{x,M'}} \arrow[r] \arrow[u, "\varphi'"] & \dfrac{\ker e^{i}_{x,M}}{\operatorname{im} e^{i-1}_{x,M}} \arrow[r] & 0
\end{tikzcd}$$

with exact rows in which φ and φ' are isomorphisms. Moreover the diagram is commutative because the injections are naturally induced by $M \to M'$. We are therefore able to complete the inductive step; and the result follows by induction.

Note that, although the proof of the above theorem relies on the ideas of Schenzel's proof of [10, 4.1], but his theorem is a particular case of ours.

Acknowledgment. The authors would like to thank the Institute for Studies in Theoretical Physics and Mathematics (IPM) for the Financial Support.

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