J. Math. Kyoto Univ. (JMKYAZ)
43-3 (2003), 609–639

# Semi-Lévy processes, semi-selfsimilar additive processes, and semi-stationary Ornstein-Uhlenbeck type processes

By

Makoto MAEJIMA and Ken-iti SATO

## 1. Introduction

The works of Wolfe [27], Jurek and Vervaat [6], Sato and Yamazato [20], [21], Sato [16], and Jeanblanc, Pitman, and Yor [4] combined show that the following three classes have one-one correspondence with each other — the class of selfsimilar additive processes, the class of stationary Ornstein-Uhlenbeck type processes, and the class of homogeneous independently scattered random measures (Lévy processes) with finite log-moment. The correspondence is given by Lamperti transformations and stochastic integrals. This correspondence gives representations of the class of selfdecomposable distributions. The aim of this paper is to give extensions of this correspondence to certain wider classes and to discuss Ornstein-Uhlenbeck type processes in a wide sense.

There are two significant classes that extend the class of stable distributions — the class of selfdecomposable distributions and the class of semi-stable distributions. The class of semi-selfdecomposable distributions is a natural extension of these two classes (see [9]). Their description in terms of Lévy measures is given in [17]. Thus we are motivated to generalize the representations of the class of selfdecomposable distributions to those of the class of semi-selfdecomposable distributions. In the case of distributions on $\mathbb{R}^d$ with $d \geqslant 2$, we will simultaneously deal with another sort of generalization. This is related to $Q$-stable and $Q$-selfdecomposable distributions (see [21]), $Q$-selfsimilar additive processes (see [16]), and Ornstein-Uhlenbeck type processes with drift $-Qx$ (see [19], [21], [26]), where $Q$ is a $d \times d$ matrix in $\mathbf{M}_d^+$ defined below.

Before going to statement of main results, let us give some definitions.

Let $\mathbf{M}_d$ be the class of $d \times d$ real matrices and $\mathbf{M}_d^+$ the class of $Q \in \mathbf{M}_d$ all of whose eigenvalues have positive real parts. Let $I$ be the identity matrix and $a^Q = \sum_{n=0}^{\infty}(n!)^{-1}(\log a)^n Q^n \in \mathbf{M}_d$ for $a > 0$ and $Q \in \mathbf{M}_d$. Sometimes we also use the class $\mathbf{M}_{l\times d}$ of $l \times d$ real matrices. Denote the transpose of $F \in \mathbf{M}_{l\times d}$ by $F'$. Let $\mathcal{L}(X)$ be the distribution of a random element $X$. When $\mathcal{L}(X) = \mathcal{L}(Y)$ for two random elements $X$ and $Y$, we write $X \overset{\mathrm{d}}{=} Y$. For

Received October 11, 2002

two stochastic processes $X = \{X_t\}$ and $Y = \{Y_t\}$, $X \stackrel{\mathrm{d}}{=} Y$ or $\{X_t\} \stackrel{\mathrm{d}}{=} \{Y_t\}$ means that they have an identical distribution as infinite-dimensional random elements, that is, have an identical system of finite-dimensional distributions, while $X_t \stackrel{\mathrm{d}}{=} Y_t$ means that $X_t$ and $Y_t$ are identically distributed for a fixed $t$. The characteristic function of a distribution $\mu$ on $\mathbb{R}^d$ is denoted by $\widehat{\mu}(z)$, $z \in \mathbb{R}^d$. For an interval $J$, $\mathcal{B}_J$ is the class of Borel sets in $J$ and $\mathcal{B}_J^0$ is the class of Borel sets whose closures in the relative topology on $J$ are compact.

A process $X = \{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$ continuous in probability with independent increments, with cadlag paths a. s., and with $X_0 = 0$ a. s. is called an *additive process* (see [17]). It is called a *Lévy process* if, in addition, $X_{t+u} - X_{s+u} \stackrel{\mathrm{d}}{=} X_t - X_s$ for all nonnegative $t$, $s$, $u$. We call an additive process satisfying the condition that $X_{t+p} - X_{s+p} \stackrel{\mathrm{d}}{=} X_t - X_s$ with a fixed $p > 0$ a *semi-Lévy process* with *period* $p$. An additive process is said to have finite log-moment if $E \log^+ |X_t| < \infty$ for all $t$. Here $\log^+ a = 0 \vee \log a$ for $0 \leqslant a < \infty$. An additive process is said to be *natural* if the location parameter $\gamma_t$ in the generating triplet $(A_t, \nu_t, \gamma_t)$ is locally of bounded variation in $t$ (see [18]). An additive process is natural if and only if it is a semimartingale. All Lévy processes are natural.

Let $Q \in \mathbf{M}_d^+$. A process $X = \{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$ is called *$Q$-selfsimilar* if $\{X_{at}\} \stackrel{\mathrm{d}}{=} \{a^Q X_t\}$ for all $a > 0$. Note that the value of $X_t$ (an element of $\mathbb{R}^d$) is always considered as a column vector. If the assumption that $\{X_{at}\} \stackrel{\mathrm{d}}{=} \{a^Q X_t\}$ is made only for a fixed $a > 1$, the process is called *$Q$-semi-selfsimilar* with *epoch* $a$. Especially $cI$-selfsimilar and $cI$-semi-selfsimilar processes with $c > 0$ are called $c$-selfsimilar (see [15], [17]) and $c$-semi-selfsimilar (see [10], [17]), respectively. In this case, $H$ is usually used instead of $c$.

Let $Q \in \mathbf{M}_d^+$. A distribution $\mu$ on $\mathbb{R}^d$ satisfying, for every $b \in (0, 1)$,

$$\widehat{\mu}(z) = \widehat{\mu}(b^{Q'} z)\widehat{\rho}_b(z) \tag{1.1}$$

with some (automatically infinitely divisible) distribution $\rho_b$ is called *$Q$-selfdecomposable.* Thus, for any $c > 0$, the $Q$-selfdecomposability and the $cQ$-selfdecomposability are equivalent. Following [11], we introduce, with $b \in (0, 1)$ fixed, the class $L_0(b, Q)$ of distributions $\mu$ on $\mathbb{R}^d$ satisfying (1.1) with some infinitely divisible distributions $\rho_b$. Distributions in $L_0(b, Q)$ are called *$(b, Q)$-decomposable.* Distributions $(b, Q)$-decomposable with some $b$ are called $Q$-semi-selfdecomposable. All $Q$-selfdecomposable and all $Q$-semi-selfdecomposable distributions are infinitely divisible. Usually $I$-selfdecomposable distributions are called *selfdecomposable* and $I$-semi-selfdecomposable distributions are called *semi-selfdecomposable* (see [9], [17]).

We use the notion of $\mathbb{R}^d$-valued independently scattered random measure (i. s. r. m.) $M = \{M(B) : B \in \mathcal{B}_J^0\}$ over an interval $J$ studied in the case $d = 1$ by Urbanik and Woyczynski [25] and Rajput and Rosinski [14]. Precise definition of this notion will be given in Section 3. For a class of $\mathbf{M}_{l \times d}$-valued functions $F(s)$ including all locally bounded measurable functions, we can define $\int_B F(s) M(ds)$ for $B \in \mathcal{B}_J^0$. A natural additive process $X$ on $\mathbb{R}^d$ induces a unique $\mathbb{R}^d$-valued i. s. r. m. $M$ over $[0,\infty)$ satisfying $M((s,t]) = X_t - X_s$ a. s. for $0 \leqslant s < t < \infty$. Any $\mathbb{R}^d$-valued i. s. r. m. over $[0,\infty)$ is obtained in this way. In this case $\int_B F(s)M(ds)$ is written also as $\int_B F(s)dX_s$. When $J$ is an interval infinite to the left, we define $\int_{-\infty}^t F(u)M(du)$ for $t \in J$ to be the limit in probability of $\int_{(s,t]} F(u)M(du)$ as $s \downarrow -\infty$ whenever this limit exists.

Given an $\mathbb{R}^d$-valued nonrandom cadlag function $y_s$ of $s \in \mathbb{R}$ and a matrix $Q \in \mathbf{M}_d$, consider the equation

$$z_{s_2} - z_{s_1} = y_{s_2} - y_{s_1} - Q\int_{s_1}^{s_2} z_u du \qquad \text{for} \quad s_1 < s_2 \tag{1.2}$$

for a nonrandom cadlag function $z_s$ of $s \in \mathbb{R}$. When the condition $z_{s_0} = \xi$ is imposed, (1.2) has a unique solution. When $\Lambda = \{\Lambda(B) \colon B \in \mathcal{B}_{\mathbb{R}}^0\}$ is an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$, we call the equation

$$Z_{s_2} - Z_{s_1} = \Lambda((s_1,s_2]) - Q\int_{s_1}^{s_2} Z_u du \tag{1.3}$$

*Langevin equation* based on $\Lambda$ and $Q$. A cadlag process $Z = \{Z_s \colon s \in \mathbb{R}\}$ which satisfies (1.3) a. s. for every $s_1, s_2$ with $s_1 < s_2$ is called a solution of (1.3) or an *Ornstein-Uhlenbeck type* (OU type) process generated by $\Lambda$ and $Q$. When the condition $Z_{s_0} = \Xi$ is imposed, its solution is unique a. s. If we introduce a cadlag process $Y = \{Y_s \colon s \in \mathbb{R}\}$ such that $Y_{s_2} - Y_{s_1} = \Lambda((s_1,s_2])$, then (1.3) is a random version of (1.2), and any solution of (1.3) is called an OU type process generated by $Y$ and $Q$. A process $Z = \{Z_s\}$ satisfying $\{Z_{s+u}\} \overset{\mathrm{d}}{=} \{Z_s\}$ for all $u$ is called *stationary*. A process $Z$ satisfying $\{Z_{s+p}\} \overset{\mathrm{d}}{=} \{Z_s\}$ for a fixed $p > 0$ is called *semi-stationary* (or periodically stationary) with *period* $p$. We say that $\Lambda$ has finite log-moment if $E \log^+ |\Lambda(B)| < \infty$ for all $B \in \mathcal{B}_{\mathbb{R}}^0$.

The following three theorems are our main results.

**Theorem 1.1.** *Let $Q \in \mathbf{M}_d^+$, $a > 1$, and $p = \log a$. Let $X = \{X_t \colon t \geqslant 0\}$ be an arbitrary $Q$-semi-selfsimilar natural additive process on $\mathbb{R}^d$ with epoch $a$. Define*

$$Z_s = e^{-sQ} X_{e^s} \qquad \textit{for} \quad s \in \mathbb{R} \tag{1.4}$$

*and*

$$\Lambda(B) = \int_{\exp B} t^{-Q} dX_t \qquad \textit{for} \quad B \in \mathcal{B}_{\mathbb{R}}^0\,, \tag{1.5}$$

*where $\exp B = \{t = e^s \colon s \in B\}$. Then $\Lambda = \{\Lambda(B) \colon B \in \mathcal{B}_{\mathbb{R}}^0\}$ is an $\mathbb{R}^d$-valued i. s. r. m. periodic with period $p$ and having finite log-moment. The process $X$ is expressed by $\Lambda$ as*

$$X_t = \int_{-\infty}^{\log t} e^{sQ} \Lambda(ds) \qquad \textit{for} \quad t > 0, \quad \textit{a. s.} \tag{1.6}$$

*The process $Z = \{Z_s\colon s \in \mathbb{R}\}$ is the unique semi-stationary OU type process with period p generated by $\Lambda$ and $Q$. It is expressible as*

$$Z_s = e^{-sQ} \int_{-\infty}^{s} e^{uQ} \Lambda(du) \qquad \text{for} \quad s \in \mathbb{R}, \quad \text{a. s.} \tag{1.7}$$

**Theorem 1.2.** *Let $Q \in \mathbf{M}_d^+$, $p > 0$, and $a = e^p$. Let $\Lambda = \{\Lambda(B)\colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ be an arbitrary $\mathbb{R}^d$-valued i. s. r. m. periodic with period p and having finite log-moment. Then $Z = \{Z_s\colon s \in \mathbb{R}\}$, a semi-stationary OU type process with period p generated by $\Lambda$ and $Q$, exists and is unique. Define*

$$\begin{cases} X_t = t^Q Z_{\log t} & \text{for} \quad t > 0, \\ X_0 = 0. \end{cases} \tag{1.8}$$

*Then $X = \{X_t\colon t \geqslant 0\}$ is a $Q$-semi-selfsimilar natural additive process on $\mathbb{R}^d$ with epoch $a$; $Z$ and $\Lambda$ are recovered from $X$ in the form of* (1.4) *and* (1.5).

**Theorem 1.3.** *Let $Q \in \mathbf{M}_d^+$ and $a > 1$. A distribution $\mu$ on $\mathbb{R}^d$ is expressible as $\mu = \mathcal{L}(X_1) = \mathcal{L}(Z_0)$ by the processes $X$ and $Z$ in Theorem* 1.1 *or* 1.2 *if and only if it is $(a^{-1}, Q)$-decomposable.*

The associated filtrations of the processes and the random measure in Theorem 1.1 or 1.2 satisfy the following:

$$\sigma(X_t\colon t \in [0, e^s]) = \sigma(Z_u\colon u \in (-\infty, s]) = \sigma(\Lambda(B)\colon B \in \mathcal{B}^0_{(-\infty,s]})\,.$$

Relations (1.4) and (1.8) between $X$ and $Z$ are generalization of the Lamperti transformation between selfsimilar processes and stationary processes introduced by Lamperti [7]. In the case of symmetric stable processes on $\mathbb{R}$, this transformation was already recognized by Doob [3, p. 368]. Between semi-selfsimilar and semi-stationary processes it was given in [10].

By Theorems 1.1–1.3, semi-selfdecomposable and $(b, Q)$-decomposable distributions have now been connected with the three classes — the class of $X$, the class of $Z$, and the class of $\Lambda$. Semi-selfdecomposable distributions are expected to have wide flexibility in modeling such as in [1].

Organization of this paper is as follows. Section 2 gives basic facts on semi-Lévy processes. Some results on i. s. r. m., stochastic integrals, and factorings are summarized in Section 3. We study in Section 4 solutions of Langevin equations on $\mathbb{R}^d$ based on $\mathbb{R}^d$-valued i. s. r. m. and matrices $Q$. The notion of $Q$-mildness at $-\infty$ or, shortly, $Q$-mildness is introduced for solutions of Langevin equations, and the existence condition for $Q$-mild solutions is given. Stationary and semi-stationary solutions are $Q$-mild. The existence condition is more analyzed in the case of periodic i. s. r. m. Using these results, we give in Section 5 proofs of Theorems 1.1 through 1.3. Formulation of results in the $Q$-selfsimilar case is given in Section 6 as consequences of Theorems 1.1–1.3. Applying the main results, we characterize factorings of $Q$-selfsimilar and $Q$-semi-selfsimilar additive processes in Section 7. It is also shown that Langevin equation based on i. s. r. m. $N$ and $R \in \mathbf{M}_d^+$ has a unique $R$-mild solution provided that $N$ has $Q$-semi-selfsimilarity on $(-\infty, 0]$. Finally Section 8 contains some results related to $(b, Q, a)$-semi-stable distributions and $(b, Q, a)$-semi-stable Lévy processes and some examples appearing in the study of diffusion processes in random environments.

Our notation and definitions follow [17]. But, in addition to the notation introduced above, we use the following: $ID = ID(\mathbb{R}^d)$ is the class of all infinitely divisible distributions on $\mathbb{R}^d$ ; $\mathcal{B}_0(\mathbb{R}^d)$ is the class of all Borel sets $B$ on $\mathbb{R}^d$ satisfying $\inf_{x \in B} |x| > 0$ ; $\delta_a$ is the distribution concentrated at a point $a$ ; p-lim stands for limit in probability; the norm of $Q \in \mathbf{M}_d$ is $\|Q\| = \sup_{|x| \leqslant 1} |Qx|$; $\mathrm{tr}A$ is the trace of a symmetric nonnegative-definite matrix $A$. A set or a function is called measurable if it is Borel measurable. For a distribution $\mu$, $\mu^n$ is the $n$-fold convolution of $\mu$. If the characteristic function $\widehat{\mu}(z)$ of a distribution $\mu$ on $\mathbb{R}^d$ vanishes nowhere, then there is a unique continuous function $f(z)$ on $\mathbb{R}^d$ such that $f(0) = 0$ and $\widehat{\mu}(z) = e^{f(z)}$. This $f(z)$ is called the distinguished logarithm of $\widehat{\mu}(z)$ and written as $f(z) = \log \widehat{\mu}(z)$ ([17, p. 33]).

Let $c(x)$ be a real-valued bounded measurable function satisfying

$$c(x) = \begin{cases} 1 + o(|x|) & \text{as} \quad |x| \to 0, \\ O(|x|^{-1}) & \text{as} \quad |x| \to \infty. \end{cases} \tag{1.9}$$

The generating triplet $(A, \nu, \gamma)_c$ of an infinitely divisible distribution $\mu$ on $\mathbb{R}^d$ is defined by the formula

$$\log \widehat{\mu}(z) = -\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d} g_c(z, x)\nu(dx) + i\langle z, \gamma\rangle \ ,$$

where $g_c(z, x) = e^{i\langle z, x\rangle} - 1 - i\langle z, x\rangle c(x)$ ; $A$ is the Gaussian covariance matrix and $\nu$ is the Lévy measure of $\mu$ ; $\gamma$ is the location parameter, which depends on the choice of $c(x)$. Standard choice of $c(x)$ is $1_{\{|x| \leqslant 1\}}(x)$ or $(1 + |x|^2)^{-1}$. In this paper we use

$$c(x) = (1 + |x|^2)^{-1}, \tag{1.10}$$

unless otherwise indicated. Thus we write $(A, \nu, \gamma)$ for $(A, \nu, \gamma)_c$ with $c(x)$ of (1.10).

## 2. Semi-Lévy processes

We will consider periodic independently scattered random measures. Semi-Lévy processes are their counterparts in stochastic processes. We gather basic properties and examples of semi-Lévy processes.

**Proposition 2.1.** *Let $X = \{X_t \colon t \geqslant 0\}$ be an additive process on $\mathbb{R}^d$ and let $\mathcal{L}(X_t) = \mu_t$. If it is a semi-Lévy process with period $p$, then*

$$\mu_{np+t} = \mu_p^n * \mu_t \tag{2.1}$$

*for all $n = 1, 2, \ldots$ and $t \geqslant 0$. If (2.1) holds for all $n = 1, 2, \ldots$ and $t \in [0, p)$, then $X$ is a semi-Lévy process with period $p$.*

*Proof.* Let $\mu_{s,t} = \mathcal{L}(X_t - X_s)$ for $0 \leqslant s \leqslant t$. Then $\mu_{0,t} = \mu_t$ and $\mu_{s,t} * \mu_{t,u} = \mu_{s,u}$ for $s \leqslant t \leqslant u$. If $X$ is semi-Lévy with period $p$, then $\mu_{s,t} = \mu_{s+p,t+p}$. $\mu_{2p} = \mu_p * \mu_{p,2p} = \mu_p^2$, and by induction $\mu_{np} = \mu_p^n$ for all $n = 1, 2 \ldots$, which implies $\mu_{np+t} = \mu_{np} * \mu_{np,np+t} = \mu_p^n * \mu_{0,t} = \mu_p^n * \mu_t$.

Conversely, assume that (2.1) holds for all $n = 1, 2, \ldots$ and $t \in [0, p)$. Then (2.1) holds for all $n = 1, 2, \ldots$ and $t \geqslant 0$. Indeed, if $kp \leqslant t < (k+1)p$, then $\mu_{np+t} = \mu_p^{n+k} * \mu_{t-kp} = \mu_p^n * \mu_p^k * \mu_{t-kp} = \mu_p^n * \mu_t$. Hence for $0 \leqslant s \leqslant t$, $\mu_{p+s} * \mu_{p+s,p+t} = \mu_{p+t} = \mu_p * \mu_t = \mu_p * \mu_s * \mu_{s,t} = \mu_{p+s} * \mu_{s,t}$. Since $\widehat{\mu}_{p+s}(z) \neq 0$, we get $\mu_{p+s,p+t} = \mu_{s,t}$. Hence $X$ is semi-Lévy with period $p$. □

**Proposition 2.2.** *If $X = \{X_t\colon t \geqslant 0\}$ is a semi-Lévy process with period $p$, then $\mu_t = \mathcal{L}(X_t)$ satisfies the following: $\mu_0 = \delta_0$, $\mu_t \in ID(\mathbb{R}^d)$, $\mu_t$ is continuous as a function of $t$, and, for any choice of $0 \leqslant s \leqslant t$, there is $\mu_{s,t} \in ID(\mathbb{R}^d)$ such that $\mu_t = \mu_s * \mu_{s,t}$.*

*In the converse direction, if a class of probability measures $\{\mu_t\colon t \in [0,p]\}$ on $\mathbb{R}^d$ satisfying these conditions for $t \in [0,p]$ is given, then there exists, uniquely in law, a semi-Lévy process $X = \{X_t\colon t \geqslant 0\}$ with period $p$ such that $\mu_t = \mathcal{L}(X_t)$ for $t \in [0,p]$.*

*Proof.* In order to see the first half, it is enough to choose $\mu_{s,t} = \mathcal{L}(X_t - X_s)$. Let us prove the second half. If $t > p$, then choose an integer $k$ such that $kp \leqslant t < (k+1)p$ and define $\mu_t = \mu_p^k * \mu_{t-kp}$. Then $\mu_t \in ID$ and, for any $0 \leqslant s \leqslant t$, there is $\mu_{s,t} \in ID$ such that $\mu_t = \mu_s * \mu_{s,t}$. We can prove that $\mu_{s,t} * \mu_{t,u} = \mu_{s,u}$ for $0 \leqslant s \leqslant t \leqslant u$, using $\widehat{\mu}_s(z) \neq 0$. Further, $\mu_0 = \delta_0$, $\mu_{s,t} \to \delta_0$ as $s \uparrow t$, and $\mu_{s,t} \to \delta_0$ as $t \downarrow s$. Thus, by [17, Theorem 9.7], there is an additive process in law $X$ such that $\mathcal{L}(X_t) = \mu_t$ and $\mathcal{L}(X_t - X_s) = \mu_{s,t}$. Then, by [17, Theorem 11.5], there is an additive process modification. It is a semi-Lévy process by Proposition 2.1. The uniqueness in law is obvious. □

**Proposition 2.3.** *Let $X = \{X_t\colon t \geqslant 0\}$ be a semi-Lévy process with period $p$. Then, $E\log^+|X_t| < \infty$ for all $t \geqslant 0$ if and only if $E\log^+|X_p| < \infty$.*

*Proof.* The Lévy measure $\nu_t$ of $X_t$ is increasing in $t$ and $\nu_{np} = n\nu_p$. By Theorem 25.3 of [17], $E\log^+|X_t|$ is finite if and only if $\int \log^+|x|\nu_t(dx) < \infty$. Hence the assertion follows. □

**Remark 2.4.** There is a semi-Lévy process $X$ with period $p$ such that $E\log^+|X_t|$ is finite for $t < p$ but infinite for $t = p$. For example, let $d = 1$, $p = 1$, and

$$\nu_t(dx) = 1_{(0,t/(1-t))}(x)\, x^{-1}(\log(2+x))^{-2}dx \qquad \text{for} \quad 0 < t \leqslant 1$$

and construct $X$, using Proposition 2.2.

**Example 2.5.** Let $X = \{X_t\colon t \geqslant 0\}$ be a semi-Lévy process on $\mathbb{R}^d$ with period $p$. Denote

$$\widetilde{\mu}_t = \mathcal{L}(X_p - X_{p-t}) \qquad \text{for} \quad 0 \leqslant t \leqslant p. \tag{2.2}$$

Then there exists, uniquely in law, a semi-Lévy process $\widetilde{X} = \{\widetilde{X}_t : t \geqslant 0\}$ with period $p$ such that $\mathcal{L}(\widetilde{X}_t) = \widetilde{\mu}_t$ for $0 \leqslant t \leqslant p$. Indeed, we can apply Proposition 2.2.

Let $Q \in \mathbf{M}_d^+$. A distribution $\mu$ on $\mathbb{R}^d$ satisfying

$$\widehat{\mu}(z)^a = \widehat{\mu}(a^{Q'} z) \tag{2.3}$$

for all $a = 1, 2, \ldots$ is called strictly $Q$-stable. If $\mu$ is strictly $Q$-stable, then $\mu \in ID$ and (2.3) holds for all $a > 0$. A Lévy process $X = \{X_t : t \geqslant 0\}$ is called strictly $Q$-stable if $\mathcal{L}(X_1)$ (and thus $\mathcal{L}(X_t)$ for all $t > 0$) is strictly $Q$-stable. (In the literature, a distribution $\mu$ on $\mathbb{R}^d$ satisfying $\widehat{\mu}(z)^a = \widehat{\mu}(a^{1/\alpha} z)$ for all $a = 1, 2, \ldots$ is called strictly $\alpha$-stable. This corresponds to the case where $Q = (1/\alpha)I$ in (2.3). Therefore, strictly $\alpha$-stable distribution is called here strictly $(1/\alpha)I$-stable distribution.)

It is easy to see that a Lévy process is strictly $Q$-stable if and only if it is $Q$-selfsimilar. We now give a new characterization of strictly $Q$-stable Lévy processes.

**Proposition 2.6.** *Let $Q \in \mathbf{M}_d^+$. If $X = \{X_t : t \geqslant 0\}$ is a $Q$-selfsimilar semi-Lévy process on $\mathbb{R}^d$, then it is a strictly $Q$-stable Lévy process.* (*The converse is trivial.*)

*Proof.* Suppose that $X$ is a $Q$-selfsimilar semi-Lévy process with period $p > 0$. Let $\mu_t = \mathcal{L}(X_t)$. Then $\widehat{\mu}_{np}(z) = \widehat{\mu}_p(z)^n$ for $n = 1, 2, \ldots$. On the other hand, by $Q$-selfsimilarity, $\widehat{\mu}_{np}(z) = \widehat{\mu}_p(n^{Q'} z)$. Hence $\widehat{\mu}_p(z)^n = \widehat{\mu}_p(n^{Q'} z)$. This means that $\mu_p$ is strictly $Q$-stable. Again by $Q$-selfsimilarity, we have $\widehat{\mu}_1(z) = \widehat{\mu}_{(1/p)p}(z) = \widehat{\mu}_p((1/p)^{Q'} z)$, and thus $\mu_1$ is also strictly $Q$-stable. Therefore $\widehat{\mu}_1(z)^a = \widehat{\mu}_1(a^{Q'} z)$ for any $a > 0$. Once again by $Q$-selfsimilarity, $\widehat{\mu}_a(z) = \widehat{\mu}_1(a^{Q'} z)$. These imply

$$\widehat{\mu}_a(z) = \widehat{\mu}_1(z)^a \qquad \text{for} \quad a > 0. \tag{2.4}$$

Let $\mu_{t,t+h} = \mathcal{L}(X_{t+h} - X_t)$. Since $\widehat{\mu}_t(z) \neq 0$, we have

$$\widehat{\mu}_{t,t+h}(z) = \widehat{\mu}_{t+h}(z)\widehat{\mu}_t(z)^{-1} = \widehat{\mu}_1(z)^{t+h}\widehat{\mu}_1(z)^{-t} = \widehat{\mu}_1(z)^h = \widehat{\mu}_h(z)$$

by using (2.4). Thus $X$ has stationary increments, and hence $X$ is a Lévy process such that $\mathcal{L}(X_t)$ is strictly $Q$-stable. □

## 3. Independently scattered random measures and stochastic integrals

We define $\mathbb{R}^d$-valued independently scattered random measures.

**Definition 3.1.** Let $J$ be an interval in $\mathbb{R}$. A family $M = \{M(B) : B \in \mathcal{B}_J^0\}$ of $\mathbb{R}^d$-valued random variables is called $\mathbb{R}^d$-valued *independently scattered random measure* (i. s. r. m.) over $J$, if the following three conditions are satisfied:

(1) for any sequence $B_1, B_2, \dots$ of disjoint sets in $\mathcal{B}_J^0$ with $B = \bigcup_{n=1}^\infty B_n \in \mathcal{B}_J^0$, $M(B) = \sum_{n=1}^\infty M(B_n)$ a. s., where the series is convergent a. s.,

(2) for any finite sequence $B_1, \dots, B_n$ of disjoint sets in $\mathcal{B}_J^0$, $M(B_1), \dots, M(B_n)$ are independent,

(3) $M(\{a\}) = 0$ a. s. for every one-point set $\{a\} \subset J$.

If, in addition, the condition

(4) $M(B) \stackrel{\mathrm{d}}{=} M(B+a)$ for every $B \in \mathcal{B}_J^0$ and $a \in \mathbb{R}$ satisfying $B + a \in \mathcal{B}_J^0$

is satisfied, then $M$ is called *homogeneous* i. s. r. m. Let $p > 0$. If $M$ satisfies (1), (2), (3), and the condition

(5) $M(B) \stackrel{\mathrm{d}}{=} M(B+p)$ for every $B \in \mathcal{B}_J^0$ satisfying $B + p \in \mathcal{B}_J^0$,

then it is called a *periodic* i. s. r. m. with period $p$ or, for short, $p$-periodic i. s. r. m.

The definitions of additive, Lévy, and semi-Lévy processes and those in law are extended to the case where the parameter set is $J = [0, t_0)$ or $[0, t_0]$. Under these names we always retain the condition that $X_0 = 0$ a. s.

The notions and the results in the rest of this section are extensions of a part of Sections 2 through 4 of [18], where only the case $J = [0, \infty)$ is studied. We omit proofs of our assertions, but they can be given either in a way similar to [18] or by reduction to the case $J = [0, \infty)$.

**Definition 3.2.** Let $J = [0, t_0)$, $[0, t_0]$, or $[0, \infty)$ and let $X = \{X_t \colon t \in J\}$ be a $J$-parameter additive process in law on $\mathbb{R}^d$. As $\mu_t = \mathcal{L}(X_t) \in ID$, the triplet of $\mu_t$ is denoted by $(A_t, \nu_t, \gamma_t)$. We say that $X$ is *natural* if $\gamma_t$ is locally of bounded variation on $J$, that is, of bounded variation on each $[t_1, t_2]$ satisfying $[t_1, t_2] \subset J$.

**Remark 3.3.** The definition above does not depend on the choice of $c(x)$ satisfying (1.9). Any $J$-parameter Lévy process in law on $\mathbb{R}^d$ is natural, since $\gamma_t = (t/t_1)\gamma_{t_1}$, where $t_1$ is positive and fixed in $J$. When $X$ is a $J$-parameter semi-Lévy process on $\mathbb{R}^d$ with period $p$, it is natural if and only if $\gamma_t$ is of bounded variation on $[0, p]$. Thus, using Proposition 2.2 or its analogue for $J = [0, t_0)$ or $[0, t_0]$, it is easy to see that there exist non-natural $J$-parameter semi-Lévy processes on $\mathbb{R}^d$. We are assuming $p < t_0$ if $J = [0, t_0)$ or $[0, t_0]$.

The connection between i. s. r. m. and additive processes in law is described in the following two propositions.

**Proposition 3.4.** *Let $J = [0, t_0)$, $[0, t_0]$, or $[0, \infty)$. If $M$ is an $\mathbb{R}^d$-valued i. s. r. m. over $J$, then the process $X$ defined by*

$$X_t = M([0, t]) \quad a.\,s. \qquad for \quad t \in J \tag{3.1}$$

*is a $J$-parameter natural additive process in law on $\mathbb{R}^d$. Conversely, if $X$ is a $J$-parameter natural additive process in law on $\mathbb{R}^d$, then there is a unique* (*in the a. s. sense*) *$\mathbb{R}^d$-valued i. s. r. m. $M$ over $J$ such that* (3.1) *holds. In this correspondence, $X$ is a Lévy process in law if and only if $M$ is homogeneous*; *$X$ is a natural semi-Lévy process in law with period $p$ if and only if $M$ is $p$-periodic.*

**Proposition 3.5.** *Let $J$ be an interval in $\mathbb{R}$.*

(i) *Suppose that $M$ is an $\mathbb{R}^d$-valued i. s. r. m. over $J$. Define, for each $s \in J$ and $t \geqslant 0$ with $s+t \in J$,*

$$X_t^{(s)} = M((s, s+t]) \quad a.\,s., \tag{3.2}$$

*where we understand that $(s,s] = \emptyset$. Then,*

(1) *for each $s \in J$, $X^{(s)} = \{X_t^{(s)} : t \in (J-s) \cap [0,\infty)\}$ is a $((J-s) \cap [0,\infty))$-parameter natural additive process in law on $\mathbb{R}^d$,*

(2) $X_{t_1}^{(s_1)} + X_{t_2}^{(s_1+t_1)} = X_{t_1+t_2}^{(s_1)}$ *a. s. if $s_1, s_1+t_1, s_1+t_1+t_2 \in J$.*

(ii) *Suppose that $X^{(s)} = \{X_t^{(s)} : t \in (J-s) \cap [0,\infty)\}$, $s \in J$, is a family of processes satisfying* (1) *and* (2) *above. Then there is a unique* (*in the a. s. sense*) *$\mathbb{R}^d$-valued i. s. r. m. $M$ over $J$ such that* (3.2) *holds for all $s \in J$ and $t \geqslant 0$ with $s+t \in J$.*

**Example 3.6.** Let $X = \{X_t : t \geqslant 0\}$ and $Y = \{Y_t : t \geqslant 0\}$ be independent additive processes in law on $\mathbb{R}^d$. Then there exists a unique $\mathbb{R}^d$-valued i. s. r. m. $M$ over $\mathbb{R}$ such that

$$M((s,t]) = \begin{cases} X_t - X_s & \text{for} \quad 0 \leqslant s < t, \\ X_t + Y_{-s} & \text{for} \quad s < 0 \leqslant t, \\ -Y_{-t} + Y_{-s} & \text{for} \quad s < t \leqslant 0. \end{cases} \tag{3.3}$$

This is proved by an application of Proposition 3.5. If $X$ is a Lévy process in law and $Y \stackrel{\mathrm{d}}{=} X$, then $M$ is homogeneous. If $X$ is a semi-Lévy process in law with period $p$ and $Y \stackrel{\mathrm{d}}{=} \widetilde{X}$, where $\widetilde{X}$ is constructed from $X$ as in Example 2.5, then $M$ is $p$-periodic.

In the rest of this section, $J$ is an arbitrary interval in $\mathbb{R}$.

**Proposition 3.7.** *Let $M$ be an $\mathbb{R}^d$-valued i. s. r. m. over $J$. Then, $\mathcal{L}(M(B)) \in ID(\mathbb{R}^d)$ for each $B$. Let $(A_B, \nu_B, \gamma_B)$ be the triplet of $\mu_B = \mathcal{L}(M(B))$. Then, $A_B$, $\gamma_B$, and $\nu_B(C)$ for each $C \in \mathcal{B}_0(\mathbb{R}^d)$ are countably additive in $B \in \mathcal{B}_J^0$.*

We use the notation $\mu_B$, $A_B$, $\nu_B$, and $\gamma_B$ as in the proposition above. The total variation measure of $\gamma_B$ is denoted by $|\gamma|_B$.

**Definition 3.8.** Let $M$ be an $\mathbb{R}^d$-valued i. s. r. m. over $J$. A pair $(\{\rho_s : s \in J\}, \sigma)$ is called a *factoring* of $M$ if the following six conditions are satisfied:

(1) $\sigma$ is a locally finite measure on $J$, that is, a measure on $J$ such that $\sigma(B) < \infty$ for all $B \in \mathcal{B}_J^0$,

(2) $\sigma$ is continuous (that is, atomless),

(3) $\rho_s \in ID(\mathbb{R}^d)$ for $s \in J$,

(4) $\log \widehat{\rho}_s(z)$ is measurable in $s \in J$ for each $z \in \mathbb{R}^d$,

(5) $\int_B |\log \widehat{\rho}_s(z)| \sigma(ds) < \infty$ for all $B \in \mathcal{B}_J^0$ and $z \in \mathbb{R}^d$,

(6) for all $B \in \mathcal{B}_J^0$ and $z \in \mathbb{R}^d$,

$$Ee^{i\langle z, M(B)\rangle} = \exp \int_B \log \widehat{\rho}_s(z)\sigma(ds). \tag{3.4}$$

The measure $\sigma$ on $J$ such that

$$\sigma(B) = \operatorname{tr}(A_B) + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu_B(dx) + |\gamma|_B \qquad \text{for} \quad B \in \mathcal{B}_J^0 \tag{3.5}$$

is called the *canonical measure* of $M$. A pair $(\{\rho_s\}, \sigma)$ is called a *canonical factoring* of $M$ if it is a factoring with $\sigma$ being the canonical measure of $M$. When $J = [0, t_0)$, $[0, t_0]$, or $[0, \infty)$ and $M$ corresponds to the $J$-parameter additive process in law $X$ by (3.1), then these notions of $M$ are sometimes considered as those of $X$.

For example, the canonical measure of a $J$-parameter Lévy process in law on $\mathbb{R}^d$ is a constant multiple of the Lebesgue measure restricted to $J$.

**Proposition 3.9.** *Let $M$ be an $\mathbb{R}^d$-valued i. s. r. m. over $J$.*
(i) *Let $(\{\rho_s\}, \sigma)$ be a factoring of $M$ and let $(A_s^\rho, \nu_s^\rho, \gamma_s^\rho)$ be the triplet of $\rho_s$. Then,*

$$A_s^\rho,\ \gamma_s^\rho,\ \textit{and}\ \nu_s^\rho(C)\ \textit{for any}\ C \in \mathcal{B}_0(\mathbb{R}^d)\ \textit{are measurable in}\ s, \tag{3.6}$$

$$\int_B \left( \operatorname{tr}(A_s^\rho) + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu_s^\rho(dx) + |\gamma_s^\rho| \right) \sigma(ds) < \infty \qquad \textit{for} \quad B \in \mathcal{B}_J^0 . \tag{3.7}$$

*Moreover, for $B \in \mathcal{B}_J^0$ and $C \in \mathcal{B}_0(\mathbb{R}^d)$,*

$$A_B = \int_B A_s^\rho \sigma(ds), \quad \nu_B(C) = \int_B \nu_s^\rho(C)\sigma(ds), \quad \gamma_B = \int_B \gamma_s^\rho \sigma(ds), \tag{3.8}$$

$$\log \widehat{\mu}_B(z) = \int_B \log \widehat{\rho}_s(z)\, \sigma(ds) . \tag{3.9}$$

(ii) *A canonical factoring $(\{\rho_s\}, \sigma)$ of $M$ exists and satisfies*

$$\operatorname*{esssup}_{s \in J} \left( \operatorname{tr}(A_s^\rho) + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu_s^\rho(dx) + |\gamma_s^\rho| \right) < \infty , \tag{3.10}$$

*where the essential supremum is with respect to $\sigma$. If $(\{\rho_s^1\}, \sigma)$ and $(\{\rho_s^2\}, \sigma)$ are canonical factorings of $M$, then $\rho_s^1 = \rho_s^2$ for $\sigma$-a. e. $s \in J$.*

Thus, when $J = [0, t_0)$, $[0, t_0]$, or $[0, \infty)$ and $X$ is a $J$-parameter additive process in law on $\mathbb{R}^d$, then a factoring of $X$ exists if and only if $X$ is natural.

Let $M$ be an $\mathbb{R}^d$-valued i. s. r. m. over $J$. We define stochastic integrals of nonrandom functions by $M$.

**Definition 3.10.** If $F(s)$ is a function on $J$ such that

$$F(s) = \sum_{j=1}^n 1_{B_j}(s) R_j , \tag{3.11}$$

where $B_1, \dots, B_n$ are disjoint Borel sets in $J$ and $R_1, \dots, R_n$ are in $\mathbf{M}_{l\times d}$, then we say that $F(s)$ is an $\mathbf{M}_{l\times d}$-valued *simple function* and define, for $B \in \mathcal{B}_J^0$,

$$\int_B F(s)M(ds) = \sum_{j=1}^{n} R_j\, M(B \cap B_j)\ . \tag{3.12}$$

An $\mathbf{M}_{l\times d}$-valued function $F(s)$ on $J$ is said to be *$M$-integrable* if it is measurable and if there is a sequence of simple functions $F_1(s), F_2(s), \dots$ on $J$ such that

(1) $F_n(s) \to F(s)$ for $\sigma$-a. e. $s \in J$, where $\sigma$ is the canonical measure of $M$,

(2) for every $B \in \mathcal{B}_J^0$, the sequence $\int_B F_n(s)M(ds)$ is convergent in probability as $n \to \infty$.

The limit in probability in (2) is denoted by $\int_B F(s)M(ds)$ and called the (stochastic) integral of $F$ over $B$ by $M$. When $J = [0, t_0)$, $[0, t_0]$, or $[0, \infty)$, then, using the $J$-parameter natural additive process in law $X$ satisfying (3.1), we sometimes write $\int_B F(s)dX_s$ for $\int_B F(s)M(ds)$.

Obviously the definition (3.12) of the integral of a simple function does not depend (in the a. s. sense) on the choice of the representation (3.11) of $F$. But the following fact, which guarantees that the integral is well-defined in $M$-integrable case, is nontrivial.

**Proposition 3.11.** *If $F(s)$ is a measurable $\mathbf{M}_{l\times d}$-valued function on $J$ and if $F_n^1(s)$ and $F_n^2(s)$, $n = 1, 2, \dots$, are sequences of simple functions satisfying* (1) *and* (2) *of Definition* 3.10 *with $F_n(s)$ replaced by $F_n^1(s)$ and $F_n^2(s)$, then, for every $B \in \mathcal{B}_J^0$,*

$$\operatorname*{p\text{-}lim}_{n\to\infty} \int_B F_n^1(s)M(ds) = \operatorname*{p\text{-}lim}_{n\to\infty} \int_B F_n^2(s)M(ds) \quad a.\,s.$$

Here are properties of integrals by $M$.

**Proposition 3.12.** *Let $F(s)$ be an $\mathbf{M}_{l\times d}$-valued measurable function bounded on each $B \in \mathcal{B}_J^0$. Then $F(s)$ is $M$-integrable. Moreover, if $F_n(s)$ is a sequence of simple functions on $J$ such that $F_n(s) \to F(s)$ $\sigma$-a. e., where $\sigma$ is the canonical measure, and if $\sup_n \sup_{s\in B} \|F_n(s)\| < \infty$ for every $B \in \mathcal{B}_J^0$, then*

$$\operatorname*{p\text{-}lim}_{n\to\infty} \int_B F_n(s)M(ds) = \int_B F(s)M(ds) \qquad for \quad B \in \mathcal{B}_J^0.$$

**Proposition 3.13.** *If $F_1(s)$ and $F_2(s)$ are $M$-integrable $\mathbf{M}_{l\times d}$-valued functions on $J$, then, for any $a_1$ and $a_2$ in $\mathbb{R}$, $a_1F_1(s)+a_2F_2(s)$ is $M$-integrable and*

$$\int_B (a_1F_1(s) + a_2F_2(s))M(ds) = a_1 \int_B F_1(s)M(ds) + a_2 \int_B F_2(s)M(ds) \quad a.\,s.$$

*for any $B \in \mathcal{B}_J^0$.*

**Proposition 3.14.** *Let $F(s)$ be an $M$-integrable $\mathbf{M}_{l\times d}$-valued function on $J$. Let $\Lambda(B)=\int_B F(s)M(ds)$ and $\lambda_B=\mathcal{L}(\Lambda(B))$ for $B\in\mathcal{B}_J^0$. Then $\Lambda$ is an $\mathbb{R}^l$-valued i. s. r. m. over $J$. If $(\{\rho_s\},\sigma)$ is a factoring of $M$, then, for $B\in\mathcal{B}_J^0$ and $z\in\mathbb{R}^l$,*

$$\int_B |\log\widehat{\rho}_s(F(s)'z)|\sigma(ds)<\infty \quad \textit{and} \quad \log\widehat{\lambda}_B(z)=\int_B \log\widehat{\rho}_s(F(s)'z)\sigma(ds).$$

*Here $\log\widehat{\rho}_s(F(s)'z)$ means $(\log\widehat{\rho}_s(w))_{w=F(s)'z}$.*

Even if $F(s)$ is $M$-integrable, we cannot always define $\int_B F(s)M(ds)$ for $B\in\mathcal{B}_J\setminus\mathcal{B}_J^0$.

**Definition 3.15.** Let $F(s)$ be an $M$-integrable $\mathbf{M}_{l\times d}$-valued function on $J$. If $J$ is infinite to the right and if, for $t\in J$, $\int_{(t,u]}F(s)M(ds)$ is convergent in probability as $u\to\infty$, then we say that $\int_t^\infty F(s)M(ds)$ is *definable* and define

$$\int_t^\infty F(s)M(ds)=\operatorname*{p-lim}_{u\to\infty}\int_{(t,u]}F(s)M(ds)\ .$$

If $J$ is infinite to the left, then the notion of definability and the definition are given similarly to $\int_{-\infty}^t F(s)M(ds)$ for $t\in J$. When $J=[0,\infty)$, then, using the natural additive process in law $X$ satisfying (3.1), write $\int_t^\infty F(s)dX_s$ for $\int_t^\infty F(s)M(ds)$.

**Remark 3.16.** Let $J$ be infinite to the right (resp. to the left). Suppose that $\int_t^\infty F(s)M(ds)$ (resp. $\int_{-\infty}^t F(s)M(ds)$) is definable for $t\in J$. Then it is a $J$-parameter stochastic process continuous in probability with independent increments. Hence it has a cadlag modification by the argument in Theorem 11.5 of [17]. Henceforth $\int_t^\infty F(s)M(ds)$ (resp. $\int_{-\infty}^t F(s)M(ds)$) denotes this modification. We also use, for a fixed $t_0\in J$, the notation

$$\int_{t_0}^t F(s)M(ds)=\begin{cases}\int_{(t_0,t]}F(s)M(ds) & \text{for}\quad t\in J\cap(t_0,\infty),\\ 0 & \text{for}\quad t=t_0,\\ -\int_{(t,t_0]}F(s)M(ds) & \text{for}\quad t\in J\cap(-\infty,t_0),\end{cases}$$

and mean a cadlag modification over $J$.

## 4. Ornstein-Uhlenbeck type processes generated by independently scattered random measures

In (i) of the following theorem, we notice that the nonrandom equation (1.2) is always solvable. This is an $\mathbb{R}^d$-version of a result of Cheridito, Kawaguchi, and Maejima [2], who consider a more general class of functions when $d=1$. In (ii) we specialize it to the case of independently scattered random measures, that is, the case of Langevin equation. There are many related papers such as Doob [3], Mikosch and Norvaiša [13], and Surgailis et al. [22].

**Theorem 4.1.** (i) *Let $Q \in \mathbf{M}_d$ and $s_0 \in \mathbb{R}$. Given a nonrandom cadlag function $y_s$ of $s \in \mathbb{R}$ and a point $\xi \in \mathbb{R}^d$, there exists a unique cadlag function $z_s$ of $s \in \mathbb{R}$ satisfying equation* (1.2) *and condition $z_{s_0} = \xi$.*

(ii) *Let $\Lambda = \{\Lambda(B) \colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ be an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$ and let $\Xi$ be an $\mathbb{R}^d$-valued random variable. Then, there exists a unique* (*in the a. s. sense*) *cadlag process $Z = \{Z_s \colon s \in \mathbb{R}\}$ such that Langevin equation* (1.3) *is satisfied a. s. for every $s_1$ and $s_2$ with $s_1 < s_2$ together with the condition that $Z_{s_0} = \Xi$ a. s. This process $Z$ is represented as*

$$Z_s = e^{(s_0-s)Q}\Xi + e^{-sQ}\int_{s_0}^{s} e^{uQ}\Lambda(du) \qquad \textit{for} \quad s \in \mathbb{R},\ \textit{a. s.} \tag{4.1}$$

Thus we get the Ornstein-Uhlenbeck type process generated by $\Lambda$ and $Q$ satisfying $Z_{s_0} = \Xi$.

*Proof of Theorem* 4.1. (i) Define

$$z_s = e^{(s_0-s)Q}\xi + y_s - e^{(s_0-s)Q}y_{s_0} - \int_{s_0}^{s} Qe^{(u-s)Q}y_u du \qquad \text{for} \quad s \in \mathbb{R}. \tag{4.2}$$

Then $z_s$ is a cadlag function with $z_{s_0} = \xi$. By a straightforward calculation we can prove that $z_s$ satisfies (1.2). In order to see the uniqueness, suppose that $z_s^{(1)}$ and $z_s^{(2)}$ are cadlag solutions of (1.2) with $z_{s_0}^{(1)} = z_{s_0}^{(2)} = \xi$. Then

$$z_s^{(1)} - z_s^{(2)} = -Q\int_{s_0}^{s} (z_u^{(1)} - z_u^{(2)})du. \tag{4.3}$$

Let $v_s = z_s^{(1)} - z_s^{(2)}$. Then we get

$$v_s = \frac{(-Q)^n}{(n-1)!}\int_{s_0}^{s} (s-u)^{n-1}v_u du \qquad \text{for} \quad n = 1, 2, \dots\ .$$

Since $((n-1)!)^{-1}(s-u)^{n-1}(-Q)^n \to 0$ uniformly in $u \in [0,s]$ as $n \to \infty$, we get $v_s = 0$. That is, $z_s^{(1)} = z_s^{(2)}$.

(ii) Define $Y_t^0$ to be $\Lambda((0,t])$ for $t > 0$, zero for $t = 0$, and $-\Lambda((t,0])$ for $t < 0$. Then, $\{Y_t^0 \colon t \in \mathbb{R}\}$ has a cadlag modification $Y = \{Y_t \colon t \in \mathbb{R}\}$ as in Theorem 11.5 of [17]. We have

$$Y_t - Y_s = \Lambda((s,t]) \quad \text{a.s.} \qquad \text{for} \quad \text{every } s,t \text{ with } s < t. \tag{4.4}$$

With this $Y_s$ replacing $y_s$, we can uniquely solve (1.2) pathwise by (i) under the condition that it equals $\Xi$ at $s = s_0$. Denoting the resulting solution by $Z = \{Z_s \colon s \in \mathbb{R}\}$, we see that $Z$ satisfies (1.3) a. s. for every $s_1, s_2$ with $s_1 < s_2$. The uniqueness is proved in the same way as in the nonrandom case, since the analogue of (4.3) holds for all $s$, a. s. It follows from the integration-by-parts formula in Corollary 4.9 of [18] that

$$\int_{s_1}^{s_2} e^{uQ}dY_u = e^{s_2Q}Y_{s_2} - e^{s_1Q}Y_{s_1} - \int_{s_1}^{s_2} Qe^{uQ}Y_u du \quad \text{a.s.} \qquad \text{for} \quad 0 \leqslant s_1 < s_2$$

and a similar expression for $s_1 < s_2 \leqslant 0$. Hence we obtain the equality in (4.1) a. s. for each $s$. Since both sides are cadlag, it holds for all $s$ a. s. $\square$

**Definition 4.2.** An OU type process $\{Z_s \colon s \in \mathbb{R}\}$ generated by $\Lambda$ and $Q$ or a solution of (1.3) is said to be $Q$-*mild* at $-\infty$ (or, simply, $Q$-mild) if $\text{p-lim}_{s\to-\infty} e^{sQ} Z_s = 0$.

**Theorem 4.3.** *Let $\Lambda$ be an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$ and $Q \in \mathbf{M}_d$. Then the following are equivalent*:

(1) *$\int_{-\infty}^0 e^{sQ}\Lambda(ds)$ is definable,*

(2) *$\text{p-lim}_{s\to-\infty} e^{sQ} Z_s$ exists for every OU type process $Z$ generated by $\Lambda$ and $Q$,*

(3) *a $Q$-mild OU type process $Z$ generated by $\Lambda$ and $Q$ exists.*

*If* (3) *holds, then a $Q$-mild OU type process $Z$ generated by $\Lambda$ and $Q$ is unique a. s. and expressed as*

$$Z_s = e^{-sQ}\int_{-\infty}^{s} e^{uQ}\Lambda(du) \qquad \textit{for} \quad s \in \mathbb{R}, \quad \textit{a. s.} \tag{4.5}$$

*Proof.* If $Z$ is an OU type process generated by $\Lambda$ and $Q$, then, by Theorem 4.1,

$$Z_s = e^{(s_0-s)Q} Z_{s_0} + e^{-sQ}\int_{s_0}^{s} e^{uQ}\Lambda(du) \qquad \text{for} \quad s_0, s \in \mathbb{R}, \quad \text{a. s.} \tag{4.6}$$

That is,

$$e^{sQ} Z_s - e^{s_0 Q} Z_{s_0} = \int_{s_0}^{s} e^{uQ}\Lambda(du) \qquad \text{for} \quad s_0, s \in \mathbb{R}, \quad \text{a. s.} \tag{4.7}$$

Letting $s = 0$ and $s_0 \to -\infty$, we get the equivalence of (1) and (2). If (3) holds, then, letting $s_0 \to -\infty$ in (4.7), we see that (1) and (4.5) are true. This shows the uniqueness of a $Q$-mild solution. If (1) holds, then by Theorem 4.1, the solution $Z$ of Langevin equation with $Z_0 = \Xi = \int_{-\infty}^0 e^{sQ}\Lambda(ds)$ a. s. satisfies

$$Z_s = e^{-sQ}\int_{-\infty}^{0} e^{uQ}\Lambda(du) + e^{-sQ}\int_{0}^{s} e^{uQ}\Lambda(du) = e^{-sQ}\int_{-\infty}^{s} e^{uQ}\Lambda(du) \quad \text{a. s.},$$

which shows that $\text{p-lim}_{s\to-\infty} e^{sQ} Z_s = 0$. Hence (1) implies (3). $\square$

**Remark 4.4.** Let $(\{\rho_s\}, \sigma)$ be a factoring of $\Lambda$. In the cases of (1) through (3) of Theorem 4.3, $\lim_{s_0\to-\infty}\int_{s_0}^{s} \log\widehat{\rho}_u(e^{uQ'}z)\sigma(du)$ exists and equals the distinguished logarithm of the characteristic function of $\int_{-\infty}^{s} e^{uQ}\Lambda(du)$. This follows from Proposition 3.14 and [17] Lemma 7.7.

We apply Theorem 4.3 to periodic i. s. r. m.

**Theorem 4.5.** *Let $\Lambda$ be an $\mathbb{R}^d$-valued $p$-periodic i. s. r. m. over $\mathbb{R}$. Let $Q \in \mathbf{M}_d^+$.*

(i) *Suppose that* $\Lambda$ *has finite log-moment. Then Langevin equation* (1.3) *based on* $\Lambda$ *and* $Q$ *has a unique semi-stationary solution* $Z$ *with period* $p$. *This solution has expression* (4.5) *and* $\mathcal{L}(Z_s) \in L_0(e^{-p}, Q)$ *for all* $s$.

(ii) *Suppose that*

$$E[\log^+ |\Lambda((0,p])|] = \infty. \tag{4.8}$$

*Then, Langevin equation* (1.3) *based on* $\Lambda$ *and* $Q$ *has no semi-stationary solution with period* $p$. *Moreover, it has no* $Q$*-mild solution.*

**Corollary 4.6.** *Let* $\Lambda$ *be an* $\mathbb{R}^d$*-valued homogeneous i. s. r. m. over* $\mathbb{R}$. *Let* $Q \in \mathbf{M}_d^+$.

(i) *Suppose that* $\Lambda$ *has finite log-moment. Then Langevin equation* (1.3) *based on* $\Lambda$ *and* $Q$ *has a unique stationary solution* $Z$. *This solution has expression* (4.5) *and* $\mathcal{L}(Z_s)$ *is* $Q$*-selfdecomposable.*

(ii) *Suppose that* $\Lambda$ *does not have finite log-moment. Then, Langevin equation* (1.3) *based on* $\Lambda$ *and* $Q$ *does not have a stationary solution.*

Actually the result in Corollary 4.6 was given in [20]. Our Theorem 4.5 is an extension of it.

In order to prove Theorem 4.5, we prepare two lemmas.

**Lemma 4.7.** *Let* $\Lambda$ *be an* $\mathbb{R}^d$*-valued i. s. r. m. over* $\mathbb{R}$ *and let* $Q \in \mathbf{M}_d^+$. *Let* $Z$ *be an OU type process generated by* $\Lambda$ *and* $Q$. *If* $Z$ *is stationary or, more generally, semi-stationary, then* $Z$ *is* $Q$*-mild at* $-\infty$.

*Proof.* Suppose that $Z$ is semi-stationary with period $p$. Let $\eta_s = \mathcal{L}(Z_s)$. It follows from (1.3) or (4.1) that $Z$ is continuous in probability. Hence $\eta_s$ is continuous in $s$. Thus we see that $\{\eta_s\colon s \in [0,p]\}$ is a compact set in the topology of the weak convergence. This set equals $\{\eta_s\colon s \in \mathbb{R}\}$ by semi-stationarity. Hence $\{\eta_s\colon s \in \mathbb{R}\}$ is tight. Since $Q \in \mathbf{M}_d^+$, we have the following estimate (see [18]): there are positive constants $c_1, \dots, c_4$ such that

$$c_4 e^{c_2 s}|x| \leqslant |e^{sQ}x| \leqslant c_3 e^{c_1 s}|x| \qquad \text{for} \quad s \leqslant 0 \text{ and } x \in \mathbb{R}^d. \tag{4.9}$$

Using this we see that, for any $\varepsilon > 0$,

$$P[\,|e^{sQ}Z_s| > \varepsilon\,] \leqslant P[\,c_3 e^{c_1 s}|Z_s| > \varepsilon\,] \leqslant \sup_u \eta_u(\{|x| > \varepsilon c_3^{-1} e^{c_1|s|}\}) \to 0$$

as $s \to -\infty$. That is, $e^{sQ}Z_s \to 0$ in probability. $\square$

**Lemma 4.8.** *Let* $\Lambda$ *be an* $\mathbb{R}^d$*-valued* $p$*-periodic i. s. r. m. over* $\mathbb{R}$. *Fix* $t_0 \in \mathbb{R}$ *and define*

$$\widetilde{\Lambda}(B) = \Lambda(t_0 - B) \qquad \textit{for} \quad B \in \mathcal{B}^0_{\mathbb{R}}\,. \tag{4.10}$$

*Then* $\widetilde{\Lambda}$ *is a* $p$*-periodic i. s. r. m. over* $\mathbb{R}$ *and* $\widetilde{\Lambda}((0,p]) \stackrel{\mathrm{d}}{=} \Lambda((0,p])$. *Let* $F(s)$ *be an* $\mathbf{M}_{l\times d}$*-valued function on* $\mathbb{R}$. *Then* $F(s)$ *is* $\widetilde{\Lambda}$*-integrable if and only if* $F(t_0 - s)$ *is* $\Lambda$*-integrable. In this case,*

$$\int_B F(s)\widetilde{\Lambda}(ds) = \int_{t_0 - B} F(t_0 - s)\Lambda(ds) \quad a.\,s. \qquad \textit{for} \quad B \in \mathcal{B}^0_{\mathbb{R}}\,. \tag{4.11}$$

*Proof.* It is easy to see that $\widetilde{\Lambda}$ is a $p$-periodic i. s. r. m. over $\mathbb{R}$. To see that $\widetilde{\Lambda}((0,p]) \stackrel{\mathrm{d}}{=} \Lambda((0,p])$, note that $\widetilde{\Lambda}((0,p]) = \Lambda([t_0 - p, t_0)) = \Lambda((t_0 - p, t_0])$ a. s. and that, choosing $n \in \mathbb{Z}$ such that $t_0 - p < np \leqslant t_0$, $\Lambda((np, t_0]) \stackrel{\mathrm{d}}{=} \Lambda((0, t_0 - np])$ and $\Lambda((t_0 - p, np]) \stackrel{\mathrm{d}}{=} \Lambda((t_0 - np, p])$. If $F(s)$ is a simple function (3.11), then $F(t_0 - u) = \sum_{j=1}^n 1_{B_j}(t_0 - u) R_j = \sum_{j=1}^n 1_{t_0 - B_j}(u) R_j$ and hence $\int_B F(s)\widetilde{\Lambda}(ds) = \sum_{j=1}^n R_j \widetilde{\Lambda}(B \cap B_j) = \sum_{j=1}^n R_j \Lambda((t_0 - B) \cap (t_0 - B_j)) = \int_{t_0 - B} F(t_0 - u)\Lambda(du)$, which is (4.11). The rest of proof is straightforward. □

*Proof of Theorem* 4.5. (i) We assume that $\Lambda$ has finite log-moment. Given $t_0 \in \mathbb{R}$, define $\widetilde{\Lambda}$ by (4.10). Then, by Lemma 4.8, $\widetilde{\Lambda}$ is a $p$-periodic i. s. r. m. with finite log-moment. By Theorem 5.2 of [18], $\int_0^\infty e^{-sQ}\widetilde{\Lambda}(ds)$ is definable. Hence, by (4.11), $\int_{-\infty}^{t_0} e^{-(t_0 - s)Q}\Lambda(ds)$ is definable and so is $\int_{-\infty}^0 e^{sQ}\Lambda(ds)$. Thus, by Theorem 4.3, there is a unique $Q$-mild OU type process $Z$ generated by $\Lambda$ and $Q$. It is expressed by (4.5). Since

$$e^{-t_0 Q}\int_{-\infty}^{t_0} e^{sQ}\Lambda(ds) = \int_0^\infty e^{-sQ}\widetilde{\Lambda}(ds) \quad \text{a. s.},$$

$\mathcal{L}(Z_{t_0}) \in L_0(e^{-p}, Q)$ for any $t_0$ by virtue of Theorem 5.2 of [18]. We have

$$Z_{s+p} = e^{-(s+p)Q}\int_{-\infty}^{s+p} e^{uQ}\Lambda(du) = e^{-sQ}\int_{-\infty}^{s} e^{vQ}\Lambda^\sharp(dv),$$

where we define $\Lambda^\sharp(B) = \Lambda(B+p)$. Since $\Lambda^\sharp(B) \stackrel{\mathrm{d}}{=} \Lambda(B)$, we get $Z_{s+p} \stackrel{\mathrm{d}}{=} e^{-sQ}\int_{-\infty}^{s} e^{vQ}\Lambda(dv) = Z_s$. Similarly, for any $s_1 < s_2 < \cdots < s_n$, $(Z_{s_j + p})_{j=1,\ldots,n} \stackrel{\mathrm{d}}{=} (Z_{s_j})_{j=1,\ldots,n}$, which is semi-stationarity with period $p$ of $Z$. By Lemma 4.7 semi-stationarity implies $Q$-mildness. Hence, by Theorem 4.3, a semi-stationary solution is unique.

(ii) We assume (4.8). Then, using Theorem 5.4 of [18] and Lemma 4.8, we see that $\int_{-\infty}^t e^{sQ}\Lambda(ds)$ is not definable. Hence, by Theorem 4.3, there is no $Q$-mild solution of Langevin equation. Lemma 4.7 tells us that, a fortiori, there is no semi-stationary solution. □

**Remark 4.9.** Let $\Lambda$ be an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$ and $Q \in \mathbf{M}_d$. If there is a semi-stationary solution $Z$ with period $p$ of Langevin equation (1.3), then $\Lambda$ is $p$-periodic. Indeed, it follows from (1.3) and $\{Z_{s+p}\} \stackrel{\mathrm{d}}{=} \{Z_s\}$ that

$$Z_{s_2+p} - Z_{s_1+p} + Q\int_{s_1}^{s_2} Z_{u+p}du \stackrel{\mathrm{d}}{=} Z_{s_2} - Z_{s_1} + Q\int_{s_1}^{s_2} Z_u du\ ,$$

that is, $\Lambda((s_1 + p, s_2 + p]) \stackrel{\mathrm{d}}{=} \Lambda((s_1, s_2])$. Similarly, if there is a stationary solution, then $\Lambda$ is homogeneous.

Theorem 4.5 shows that, when we restrict our attention to $p$-periodic i. s. r. m., the integrals $\int_{-\infty}^0 e^{sQ}\Lambda(ds)$ (if definable) with $Q \in \mathbf{M}_d^+$ have distributions in a restricted class. But, in the case of general i. s. r. m., the integrals can have arbitrary distributions. In fact, we can show the following.

**Proposition 4.10.** *Let $F(s)$ be an $\mathbf{M}_d$-valued continuous function on $(-\infty, 0]$ such that, for every $s$, $F(s)$ is an invertible matrix. Then, for any $\mu \in ID(\mathbb{R}^d)$, we can choose an $\mathbb{R}^d$-valued i. s. r. m. $\Lambda$ over $(-\infty, 0]$ in such a way that $\int_{-\infty}^0 F(s)\Lambda(ds)$ is definable and has distribution $\mu$.*

*Proof.* Let $Y^\sharp = \{Y_s^\sharp \colon s \geqslant 0\}$ be a Lévy process with $\mathcal{L}(Y_1^\sharp) = \mu$. Define

$$\Lambda(B) = \int_{\exp B} (F(\log u))^{-1} dY_u^\sharp \qquad \text{for} \quad B \in \mathcal{B}^0_{(-\infty,0]} \,.$$

Then $\Lambda$ is an $\mathbb{R}^d$-valued i. s. r. m. over $(-\infty, 0]$. We have, by Proposition 3.14,

$$\begin{aligned} Ee^{i\langle z,\Lambda(B)\rangle} &= \exp \int_{\exp B} \log \widehat{\mu}((F(\log u)')^{-1} z) du \\ &= \exp \int_B e^v \log \widehat{\mu}((F(v)')^{-1} z) dv \,. \end{aligned}$$

Thus we can choose a factoring $(\{\rho_s\}, \sigma)$ of $\Lambda$ such that $\widehat{\rho_s}(z) = \widehat{\mu}((F(s)')^{-1} z)^{e^s}$ and $\sigma =$ Lebesgue. Hence, by Proposition 3.14,

$$\begin{aligned} E \exp \left[ i \left\langle z, \int_{s_1}^{s_2} F(s)\Lambda(ds) \right\rangle \right] &= \exp \int_{s_1}^{s_2} \log \widehat{\rho_s}(F(s)' z) ds \\ &= \exp \left[ \int_{s_1}^{s_2} e^s ds \log \widehat{\mu}(z) \right], \end{aligned}$$

which tends to 1 as $s_1, s_2 \to -\infty$. It follows that $\int_{-\infty}^0 F(s)\Lambda(ds)$ is definable and that

$$E \exp \left[ i \left\langle z, \int_{-\infty}^{0} F(s)\Lambda(ds) \right\rangle \right] = \exp \left[ \int_{-\infty}^{0} e^s ds \log \widehat{\mu}(z) \right] = \widehat{\mu}(z),$$

that is, $\mathcal{L}\left( \int_{-\infty}^0 F(s)\Lambda(ds) \right) = \mu$. $\square$

## 5. Proofs of main results

Let us prove the three theorems formulated in Section 1.

*Proof of Theorem* 1.1. Let $M$ be the $\mathbb{R}^d$-valued i. s. r. m. over $[0, \infty)$ induced by the process $X$ (Proposition 3.4). Let $M_0(B) = M(B)$ for $B \in \mathcal{B}^0_{(0,\infty)}$. Then $M_0$ is an i. s. r. m. over $(0, \infty)$, which is a restriction of $M$. The function $t^{-Q}$ is $M_0$-integrable by Proposition 3.12. If $B \in \mathcal{B}^0_{\mathbb{R}}$, then $\exp B \in \mathcal{B}^0_{(0,\infty)}$ and hence we can define $\int_{\exp B} t^{-Q} M_0(dt)$. We denote this integral by $\int_{\exp B} t^{-Q} dX_t$. The right-hand side of (1.5) means this integral. By Proposition 3.14, $\Lambda = \{\Lambda(B) \colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ thus defined by (1.5) is an $\mathbb{R}^d$-valued i. s. r. m.

over $\mathbb{R}$. Using Proposition 3.12 again, we can prove that, if $\varepsilon > 0$, then, for all $B \in \mathcal{B}^0_{\mathbb{R}}$ satisfying $\exp B \subset [\varepsilon, \infty)$, $\int_{\exp B} t^{-Q} M_0(dt) = \int_{\exp B} F(t) dX_t$ a. s., where $F(t)$ is a continuous function on $[0, \infty)$ satisfying $F(t) = t^{-Q}$ on $[\varepsilon/2, \infty)$. Let $X^\sharp_t = X_{at}$. Using Theorem 4.10 of [18] and recalling $\{X^\sharp_t\} \overset{\mathrm{d}}{=} \{a^Q X_t\}$, we get

$$\Lambda(B+p) = \int_{a \exp B} t^{-Q} dX_t = \int_{\exp B} (at)^{-Q} dX^\sharp_t \overset{\mathrm{d}}{=} \int_{\exp B} t^{-Q} dX_t = \Lambda(B) .$$

Hence $\Lambda$ is $p$-periodic. Define $\Lambda^\sharp(B) = \Lambda(\log B)$ for $B \in \mathcal{B}^0_{(0,\infty)}$. Then $\Lambda^\sharp$ is an i. s. r. m. over $(0, \infty)$ and $\Lambda^\sharp(B) = \int_B t^{-Q} dX_t$ under similar interpretation of the integral. Use of analogues of Theorems 4.6 and 4.10 of [18] gives, for $0 < t_1 < t_2$,

$$\int_{\log t_1}^{\log t_2} e^{uQ} \Lambda(du) = \int_{t_1}^{t_2} u^Q \Lambda^\sharp(du) = \int_{t_1}^{t_2} u^Q u^{-Q} dX_u = X_{t_2} - X_{t_1} .$$

As $t_1 \downarrow 0$, $X_{t_1} \to 0$ a. s. Hence $\int_{-\infty}^{\log t_2} e^{uQ} \Lambda(du)$ is definable. It follows from Theorem 4.5 (ii) that $\Lambda$ has finite log-moment. We get also the expression (1.6).

By (1.4) and (1.6), we get (1.7). Hence, by Theorem 4.5, $Z$ is the semi-stationary OU type process with period $p$ generated by $\Lambda$ and $Q$. □

*Proof of Theorem* 1.2. Existence and uniqueness of the semi-stationary OU type process $Z$ with period $p$ generated by $\Lambda$ and $Q$, are shown in Theorem 4.5. It is expressed by (4.5). Hence $X_t$ has the expression (1.6) for $t > 0$. As $t \downarrow 0$, $X_t = \int_{-\infty}^{\log t} e^{sQ} \Lambda(ds) \to 0$ in probability. It follows from (1.6) that $X$ has independent increments. Since $X$ is continuous in probability for $t \geqslant 0$, it is an additive process in law and thus has a cadlag modification ([17, Theorem 11.5]). On the other hand, $X$ is itself cadlag for $t > 0$ a. s., since $Z$ is cadlag. It follows that $X$ is cadlag for $t \geqslant 0$ a. s. Let $\Lambda^\sharp(B) = \Lambda(B + p)$. We have

$$X_{at} = \int_{-\infty}^{p+\log t} e^{sQ} \Lambda(ds) = \int_{-\infty}^{\log t} e^{(s+p)Q} \Lambda^\sharp(ds) \overset{\mathrm{d}}{=} a^Q \int_{-\infty}^{\log t} e^{sQ} \Lambda(ds) = a^Q X_t,$$

and similarly for joint distributions. Thus $\{X_{at}\} \overset{\mathrm{d}}{=} \{a^Q X_t\}$. Hence $X$ is a $Q$-semi-selfsimilar additive process with epoch $a$. Define $X^\sharp_t = X_{1+t} - X_1$ and $X^{\sharp\sharp}_t = X^\sharp_{e^t - 1}$ for $t \geqslant 0$. Since $X^\sharp_t = \int_0^{\log(1+t)} e^{sQ} \Lambda(ds)$, $X^\sharp$ is a natural additive process, by Propositions 3.4 and 3.14. Then $X^{\sharp\sharp}_t = X_{e^t} - X_1 = \int_0^t e^{sQ} \Lambda(ds)$. If $s \geqslant 0$, then

$$\int_1^{e^s} t^{-Q} dX_t = \int_0^{e^s - 1} (1+t)^{-Q} dX^\sharp_t = \int_0^s (e^t)^{-Q} dX^{\sharp\sharp}_t = \Lambda((0, s]),$$

where the second equality is by Theorem 4.10 of [18] and the third is by Theorem 4.6 of [18]. If $s < 0$, then

$$\int_1^{e^s} t^{-Q} dX_t = -\int_{e^s}^1 t^{-Q} dX_t = -\Lambda((s, 0])$$

similarly. Hence we obtain (1.5). That is, $\Lambda$ is recovered from $X$ as in Theorem 1.1. The expression (1.4) of $Z$ by $X$ follows from (1.8). □

The following lemma is an extension of Theorem 10 of [10].

**Lemma 5.1.** *Let $Q \in \mathbf{M}_d^+$ and $b \in (0,1)$. A distribution $\mu$ is in $L_0(b,Q)$ if and only if there exists a $Q$-semi-selfsimilar natural additive process $X = \{X_t \colon t \geqslant 0\}$ with epoch $b^{-1}$ such that $\mathcal{L}(X_1) = \mu$.*

*Proof.* We write $a = b^{-1}$.

*The 'if' part.* We have $\{X_{at}\} \stackrel{\mathrm{d}}{=} \{a^Q X_t\}$. Hence $b^Q X_1 \stackrel{\mathrm{d}}{=} X_b$. It follows that

$$Ee^{i\langle z,X_1\rangle} = Ee^{i\langle z,X_b\rangle}\, Ee^{i\langle z,X_1-X_b\rangle} = Ee^{i\langle b^{Q'}z,X_1\rangle}\, Ee^{i\langle z,X_1-X_b\rangle}.$$

Since $\mathcal{L}(X_1 - X_b)$ is infinitely divisible, this means $\mathcal{L}(X_1) \in L_0(b,Q)$. (Here we do not use naturalness. Similarly we can prove $\mathcal{L}(X_t) \in L_0(b,Q)$ for all $t$.)

*The 'only if' part.* If we construct from $\mu \in L_0(b,Q)$ a system of distributions $\{\mu_t \colon 1 \leqslant t \leqslant a\}$ on $\mathbb{R}^d$ such that (1) $\mu_1 = \mu$, (2) $\widehat{\mu}_a(z) = \widehat{\mu}(a^{Q'}z)$, (3) there is a distribution $\mu_{s,t}$ for $1 \leqslant s \leqslant t \leqslant a$ such that $\mu_t = \mu_s * \mu_{s,t}$ , and (4) $\widehat{\mu}_t(z)$ is continuous in $t \in [1,a]$, then there is, uniquely in law, a $Q$-semi-selfsimilar additive process $X$ with epoch $a$ such that $\mathcal{L}(X_t) = \mu_t$ for $t \in [1,a]$. This is verified in the same way as the proof of Theorem 7 of [10]. A construction of such a system $\{\mu_t\}$ is as follows. Recall that $\mu \in ID$ (see [11]). Define $\mu_t$ for $1 \leqslant t \leqslant a$ by

$$\widehat{\mu}_t(z) = \widehat{\mu}(z)^{1-h(t)}\widehat{\mu}(a^{Q'}z)^{h(t)} \tag{5.1}$$

with a continuous increasing function $h(t)$ satisfying $h(1) = 0$ and $h(a) = 1$. Then $\{\mu_t\}$ satisfies conditions (1) through (4) above. Indeed, (1), (2), and (4) are obvious. To see (3), let $1 \leqslant s \leqslant t \leqslant a$. Notice that

$$\begin{aligned}
\widehat{\mu}_t(z) &= \widehat{\mu}(z)^{1-h(t)}\widehat{\mu}(a^{Q'}z)^{h(s)}\widehat{\mu}(a^{Q'}z)^{h(t)-h(s)},\\
\widehat{\mu}_s(z) &= \widehat{\mu}(z)^{1-h(t)}\widehat{\mu}(z)^{h(t)-h(s)}\widehat{\mu}(a^{Q'}z)^{h(s)}.
\end{aligned}$$

Since $\mu \in L_0(a^{-1},Q)$, there is $\rho \in ID$ such that $\widehat{\mu}(z) = \widehat{\mu}(a^{-Q'}z)\widehat{\rho}(z)$, that is, $\widehat{\mu}(a^{Q'}z) = \widehat{\mu}(z)\widehat{\rho}(a^{Q'}z)$. Hence $\widehat{\mu}_t(z) = \widehat{\mu}_s(z)\widehat{\rho}(a^{Q'}z)^{h(t)-h(s)}$, which shows that condition (3) is satisfied. It follows from (5.1) that the location parameter in the triplet $(A_t, \nu_t, \gamma_t)$ of $\mu_t$ satisfies $\gamma_t = (1-h(t))\gamma_1 + h(t)\gamma_a$, which is of bounded variation in $t \in [1,a]$. Hence the process $X$ constructed is natural by Theorem 2.13 of [18]. □

*Proof of Theorem* 1.3. *The 'only if' part.* Let $\mu = \mathcal{L}(X_1) = \mathcal{L}(Z_0)$, where $X$ and $Z$ are the processes in Theorem 1.1 or 1.2. Then $\mu \in L_0(a^{-1},Q)$ by Lemma 5.1.

*The 'if' part.* Given $\mu \in L_0(a^{-1},Q)$, use the process $X$ in Lemma 5.1 for the process in Theorem 1.1. □

**Remark 5.2.** The 'only if' part of Theorem 1.3 can be strengthened as follows: the distributions $\mathcal{L}(X_t)$ for all $t \geqslant 0$ and $\mathcal{L}(Z_s)$ for all $s \in \mathbb{R}$ are $(a^{-1}, Q)$-decomposable.

**Remark 5.3.** In the proof of the 'only if' part of Lemma 5.1, the construction of $X$ has freedom of choice of the function $h(t)$ on $[1, a]$. Freedom of choice of systems $\{\mu_t \colon 1 \leqslant t \leqslant a\}$ is even larger, since there exist systems not of the form (5.1). See examples in [10] in the case $Q = cI$ with $c > 0$. This corresponds to the variety of processes $X$ and $Z$ that express the same $\mu$ in Theorem 1.3. See also Remark 8.5. This is in contrast to the situation in the $Q$-selfsimilar case, which we will formulate in Section 6.

**Corollary 5.4.** *Let $Q \in \mathbf{M}_d^+$ and $a > 1$. A distribution $\mu$ on $\mathbb{R}^d$ is $(a^{-1}, Q)$-decomposable if and only if $\mu$ is expressible as*

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-tQ} dY_t\right) \tag{5.2}$$

*by a natural semi-Lévy process $Y = \{Y_t \colon t \geqslant 0\}$ with period $\log a$ with finite log-moment. In particular, a distribution $\mu$ on $\mathbb{R}^d$ is semi-selfdecomposable if and only if $\mu$ is expressible as $\mu = \mathcal{L}\left(\int_0^\infty e^{-tI} dY_t\right)$ by a natural semi-Lévy process $\{Y_t \colon t \geqslant 0\}$ with finite log-moment.*

*Proof.* Use Theorem 1.3 and Lemma 4.8. ☐

## 6. Selfsimilar additive processes, stationary OU type processes, and homogeneous independently scattered random measures

Relations of the three objects in the title of this section are formulated below. These are consequences of Theorems 1.1 through 1.3 except the uniqueness assertions in Theorem 6.3 and Corollary 6.4. Note that any $Q$-selfsimilar additive process is natural (Theorem 2.14 of [18]). When the basic matrix $Q$ equals the identity matrix $I$, these are new formulations of essentially known results.

**Theorem 6.1.** *Let $Q \in \mathbf{M}_d^+$. Let $X = \{X_t \colon t \geqslant 0\}$ be an arbitrary $Q$-selfsimilar additive process on $\mathbb{R}^d$. Define $Z = \{Z_s \colon s \in \mathbb{R}\}$ and $\Lambda = \{\Lambda(B) \colon B \in \mathcal{B}_\mathbb{R}^0\}$ by* (1.4) *and* (1.5)*, respectively. Then $\Lambda$ is an $\mathbb{R}^d$-valued homogeneous i. s. r. m. over $\mathbb{R}$ with finite log-moment. The process $X$ is expressed by $\Lambda$ in the form of* (1.6)*. The process $Z$ is the unique stationary OU type process generated by $\Lambda$ and $Q$; it is expressible in the form of* (1.7)*.*

**Theorem 6.2.** *Let $Q \in \mathbf{M}_d^+$ and let $\Lambda = \{\Lambda(B) \colon B \in \mathcal{B}_\mathbb{R}^0\}$ be an arbitrary $\mathbb{R}^d$-valued homogeneous i. s. r. m. over $\mathbb{R}$ with finite log-moment. Let $Z = \{Z_s \colon s \in \mathbb{R}\}$ be the unique stationary OU type process generated by $\Lambda$ and $Q$. Define $X = \{X_t \colon t \geqslant 0\}$ by* (1.8)*. Then $X$ is a $Q$-selfsimilar additive process on $\mathbb{R}^d$; $Z$ and $\Lambda$ are recovered from $X$ in the form of* (1.4) *and* (1.5)*.*

**Theorem 6.3.** *Fix $Q \in \mathbf{M}_d^+$. A distribution $\mu$ on $\mathbb{R}^d$ given by $\mu = \mathcal{L}(X_1) = \mathcal{L}(Z_0)$ in Theorem* 6.1 *or* 6.2 *is $Q$-selfdecomposable. Conversely, for any $Q$-selfdecomposable distribution $\mu$ on $\mathbb{R}^d$, there is, uniquely in law, an $\mathbb{R}^d$-valued homogeneous i. s. r. m. $\Lambda$ over $\mathbb{R}$ with finite log-moment in Theorem* 6.2 *such that $\mu = \mathcal{L}(X_1) = \mathcal{L}(Z_0)$.*

The relation of $Z$ and $\mu$ in Theorem 6.3 was proved by [20] and [21]; the relation of $X$ and $\mu$ there was proved by [16].

**Corollary 6.4.** *Fix $Q \in \mathbf{M}_d^+$. A distribution $\mu$ on $\mathbb{R}^d$ is $Q$-selfdecomposable if and only if*

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-tQ} dY_t\right) \tag{6.1}$$

*with $Y = \{Y_t \colon t \geqslant 0\}$ being a Lévy process on $\mathbb{R}^d$ with finite log-moment. In this case, $Y$ is determined by $\mu$ uniquely in law.*

This result was directly proved by Wolfe [28] and Jurek [5].

For completeness, we give a proof of the uniqueness assertion in Theorem 6.3. Let $\Lambda$ be an $\mathbb{R}^d$-valued homogeneous i. s. r. m. over $\mathbb{R}$ and let us define $Z$ and $X$ as in Theorem 6.2. Let $\mu = \mathcal{L}(X_1) = \mathcal{L}(Z_0)$. Since $X$ is a $Q$-selfsimilar additive process, its distribution as a stochastic process is determined by $\mu$. Hence, by (1.5), the distribution of $\Lambda$ is determined by $\mu$.

## 7. Further results on selfsimilar and semi-selfsimilar additive processes

Applying Theorems 1.1 and 1.2 to $Q$-selfsimilar and $Q$-semi-selfsimilar additive processes on $\mathbb{R}^d$, we can give characterization of their factorings and provide new examples of $Q$-mild OU type processes.

The following theorem is concerned with $Q$-selfsimilar additive processes.

**Theorem 7.1.** *Let $Q \in \mathbf{M}_d^+$.*

(i) *Let $\rho^0$ be in $ID(\mathbb{R}^d)$ with finite log-moment. Then $(\{\rho_s \colon s \geqslant 0\}, \sigma)$ defined by*

$$\log \widehat{\rho}_s(z) = s^{-1} \log \widehat{\rho^0}(s^{Q'} z) \text{ for } s > 0 \text{ and } \rho_0 = \delta_0 \tag{7.1}$$

*and $\sigma$ = Lebesgue on $[0,\infty)$ is a factoring of a $Q$-selfsimilar natural additive process $X = \{X_t \colon t \geqslant 0\}$.*

(ii) *Any $Q$-selfsimilar additive process $X = \{X_t \colon t \geqslant 0\}$ on $\mathbb{R}^d$ has a factoring $(\{\rho_s\}, \sigma)$ described in* (i) *with some $\rho^0 \in ID(\mathbb{R}^d)$ having finite log-moment.*

*Sketch of proof.* (i) By Proposition 2.11 of [18], $(\{\rho_s\}, \sigma)$ is a factoring of some additive process in law $X^0$ if (1), (2), (3) of Definition 3.8, (3.6), and

(3.7) are satisfied. Among them (1), (2), and (3) are obvious. The triplet $(A^\rho_s, \nu^\rho_s, \gamma^\rho_s)$ of $\rho_s$ for $s > 0$ is expressed as $A^\rho_s = s^{-1}s^Q A^0 s^{Q'}$, $\nu^\rho_s(B) = s^{-1}\int 1_B(s^Q x)\nu^0(dx)$, and

$$\gamma^\rho_s = s^{-1}s^Q \left(\gamma^0 + \int x\, r_s(x)\nu^0(dx)\right),$$

where $(A^0, \nu^0, \gamma^0)$ is the triplet of $\rho^0$ and

$$r_s(x) = c(s^Q x) - c(x) = (|x|^2 - |s^Q x|^2)(1 + |s^Q x|^2)^{-1}(1 + |x|^2)^{-1}.$$

Hence (3.6) holds. Further we can check (3.7) by using (4.9). The process $X^0$ is $Q$-selfsimilar, since

$$\begin{aligned} Ee^{i\langle z, X^0_{at}\rangle} &= \exp \int_0^{at} \log \widehat{\rho^0}(s^{Q'} z)s^{-1}ds \\ &= \exp \int_0^{t} \log \widehat{\rho^0}(s^{Q'} a^{Q'} z))s^{-1}ds = Ee^{i\langle z, a^Q X^0_t\rangle} \end{aligned}$$

for any $a > 0$. Now let $X$ be the cadlag modification of $X^0$.

(ii) By Theorem 6.1, $X_t = \int_{-\infty}^{\log t} e^{sQ}\Lambda(ds)$ a. s. with some $\mathbb{R}^d$-valued homogeneous i. s. r. m. $\Lambda$ over $\mathbb{R}$ with finite log-moment. Let $\rho^0 = \mathcal{L}(\Lambda((0,1]))$. Then, there is a constant $c > 0$ such that $(\{(\rho^0)^c\}, c^{-1}ds)$ is the canonical factoring of $\Lambda$. Hence, using Theorem 5.2 of [18], we can show that $(\{\rho_s\}, \sigma)$ defined in (i) is a factoring of $X$. □

**Example 7.2.** Let $\mu$ be a selfdecomposable distribution on $\mathbb{R}$ with support $[0,\infty)$. Then

$$\widehat{\mu}(z) = \exp \int_0^\infty (e^{izx} - 1)k(x)x^{-1}dx \qquad \text{for} \quad z \in \mathbb{R},$$

where $k(x)$ is a nonnegative decreasing right-continuous function on $(0,\infty)$ with $\int_0^\infty (1 \wedge x^{-1})k(x)dx < \infty$. Let $c > 0$. Let $X = \{X_t : t \geqslant 0\}$ be the $c$-selfsimilar additive process on $\mathbb{R}$ with $\mathcal{L}(X_1) = \mu$. Define a measure $\eta$ on $(0,\infty)$ by $\eta((y,\infty)) = k(y)$ and $\rho^0 \in ID(\mathbb{R})$ by

$$\log \widehat{\rho^0}(z) = c\int_0^\infty (e^{izy} - 1)\eta(dy) .$$

Note that $\int_{(0,2]} y\eta(dy) + \int_{(2,\infty)} \log y\eta(dy) < \infty$, which follows from $\int_0^1 k(x)dx + \int_1^\infty k(x)x^{-1}dx < \infty$. Then

$$\begin{aligned} Ee^{izX_t} = Ee^{izt^c X_1} &= \exp \int_0^\infty (e^{izx} - 1)k(t^{-c}x)x^{-1}dx \\ &= \exp \int_0^\infty (e^{izx} - 1)x^{-1}dx \int_0^\infty 1_{\{y > t^{-c}x\}}\eta(dy) \end{aligned}$$

$$= \exp \int_0^\infty \eta(dy) \int_0^t (e^{izs^c y} - 1) cs^{-1} ds = \exp \int_0^t s^{-1} \log \widehat{\rho^0}(s^c z) ds \,.$$

This is exactly the relation between $\rho^0$ and $X$ in Theorem 7.1.

Next, let us study $Q$-semi-selfsimilar additive processes.

**Theorem 7.3.** *Let $a > 1$ and $Q \in \mathbf{M}_d^+$.*

(i) *Let $X = \{X_t : t \geqslant 0\}$ be a natural additive process on $\mathbb{R}^d$ with a factoring $(\{\rho_s\}, \sigma)$ satisfying the following two conditions*:

$$\log \widehat{\rho}_{as}(z) = a^{-1} \log \widehat{\rho}_s(a^{Q'} z) \qquad \textit{for} \quad s > 0, \tag{7.2}$$

$$\sigma(ds) = s\sigma^\sharp(ds), \tag{7.3}$$

*where $\sigma^\sharp$ is a locally finite continuous measure on $[0, \infty)$ such that*

$$\int f(as)\sigma^\sharp(ds) = \int f(s)\sigma^\sharp(ds) \tag{7.4}$$

*for all nonnegative measurable $f$. Then, $X$ is $Q$-semi-selfsimilar with epoch $a$.*

(ii) *Any $Q$-semi-selfsimilar natural additive process $X = \{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$ with epoch $a$ has a factoring $(\{\rho_s\}, \sigma)$ satisfying the conditions above.*

*Sketch of proof.* (i) We can prove $Ee^{i\langle z, X_{at}\rangle} = Ee^{i\langle z, a^Q X_t\rangle}$ from (7.2) and (7.3).

(ii) Let $p = \log a$. Use the expression of $X$ in Theorem 1.1 by an $\mathbb{R}^d$-valued $p$-periodic i. s. r. m. $\Lambda$ over $\mathbb{R}$ with finite log-moment. The canonical factoring $(\{\rho_s^\Lambda\}, \sigma^\Lambda)$ of $\Lambda$ is $p$-periodic. Define $\sigma^\sharp$ by $\sigma^\sharp(B) = \int 1_B(e^s)\sigma^\Lambda(ds)$. Define $\sigma$ and $\rho_s$ by $\sigma(ds) = s\sigma^\sharp(ds)$ and by $\log \widehat{\rho}_s(z) = s^{-1} \log \widehat{\rho}^\Lambda_{\log s}(s^{Q'} z)$. Then (7.2) and (7.4) are satisfied. By these properties and Theorem 5.2 of [18], we can show that $(\{\rho_s\}, \sigma)$ is a factoring of $X$. □

Let us show that $Q$-semi-selfsimilar additive processes induce $R$-mild OU type processes for any $R \in \mathbf{M}_d^+$.

**Theorem 7.4.** *Let $N = \{N(B) : B \in \mathcal{B}_{\mathbb{R}}^0\}$ be an $\mathbb{R}^d$-valued i. s. r. m. over $\mathbb{R}$. Suppose that the process $X = \{X_t : t \geqslant 0\}$ defined by $X_t = N((-t, 0])$ is, for some $Q \in \mathbf{M}_d^+$, a $Q$-semi-selfsimilar additive process in law with some epoch $a$. Then, for any $R \in \mathbf{M}_d^+$, Langevin equation based on $N$ and $R$ has a unique $R$-mild solution.*

This result is new even if $X$ is a $cI$-selfsimilar additive process with $c > 0$.

*Sketch of proof of Theorem* 7.4. By Theorem 4.3, an $R$-mild solution is unique if it exists. By the same theorem, in order to prove our assertion, it is enough to show that $\int_{-\infty}^0 e^{sR} N(ds)$ is definable. Let $(\{\rho_s\}, \sigma)$ be the factoring of $X$ given in the proof of Theorem 7.3 (ii). Then it suffices to prove that

$$\int_0^\infty |\log \widehat{\rho}_s(e^{-sR'} z)| \, \sigma(ds) < \infty \qquad \text{for} \quad z \in \mathbb{R}^d.$$

Let $(A_s^\rho, \nu_s^\rho, \gamma_s^\rho)$ be the triplet of $\rho_s$. As in the proof of Theorem 7.3 (ii), let $\Lambda$ be an $\mathbb{R}^d$-valued $p$-periodic i. s. r. m. over $\mathbb{R}$ such that (1.6) holds. Let $(\{\rho_s^\Lambda\}, \sigma^\Lambda)$ be the $p$-periodic canonical factoring of $\Lambda$. For any positive integer $m$,

$$\int_{e^{mp}}^{\infty} |\log \widehat{\rho}_s(e^{-sR'}z)|\, \sigma(ds) = \sum_{n=m}^{\infty} \int_0^p |\log \widehat{\rho}_s^\Lambda(e^{(np+s)Q'} e^{-e^{np+s}R'} z)|\, \sigma^\Lambda(ds)\ .$$

For any $F \in \mathbf{M}_d$,

$$\begin{aligned}|\log \widehat{\rho}_s^\Lambda(F'z)| \leqslant \frac{1}{2}|z|^2\|FA_s^{\rho\Lambda}F'\| + |z|\,|F\gamma_s^{\rho\Lambda}| + C_z \int_{\mathbb{R}^d} \frac{|Fx|^2}{1+|Fx|^2}\nu_s^{\rho\Lambda}(dx)\\ + |z| \int_{\mathbb{R}^d} \frac{|Fx|\,\big||x|^2 - |Fx|^2\big|}{(1+|Fx|^2)(1+|x|^2)}\nu_s^{\rho\Lambda}(dx)\ ,\end{aligned}$$

where $(A_s^{\rho\Lambda}, \nu_s^{\rho\Lambda}, \gamma_s^{\rho\Lambda})$ is the triplet of $\rho_s^\Lambda$ and $C_z$ is a constant depending on $z$ but independent of $s$ and $F$. Now use the estimate (4.9) for $Q$ and a similar estimate for $R$. We also use the decomposition of the Lévy measure of the semi-Lévy process $Y$ defined by $Y_s = \Lambda((0,s])$ into $\nu^{*\Lambda}$ and $\sigma_x^{*\Lambda}$ as in Lemma 5.3 of [18]. In this way we can show that $\int_{e^{mp}}^\infty |\log \widehat{\rho}_s(e^{-sR'}z)|\,\sigma(ds)$ is finite for some $m$. □

**Remark 7.5.** Notice that, in Theorem 7.4, the restriction of $N$ to $(-\infty, 0]$ may or may not have finite log-moment. Thus the finite log-moment property of the underlying i. s. r. m. is not a necessary condition for the existence of $R$-mild OU type processes for $R \in \mathbf{M}_d^+$. On the other hand, it is a simple consequence of Theorem 4.3 that an $R$-mild solution of Langevin equation based on i. s. r. m. $N$ and $R \in \mathbf{M}_d^+$ may not exist even if $N$ has finite log-moment.

## 8. Results and examples related to semi-stability

In this section, let $Q \in \mathbf{M}_d^+$ and $b \in (0,1)$. For $\alpha > 0$, a distribution $\mu$ on $\mathbb{R}^d$ is called semi-stable with index $\alpha$ and span $b^{-1}$ if $\mu \in ID$ and

$$\widehat{\mu}(z)^{b^\alpha} = \widehat{\mu}(bz)e^{i\langle\gamma,z\rangle} \qquad \text{for} \quad z \in \mathbb{R}^d \tag{8.1}$$

for some $\gamma \in \mathbb{R}^d$. In order that such a nontrivial (that is, not concentrated at a point) distribution $\mu$ exists, we must have $\alpha \leqslant 2$. We extend this notion. Considering the definition of the class $OSS(b,Q)$ of operator semi-stable distributions in [12, p. 236], we call a distribution $\mu$ on $\mathbb{R}^d$ $(b,Q)$-*semi-stable* if $\mu \in ID$ and, for some $a \in (0,1)$ and $\gamma \in \mathbb{R}^d$,

$$\widehat{\mu}(z)^a = \widehat{\mu}(b^{Q'}z)e^{i\langle\gamma,z\rangle} \qquad \text{for} \quad z \in \mathbb{R}^d. \tag{8.2}$$

Expressing $a$ explicitly, we say that $\mu$ is $(b,Q,a)$-*semi-stable* if $\mu \in ID$ and (8.2) holds with some $\gamma$. If $\mu \in ID$ and (8.2) holds with $\gamma = 0$, we say that $\mu$ is *strictly* $(b,Q,a)$-*semi-stable*. An additive or Lévy process $X = \{X_t : t \geqslant 0\}$ is said to be $(b, Q, a)$-semi-stable (resp. strictly $(b, Q, a)$-semi-stable) if $\mathcal{L}(X_t)$ is $(b, Q, a)$-semi-stable (resp. strictly $(b, Q, a)$-semi-stable) for all $t$. In this section we give some remarks on representations of $(b, Q, a)$-semi-stable distributions in application of our main theorems. We also give examples of $Q$-semi-selfsimilar processes connected with processes in the study of diffusion processes in semi-stable random environments.

We give two basic lemmas.

**Lemma 8.1.** *If $\mu$ is $(b, Q, a)$-semi-stable on $\mathbb{R}^d$, then $\mu \in L_0(b, Q)$.*

*Proof.* It follows from (8.2) that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{Q'} z)^{a^{-1}} e^{i\langle a^{-1}\gamma, z\rangle} = \widehat{\mu}(b^{Q'} z)\widehat{\mu}(b^{Q'} z)^{a^{-1}-1} e^{i\langle a^{-1}\gamma, z\rangle} .$$

Since $\widehat{\mu}(b^{Q'} z)^{a^{-1}-1} e^{i\langle a^{-1}\gamma, z\rangle}$ is infinitely divisible, we have the decomposition (1.1) with $\rho_b \in ID$. □

**Lemma 8.2.** *If $\mu$ is $(b, Q, a)$-semi-stable on $\mathbb{R}^d$, then there is $c \in (0, \infty)$ such that $\int_{\mathbb{R}^d} |x|^c \mu(dx) < \infty$.*

*Proof.* See Łuczak [8]. The special case of $Q = I$ is treated in [17]. □

It follows from this lemma that any $(b, Q, a)$-semi-stable distribution has finite log-moment.

**Proposition 8.3.** *Let $Y = \{Y_t \colon t \geqslant 0\}$ be a Lévy process on $\mathbb{R}^d$ with finite log-moment. Then $\mathcal{L}\left(\int_0^\infty e^{-tQ} dY_t\right)$ is $(b, Q, a)$-semi-stable if and only if $Y$ is a $(b, Q, a)$-semi-stable Lévy process. The statement with the word 'strictly' added in both conditions is also true.*

*Proof.* Let $\rho = \mathcal{L}(Y_1)$ and $\mu = \mathcal{L}\left(\int_0^\infty e^{-tQ} dY_t\right)$. Since $Y$ is a Lévy process, it is $(b, Q, a)$-semi-stable if $\rho$ is $(b, Q, a)$-semi-stable. We have, by Theorem 5.2 of [18],

$$\int_0^\infty \sup_{|z| \leqslant a} |\log \widehat{\rho}(e^{-tQ'} z)| dt < \infty \qquad \text{for} \quad a \in (0, \infty)$$

and $\log \widehat{\mu}(z) = \int_0^\infty \log \widehat{\rho}(e^{-tQ'} z) dt$.

If $\rho$ is $(b, Q, a)$-semi-stable, then, with some $\gamma$,

$$\begin{aligned}\log \widehat{\mu}(b^{Q'} z) &= \int_0^\infty \log \widehat{\rho}(e^{-tQ'} b^{Q'} z) dt = \int_0^\infty (a \log \widehat{\rho}(e^{-tQ'} z) - i\langle \gamma, e^{-tQ'} z\rangle) dt \\ &= a \log \widehat{\mu}(z) - i\langle Q^{-1}\gamma, z\rangle,\end{aligned}$$

that is, $\mu$ is $(b, Q, a)$-semi-stable.

Conversely, assume that $\mu$ is $(b, Q, a)$-semi-stable. Then, with some $\gamma$,

$$\int_0^\infty \log \widehat{\rho}(e^{-tQ'} b^{Q'} z) dt = a \int_0^\infty \log \widehat{\rho}(e^{-tQ'} z) dt - i\langle \gamma, z\rangle .$$

Since $z$ is arbitrary, we have

$$\int_0^\infty \log \widehat{\rho}(b^{Q'} e^{-(t+u)Q'} z)dt = a \int_0^\infty \log \widehat{\rho}(e^{-(t+u)Q'} z)dt - i\langle \gamma, e^{-uQ'} z\rangle$$

for $u \in \mathbb{R}$. That is,

$$\int_u^\infty \log \widehat{\rho}(b^{Q'} e^{-tQ'} z)dt = a \int_u^\infty \log \widehat{\rho}(e^{-tQ'} z)dt - i\langle \gamma, e^{-uQ'} z\rangle \ .$$

Differentiating in $u$ and letting $u = 0$, we obtain

$$\log \widehat{\rho}(b^{Q'} z) = a \log \widehat{\rho}(z) - i\langle Q\gamma, z\rangle \ ,$$

which shows that $\rho$ is $(b, Q, a)$-semi-stable. The assertion for strict $(b, Q, a)$-semi-stability is proved with $\gamma = 0$. □

The class of distributions on $\mathbb{R}^d$ which are $(b, Q, a)$-semi-stable with some $b$ and $a$ neither includes, nor is included by, the class of $Q$-selfdecomposable distributions. Concerning the intersection of the two classes we have the following assertion.

**Proposition 8.4.** *A distribution $\mu$ on $\mathbb{R}^d$ is $Q$-selfdecomposable and $(b, Q, a)$-semi-stable if and only if there is a $(b, Q, a)$-semi-stable Lévy process $Y$ on $\mathbb{R}^d$ such that*

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-tQ} dY_t\right) \ . \tag{8.3}$$

*The statement with the word 'strictly' added in both conditions is also true.*

*Proof.* *The 'if' part.* By Lemma 8.2, the integral in (8.3) is definable. It follows from (8.3) that $\mu$ is $Q$-selfdecomposable by Corollary 6.4 and that $\mu$ is $(b, Q, a)$-semi-stable by Proposition 8.3.

*The 'only if' part.* By $Q$-selfdecomposability, $\mu$ is represented in the form of (8.3) with a unique (in law) Lévy process $Y$ with finite log-moment by Corollary 6.4. Then, using Proposition 8.3, we see that $Y$ is $(b, Q, a)$-semi-stable.

The case of strict $(b, Q, a)$-semi-stability is similar. □

**Remark 8.5.** Let $\mu$ be $Q$-selfdecomposable and $(b, Q, a)$-semi-stable on $\mathbb{R}^d$. Then $\mu$ is $(c, Q)$-decomposable for any $c \in (0, 1)$. Thus $\mu$ has a unique representation (8.3) with a Lévy process $Y$ on one hand and representation (5.2) on the other. Let us denote by $Y^\sharp = \{Y_t^\sharp\}$ the natural semi-Lévy process $\{Y_t\}$ with period $\log(1/c)$ appearing in (5.2). Of course (8.3) is a special case of (5.2). That is, $Y$ in (8.3) is one of many choices of $Y^\sharp$. There is a unique (in law) $Q$-selfsimilar additive process $X$ with $\mathcal{L}(X_1) = \mu$; the process $Y$ is connected with this $X$ (Theorems 6.1 through 6.3 and Lemma 4.8). We can construct $Q$-semi-selfsimilar natural additive processes $X^\sharp$ with epoch $c^{-1}$ with $\mathcal{L}(X_1^\sharp) = \mu$ (see Lemma 5.1) in many ways and the processes $Y^\sharp$ are connected with these $X^\sharp$ (Corollary 5.4). Thus $X$ is a special case of the processes $X^\sharp$. But, in general, no choice of the function $h(t)$ on $[1, c^{-1}]$ in the Proof of Lemma 5.1 gives the $Q$-selfsimilar process $X$. Let us see this fact when $c = b$. It follows from (8.2) that the distribution $\mu_t$ in (5.1) satisfies

$$\widehat{\mu}_t(z) = \widehat{\mu}(z)^{1-h(t)}\widehat{\mu}(b^{-Q'}z)^{h(t)} = \widehat{\mu}(z)^{1-h(t)+a^{-1}h(t)}e^{i\langle a^{-1}h(t)b^{-Q}\gamma, z\rangle}$$

for $1 \leqslant t \leqslant b^{-1}$, since $\widehat{\mu}(b^{-Q'}z) = \widehat{\mu}(z)^{a^{-1}}e^{i\langle a^{-1}b^{-Q}\gamma, z\rangle}$. If this system $\{\mu_t\}$ satisfies $\mu_t = \mathcal{L}(X_t)$ for a $Q$-selfsimilar additive process $X$, then $\widehat{\mu}_{rt}(z) = \widehat{\mu}_t(r^{Q'}z)$ for $t > 0$ and $r > 0$ and hence $\widehat{\mu}_t(z) = \widehat{\mu}(t^{Q'}z)$. Thus, in this case, the Lévy measure $\nu$ of $\mu$ satisfies, for $1 \leqslant t \leqslant b^{-1}$,

$$\nu(t^{-Q}B) = \nu_t(B) = (1 + (a^{-1} - 1)h(t))\,\nu(B) \qquad \text{for} \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where $\nu_t$ is the Lévy measure of $\mu_t$. In general, no choice of the function $h(t)$ validates this relation. For example, consider a $(b, I, b^\alpha)$-semi-stable distribution $\mu$ with Lévy measure $\nu = \sum_{n=-\infty}^{\infty} b^{n\alpha}\delta_{b^{-n}c}$ with $0 < \alpha < 2$ and $1 < |c| \leqslant b^{-1}$ in Remark 14.4 of [17]. Then

$$\nu(t^{-I}B) = \sum_{n=-\infty}^{\infty} b^{n\alpha}\delta_{tb^{-n}c}(B)$$

while

$$(1 + (b^{-\alpha} - 1)h(t))\,\nu(B) = (1 + (b^{-\alpha} - 1)h(t))\sum_{n=-\infty}^{\infty} b^{n\alpha}\delta_{b^{-n}c}(B).$$

In the rest of this section we consider some examples appearing in the study of diffusion processes in semi-stable random environments. It consists of two parts.

*Part* 1. Let $X = \{X_t \colon t \geqslant 0\}$ be a $c$-semi-selfsimilar process on $\mathbb{R}$ with epoch $a$, where $c > 0$ and $a > 1$. Assume that $X$ has cadlag paths and that

$$\limsup_{t\to\infty}\left(X_t - \inf_{s\leqslant t} X_s\right) = \infty \quad \text{a. s.} \tag{8.4}$$

Define, for $t \geqslant 0$,

$$\begin{aligned} M_t &= \inf\left\{u \geqslant 0 \colon X_u - \inf_{s\leqslant u} X_s \geqslant t\right\}\,, \\ V_t &= -\inf\{X_s \colon s \leqslant M_t\}\,, \\ N_t &= \inf\{u \in [0, M_t] \colon X_u \wedge X_{u-} = -V_t\}\,, \end{aligned}$$

where we understand $X_{0-} = X_0$. Denote by $\operatorname{diag}(a_1, \ldots, a_d)$ a $d \times d$ diagonal matrix with $(j, j)$-entry equal to $a_j$.

**Proposition 8.6.** *Let $Q = \operatorname{diag}(c^{-1}, 1, c^{-1})$. Under the assumptions above,*

$$\{(M_{a^ct}, V_{a^ct}, N_{a^ct})' \colon t \geqslant 0\} \stackrel{\mathrm{d}}{=} \{a^{cQ}(M_t, V_t, N_t)' \colon t \geqslant 0\}\,, \tag{8.5}$$

*that is, the process $\{(M_t, V_t, N_t)' \colon t \geqslant 0\}$ is $Q$-semi-selfsimilar with epoch $a^c$.*

*Proof.* Notice that $\{a^{-c}X_t\} \stackrel{\mathrm{d}}{=} \{X_{a^{-1}t}\}$. We get $M_{a^c t} \stackrel{\mathrm{d}}{=} aM_t$, since

$$\begin{aligned} M_{a^c t} &= \inf\left\{u \geqslant 0\colon a^{-c}\left(X_u - \inf_{s\leqslant u} X_s\right) \geqslant t\right\} \\ &\stackrel{\mathrm{d}}{=} \inf\left\{u \geqslant 0\colon X_{a^{-1}u} - \inf_{s\leqslant u} X_{a^{-1}s} \geqslant t\right\} \\ &= \inf\left\{u \geqslant 0\colon X_{a^{-1}u} - \inf_{s\leqslant a^{-1}u} X_s \geqslant t\right\} \\ &= a\inf\left\{u \geqslant 0\colon X_u - \inf_{s\leqslant u} X_s \geqslant t\right\} = aM_t\ . \end{aligned}$$

Similarly we have $V_{a^c t} \stackrel{\mathrm{d}}{=} a^c V_t$ and $N_{a^c t} \stackrel{\mathrm{d}}{=} aN_t$ in the following way:

$$\begin{aligned} V_{a^c t} &= -\inf\{X_s\colon s \leqslant M_{a^c t}\} = -a^c \inf\{a^{-c}X_s\colon s \leqslant M_{a^c t}\} \\ &\stackrel{\mathrm{d}}{=} -a^c\{X_{a^{-1}s}\colon s \leqslant aM_t\} = a^c V_t\ , \\ N_{a^c t} &= \inf\{u \in [0, M_{a^c t}]\colon X_u \wedge X_{u-} = -V_{a^c t}\} \\ &= \inf\{u \in [0, M_{a^c t}]\colon (a^{-c}X_u) \wedge (a^{-c}X_{u-}) = -a^{-c}V_{a^c t}\} \\ &\stackrel{\mathrm{d}}{=} \inf\{u \in [0, aM_t]\colon X_{a^{-1}u} \wedge X_{a^{-1}u-} = -V_t\} = aN_t\ . \end{aligned}$$

In checking the identities in law above, the only transformation involved is that of $\{a^{-c}X_t\} \stackrel{\mathrm{d}}{=} \{X_{a^{-1}t}\}$. Hence the same proof applies to the joint distributions of the three processes $\{M_t\}$, $\{X_t\}$, and $\{N_t\}$. Thus we get $\{(M_{a^c t}, V_{a^c t}, N_{a^c t})'\}$ $\stackrel{\mathrm{d}}{=} \{(aM_t, a^c V_t, aN_t)'\}$. Since $a^{cQ} = \mathrm{diag}(a, a^c, a)$, this means (8.5). ☐

*Part* 2. Suppose that $X = \{X_t\colon t \geqslant 0\}$ is a strictly $(b, I, b^\alpha)$-semi-stable Lévy process on $\mathbb{R}$ with $0 < b < 1$ and $0 < \alpha \leqslant 2$. That is, $X$ is a Lévy process satisfying $\{X_{b^\alpha t}\} \stackrel{\mathrm{d}}{=} \{bX_t\}$. Hence, $X$ is a $c$-semi-selfsimilar Lévy process on $\mathbb{R}$ with epoch $a$, where $c = \alpha^{-1}$ and $a = b^{-\alpha}$. Approaching a generalization of Tanaka's paper [24] on diffusion processes in Brownian or symmetric stable environments, Takahashi [23] obtains the following results for this process.

**Proposition 8.7.** *Assume that $X$ satisfies* (8.4). *Then $\{N_t\}$ is an additive process* (*hence, it is an $\alpha$-semi-selfsimilar additive process with epoch $b^{-1}$*).

**Proposition 8.8.** *Assume, in addition to* (8.4), *that $X$ does not have positive jumps. Then $\{M_t\}$ and $\{V_t\}$ are also additive processes* (*hence, $\{M_t\}$ and $\{V_t\}$ are, respectively, $\alpha$-semi-selfsimilar and* 1*-semi-selfsimilar additive processes with epoch $b^{-1}$*).

A process $X$ on $\mathbb{R}$ satisfies the assumptions in Proposition 8.8 if and only if $X$ is either a nonzero constant multiple of Brownian motion or a nonzero strictly $(b, I, b^\alpha)$-semi-stable Lévy process with $0 < b < 1$ and $1 < \alpha < 2$ having Lévy measure concentrated on the negative axis.

K. Kawazu finds that, in the case of Brownian motion on $\mathbb{R}$, the process $\{(V_t, N_t)'\}$ is an additive process on $\mathbb{R}^2$ but the process $\{(M_t, V_t, N_t)'\}$ is not an additive process on $\mathbb{R}^3$ (see Example 3.3 of [16]). We do not know to what extent this fact can be generalized.

Makoto Maejima
Department of Mathematics
Keio University
Hiyoshi, Yokohama 223-8522, Japan
e-mail: maejima@math.keio.ac.jp

Ken-iti Sato
Hachiman-yama 1101-5-103
Tenpaku-ku, Nagoya 468-0074, Japan
e-mail: ken-iti.sato@nifty.ne.jp

## References

[1] O. E. Barndorff-Nielsen and N. Shephard, *Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics* (*with discussion*), J. Royal Statist. Soc. Series B **63** (2001), 167–241.

[2] P. Cheridito, H. Kawaguchi and M. Maejima, *Fractional Ornstein-Uhlenbeck processes*, Electric Journal of Probability **8** (2003), Paper 3, 1–14.

[3] J.L. Doob, *The Brownian movement and stochastic equations*, Ann. of Math. **43** (1942), 351–369.

[4] M. Jeanblanc, J. Pitman and M. Yor, *Self-similar processes with independent increments associated with Lévy and Bessel processes*, Stochastic Process. Appl. **100** (2002), 223–231.

[5] Z. J. Jurek, *An integral representation of operator-selfdecomposable random variables*, Bull. Acad. Polonaise Sci. Sér. Sci. Math. **30** (1982), 385–393.

[6] Z. J. Jurek and W. Vervaat, *An integral representation for self-decomposable Banach space valued random variables*, Zeit. Wahrsch. Verw. Gebiete **62** (1983), 247–262.

[7] J. Lamperti, *Semi-stable stochastic processes*, Trans. Amer. Math. Soc. **104** (1962), 62–78.

[8] A. Łuczak, *Operator semi-stable probability measures on* $\mathbf{R}^N$, Colloq. Math. **45** (1981), 287–300; Corrigenda **52** (1987), 167–169.

[9] M. Maejima and Y. Naito, *Semi-selfdecomposable distributions and a new class of limit theorems*, Probab. Theory Related Fields **112** (1998), 13–31.

[10] M. Maejima and K. Sato, *Semi-selfsimilar processes*, J. Theoret. Probab. **12** (1999), 347–373.

[11] M. Maejima, K. Sato and T. Watanabe, *Operator semi-selfdecomposability, $(C, Q)$-decomposability and related nested classes*, Tokyo J. Math. **22** (1999), 473–509.

[12] ———, *Completely operator semi-selfdecomposable distributions*, Tokyo J. Math. **23** (2000), 235–253.

[13] T. Mikosch and R. Norvaiša, *Stochastic integral equations without probability*, Bernoulli **6** (2000), 401–434.

[14] B. S. Rajput and J. Rosinski, *Spectral representations of infinitely divisible processes*, Probab. Theory Related Fields **82** (1989), 451–487.

[15] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman & Hall, New York, 1994.

[16] K. Sato, *Self-similar processes with independent increments*, Probab. Theory Related Fields **89** (1991), 285–300.

[17] ———, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, Cambridge, 1999.

[18] ———, *Stochastic integrals in additive processes and application to semi-Lévy processes*, Osaka J. Math. to appear.

[19] K. Sato, T. Watanabe, K. Yamamuro and M. Yamazato, *Multidimensional process of Ornstein-Uhlenbeck type with nondiagonalizable matrix in linear drift terms*, Nagoya Math. J. **141** (1996), 45–78.

[20] K. Sato and M. Yamazato, *Stationary processes of Ornstein-Uhlenbeck type*, Probability Theory and Mathematical Statistics, Fourth USSR-Japan Symp., Proc. 1982, Lecture Notes in Math. **1021**, Springer, Berlin, 1983, pp. 541–551.

[21] K. Sato and M. Yamazato, *Operator-selfdecomposable distributions as limit distributions of processes of Ornstein-Uhlenbeck type*, Stochastic Process. Appl. **17** (1984), 73–100.

[22] D. Surgailis, J. Rosinski, V. Mandrekar and S. Cambanis, *On the mixing structure of stationary increment and self-similar $S\alpha S$ processes*, preprint.

[23] H. Takahashi, *One dimensional diffusion processes in semi-selfsimilar random environments*, preprint.

[24] H. Tanaka, *Limit distributions for one-dimensional diffusion processes in self-similar random environments*, Hydrodynamic Behavior and Interacting Particle Systems, IMA Volumes **9**, Springer, 1987, pp. 189–210.

[25] K. Urbanik and W. A. Woyczynski, *Random integrals and Orlicz spaces*, Bull. Acad. Polon. Sci. **15** (1967), 161–169.

[26] T. Watanabe, *Sato's conjecture on recurrence conditions for multidimensional processes of Ornstein-Uhlenbeck type*, J. Math. Soc. Japan **50** (1998), 155–168.

[27] S. J. Wolfe, *On a continuous analogue of the stochastic difference equation* $X_n = \rho X_{n-1} + B_n$, Stochastic Process. Appl. **12** (1982), 301–312.

[28] ———, *A characterization of certain stochastic integrals*, (Tenth Conference on Stochastic Processes and Their Applications, Contributed Papers) Stochastic Process. Appl. **12** (1982), 136–136.