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MAHARAM-TYPES AND LYAPUNOV'S THEOREM FOR VECTOR MEASURES ON BANACH SPACES

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This paper is respectfully dedicated to the memory of Professor Jerry J. Uhl, Jr. whose interest in the interplay between measure theory and the geometry of Banach spaces is a constant inspiration

ABSTRACT. This paper offers a sufficient condition, based on Maharam (*Proc. Natl. Acad. Sci. USA* **28** (1942) 108–111) and re-emphasized by Hoover and Keisler (*Trans. Amer. Math. Soc.* **286** (1984) 159–201), for the validity of Lyapunov's theorem on the range of a nonatomic vector measure taking values in an infinite-dimensional Banach space that is not necessarily separable nor has the Radon–Nikodym property (RNP). In particular, we obtain an extension of a corresponding result due to Uhl (*Proc. Amer. Math. Soc.* **23** (1969) 158–163). The proposed condition is also shown to be necessary in the sense formalized by Keisler and Sun (*Adv. Math.* **221** (2009) 1584–1607), and thereby closes a question of long-standing as regards an infinite-dimensional generalization of the theorem. The result is applied to obtain short simple proofs of recent results on the convexity of the integral of a set-valued function, and on the characterization of restricted cores of a saturated economy.

1. Introduction

Along with the Brouwer–Kakutani–Fan–Glicksberg fixed point theorems, and versions of the Hahn–Banach theorem, Lyapunov's theorem on the range of a nonatomic finite-dimensional vector measure is a staple of modern mathematical economics, specifically general equilibrium and game theory; see the references furnished in [22]. To be sure, the relevance of the result goes be-

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yond mathematical economics to several areas in applied mathematics, including statistical decision and optimal control theory. Indeed, the impetus for an infinite-dimensional generalization of the theorem has initially come from these quarters; see [26, Chapters V and VI] and their references [25], [27], [40], [46]. These references spell out the trajectory of how, with an additional condition, the closure operation on the range in Uhl's approximate version of the Lyapunov theorem can be dropped, and the range space generalized from a Banach space with the Radon–Nikodym property (RNP) to a general Banach space. This additional condition used the concept of a *thin* set, based on a notion that can be traced to [25] in its formulation of a requirement on the noninjectivity of an integration operator; see Knowles' theorem in [26, Theorem V.1], in [2], and the discussion in Section 3 below.

It bears emphasis, however, that in mathematical economics, it is not so much Lyapunov's theorem itself that has proved to be the result of substantive consequence and use, but rather its straightforward corollary pertaining to the integral of a set-valued mapping (synonymously, multifunction) on a nonatomic probability space. It is this result that is *the* vehicle for the formalization of the intuition that "aggregation eliminates nonconvexity," and thereby allows a substitution of a nonatomic multiplicity for the convexity assumption. However, this convexity property, while true for a multifunction on a nonatomic probability space with a finite-dimensional Euclidean space as its range, is only approximately true in general if (i) the range space is infinite-dimensional, or (ii) the domain consists of a "large but finite" index set; see [20], [21] and their references. This approximation testifies to the fact that Lyapunov's theorem is false if either of the twin assumptions of nonatomicity and finite-dimensionality are dispensed with; see [6], [46]. These approximation results rely on the Shapley–Folkman theorem in the first instance, and on Uhl's approximate Lyapunov theorem in the second where it is the closure of the integral, rather than the integral itself, that is shown to be convex: see [21] for the latter case and [22] for references to the former; [33] is the original statement.

It is thus not surprising that in the last decade and a half, there has been a sustained attempt in mathematical economics to eliminate the closure operation in the theory of integration of a multifunction defined on a nonatomic probability space and taking values in an infinite-dimensional Banach space. This has proceeded in what can now be seen as two well-identified steps: work on multifunctions defined on (i) a Loeb probability space, first introduced in Loeb [31], (ii) a saturated probability space, first introduced in Hoover and Keisler [14] (also see the comprehensive exposition in [11]) in the form of a saturated filtration for the systematic study of the existence of strong solutions for stochastic integral equations. There are two observations from [14] that are relevant in this connection: first, the saturation property can be directly connected to Maharam's [34] classification of measure algebras; and second, in the context of single measure space, Loeb spaces satisfy the property, and therefore can be regarded as a special case of a saturated space. Moreover, as stressed in [18, Acknowledgement], and also in [45] and [32, Remarks 2.5 and 2.6], results on Loeb spaces can be transferred in a straightforward way from Loeb spaces to saturated spaces. Indeed, Loeb spaces can be eschewed altogether, and results on integration such as those in [44] proved directly from the results furnished in [18] on distributions of multifunctions defined on saturated spaces. To be sure, once the connection to Maharam's work is made, and the saturated property identified, loosely speaking, as the requirement that the restriction of the σ-algebra to any set of positive measure be *not* countably-generated, one can ask whether results such as those in [43], [44] proved solely from this identification, which is to say without relying on the Hoover–Keisler definition at all. Such alternative direct proofs have been systematically presented in [35]. Thus, whatever the route, all approaches led to the same consequence and we now have a well-articulated theory of the convexity of the integral, rather than the closure of the integral, of a set-valued mapping taking values in an infinite-dimensional Banach space. What is of especial and key interest in all of this work is the demonstration that a saturated space is not only sufficient but also *necessary* for the result, as summed up by Keisler and Sun [18] in their pithy aphorism that "any probability space that 'out-performs' the Lebesgue unit interval in almost any way at all is already saturated"(see [18, p. 1585]).

A natural question pertains to the Lyapunov theorem itself: do its conclusions regarding the convexity and closure of the range hold for a vector measure taking values in an infinite-dimensional Banach space but defined on a saturated space, and dispensing with the noninjectivity condition of the earlier control-theory literature? We answer this question in this paper. We give a complete characterization of Lyapunov's theorem, one that identifies a saturated measure space to be necessary and sufficient for the validity of its conclusions in the case of an infinite-dimensional Banach space that is not necessarily separable nor has the RNP. To be sure, the equivalence of the exact convexity results for the range of a vector measure with the RNP and the integral of multifunction is quite standard; see [44, p. 132] for one precise reference. Thus, it is clear that the exact version of Lyapunov's Theorem holds for a saturated vector measure with the RNP, and thus for any vector measure with bounded variation taking values in a Banach space with the RNP; for the case for vector Loeb measures, see [42, Proposition 4.5] in addition to [44]. In sum, the principal contribution of the work reported here is to show that the exact version of Lyapunov's Theorem holds for a saturated vector measure that may not have a Radon–Nikodym derivative, and thus holds for vector measures in a Banach space without the RNP.

Our work takes off from the Knowles–Rudin condition on the noninjectivity of an integration operator, as further utilized and emphasized in [35], [39] in terms of a requirement on the measure space. However this requirement has provoked the observation that "there is no natural measure space satisfying the required condition," see [44, Remark on pp. 140–141]. Whereas this is incontrovertibly true, and the requirement does not have direct relevance to saturated spaces, it has an affinity to condition (3.1) below on higher order Maharam-types that we utilize to derive the Lyapunov property in a setting that goes beyond separable Banach spaces; also see Remarks 3.2 and 5.1 below in this connection. We may also note here that characterizations of the Lyapunov property in terms of sign embedding operators and "small enough" atoms are respectively available in [15], [16], [17], [40] and in [8]; and those of nonatomicity in terms of strong continuity of a vector measure taking values in a Fréchet space in [47] and his references; but they are all phrased in terms of the closure operation. The point is that conditions on the measure space itself that are necessary and sufficient for the Lyapunov property to hold without this operator, as are furnished in this paper, have proved elusive, and the condition that we offer here in this regard shifts attention from the uncountable cardinality of the ambient space to the uncountability of the sets generating its σ-algebra.

We also present two applications of our principal result. The first focuses on the integration of multifunctions, and offers a short direct proof of the convexity result recently established in [35], [45]. It is satisfying that the proof simply rehearses the standard argumentation for the finite-dimensional Euclidean case; see [5, pp. 369–370] and his references. The second focuses on an application of Lyapunov's theorem to show the irrelevance of a substantive restriction on the *core*, a solution concept for the allocation of resources of an exchange economy with a nonatomic continuum of agents; see [41] and the references to subsequent work in [10], [19]. Our result allows us to recover in its entirety the finite-dimensional, purely measure-theoretic intuition in an infinite-dimensional setting, and here again enables a rehearsal of the standard arguments without extraneous assumptions involving the closure operation through the assumption of continuous preferences.

The paper proceeds as follows: after collecting preliminaries in Section 2, and some preparatory results in Section 3, the main results are presented in Sections 4 and 5, with the two applications in Section 6.

2. Preliminaries

This section collects some basic notions and results employed in the sequel. We begin with the saturation property for finite measure spaces in terms of measure algebras, and observe that it reinforces the more conventional notion of nonatomicity. Next, we define the notion of saturation for vector measures via their control measures with the saturated property. A useful reference for measure algebras is [12].

2.1. Saturated measure spaces. A *measure algebra* is a pair $(\mathcal{F},\mu)$, where $\mathcal{F}$ is a Boolean σ-algebra with binary operations $\vee$ and $\wedge$, a unary operation ^{c} and a real-valued function $\mu:\mathcal{F}\to\mathbb{R}$ satisfying the following conditions: (i) $\mu(A)=0$ if and only if $A=\emptyset$, where $\emptyset=\Omega^c$ and $\Omega=\emptyset^c$ are the smallest and largest elements in $\mathcal{F}$, respectively; (ii) $\mu(\bigvee_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\mu(A_n)$ for every sequence $\{A_n\}$ in $\mathcal{F}$ such that $A_n\wedge A_m=\emptyset$ whenever $m\neq n$.

Let $(\mathcal{F},\mu)$ and $(\mathcal{G},\nu)$ be measure algebras. A mapping $\Phi:\mathcal{F}\to\mathcal{G}$ is an *isomorphism* if it is a bijection satisfying the following conditions: (i) $\Phi(A\vee B)=\Phi(A)\vee\Phi(B)$ for every $A,B\in\mathcal{F}$; (ii) $\Phi(A\wedge B^c)=\Phi(A)\wedge\Phi(B)^c$ for every $A,B\in\mathcal{F}$; (iii) $\mu(A)=\nu(\Phi(A))$ for every $A\in\mathcal{F}$. When such an isomorphism exists, $(\mathcal{F},\mu)$ is said to be *isomorphic* to $(\mathcal{G},\nu)$.

A *subalgebra* of $\mathcal{F}$ is a subset of $\mathcal{F}$ that contains Ω and is closed under the Boolean operations $\vee$, $\wedge$ and ^{c}. A subalgebra $\mathcal{U}$ of $\mathcal{F}$ is *order-closed* with respect to the order $\le$ given by $A\le B\Longleftrightarrow A=A\wedge B$ if any nonempty upwards directed subset of $\mathcal{U}$ with its supremum in $\mathcal{F}$ has the supremum in $\mathcal{U}$. A subset $\mathcal{U}\subset\mathcal{F}$ *completely generates* $\mathcal{F}$ if the smallest order closed subalgebra in $\mathcal{F}$ containing $\mathcal{U}$ is $\mathcal{F}$ itself. The *Maharam type* of $(\mathcal{F},\mu)$ is the smallest cardinal of any subset $\mathcal{U}\subset\mathcal{F}$ which completely generates $\mathcal{F}$.

Let $(\Omega,\mathcal{F},\mu)$ be a finite measure space. Denote by $L^1(\mu)$ the space of μ-integrable functions on Ω and by $L^\infty(\mu)$ the space of μ-essentially bounded functions on Ω. Let χ_E be the characteristic function of $E\in\mathcal{F}$. Denote by $L^1_E(\mu)=\{f\chi_E\mid f\in L^1(\mu)\}$ the vector subspace of $L^1(\mu)$ consisting of μ-integrable functions restricted to E and similarly, by $L^\infty_E(\mu)=\{f\chi_E\mid f\in L^\infty(\mu)\}$ the vector subspace of $L^\infty(\mu)$ consisting of μ-essentially bounded functions on Ω restricted to E.

For a finite measure space $(\Omega,\mathcal{F},\mu)$, an equivalence relation $\sim$ on $\mathcal{F}$ is given by $A\sim B$ if and only if $\mu(A\triangle B)=0$, where $A\triangle B$ is the symmetric difference of A and B in $\mathcal{F}$. The collection of equivalence classes is denoted by $\widehat{\mathcal{F}}=\mathcal{F}/\sim$ and its generic element $\widehat{A}$ is the equivalence class of $A\in\mathcal{F}$. The lattice operations $\vee$ and $\wedge$ in $\widehat{\mathcal{F}}$ are given in a usual way by $\widehat{A}\vee\widehat{B}=\widehat{A\cup B}$ and $\widehat{A}\wedge\widehat{B}=\widehat{A\cap B}$. The unary operation ^{c} in $\widehat{\mathcal{F}}$ is obtained for taking complements in $\widehat{\mathcal{F}}$ by $\widehat{A}^c=\widehat{(A^c)}$. Under these operations $\widehat{\mathcal{F}}$ is a partially ordered set furnished with the order $\le$, and hence, a Boolean σ-algebra. Define the real-valued function $\hat{\mu}:\widehat{\mathcal{F}}\to[0,\infty)$ by $\hat{\mu}(\widehat{A})=\mu(A)$ for $\widehat{A}\in\widehat{\mathcal{F}}$. Then the pair $(\widehat{\mathcal{F}},\hat{\mu})$ is a measure algebra associated to $(\Omega,\mathcal{F},\mu)$. The Maharam type of $(\Omega,\mathcal{F},\mu)$ is defined to be that of $(\widehat{\mathcal{F}},\hat{\mu})$.

We define a metric ρ on $\widehat{\mathcal{F}}$ by $\rho(\widehat{A},\widehat{B})=\mu(A\triangle B)$. Then $(\widehat{\mathcal{F}},\rho)$ is a complete metric space (see [1, Lemma 13.13] or [7, Lemma III.7.1]). A measure algebra $(\widehat{\mathcal{F}},\hat{\mu})$ is *separable* if $(\widehat{\mathcal{F}},\rho)$ is a separable metric space. It is well known that $(\widehat{\mathcal{F}},\hat{\mu})$ is separable if and only if $L^1(\mu)$ is separable (see [1, Lemma 13.14]).

Let $\mathcal{F}_E=\{A\cap E\mid A\in\mathcal{F}\}$ be a σ-algebra of $E\in\mathcal{F}$ inherited from $\mathcal{F}$ and define the subspace measure $\mu_E:\mathcal{F}_E\to[0,\infty)$ by $\mu_E(A)=\mu(A)$ for $A\in\mathcal{F}_E$.

A finite measure space $(\Omega, \mathcal{F}, \mu)$ is (*Maharam-type-*)*homogeneous* if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is equal to that of $(\Omega, \mathcal{F}, \mu)$.

By the Maharam theorem (see [34, Theorem 1] or [12, Theorem 331I]), if $(\widehat{\mathcal{F}}, \hat{\mu})$ and $(\widehat{\mathcal{G}}, \hat{\nu})$ are homogeneous measure algebras of finite measure spaces with the same Maharam-type and the same mass, then $(\widehat{\mathcal{F}}, \hat{\mu})$ and $(\widehat{\mathcal{G}}, \hat{\nu})$ are isomorphic. The classical isomorphism theorem cited below (see [4], [13] and [37, Theorem 15.3.4]) is a special case of the Maharam theorem.

Isomorphism Theorem. *Every separable measure algebra of a nonatomic probability space is isomorphic to the measure algebra of the Lebesgue unit interval.*

The notion of the saturation of measure spaces introduced in [14] is a formalization of the property embodied by nonatomic Loeb probability spaces and the product spaces of the form $\{0,1\}^{\mathfrak{m}}$ and $[0,1]^{\mathfrak{m}}$, where $\mathfrak{m}$ is an uncountable cardinal, $\{0,1\}$ has the uniform measure and $[0,1]$ has the Lebesgue measure (see [11, Theorem 3B.12] and [12, Theorem 331K]).

Definition 2.1. A finite measure space $(\Omega, \mathcal{F}, \mu)$ is *saturated* if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is uncountable. A measure μ is *saturated* if $(\Omega, \mathcal{F}, \mu)$ is saturated.

Measure algebras of $\{0,1\}^{\mathfrak{m}}$ and $[0,1]^{\mathfrak{m}}$ are Maharam type homogeneous of $\mathfrak{m}$ and isomorphic whenever $\mathfrak{m}$ is an infinite cardinal (see [12, Theorems 331I and 331K]), and they are separable if and only if $\mathfrak{m}$ is countable. Thus, the countable products of $\{0,1\}$ and $[0,1]$ are typical examples of nonsaturated nonatomic probability spaces. It is known in the literature that saturation is a much stronger condition on measure spaces than nonatomicity. It is called "$\aleph_1$-atomless" in [14], "nowhere separable" in [9], "super-atomless" in [35] and "nowhere countably-generated" in [32]. For equivalent conditions on saturation, see [18] and their references.

Here, we employ the following useful characterization on nonatomicity and saturation (see [12, 365X(p)] and [35, Section 2.2(c) and Fact]).

Proposition 2.1.

(i) *A finite measure space $(\Omega, \mathcal{F}, \mu)$ is nonatomic if and only if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is infinite.*

(ii) *An finite measure space $(\Omega, \mathcal{F}, \mu)$ is saturated if and only if $L^1_E(\mu)$ is nonseparable for every $E \in \mathcal{F}$ with $\mu(E) > 0$.*

Every countably-generated, nonatomic, finite measure space induces a homogeneous measure algebra of Maharam-type $\aleph_0$. If μ and ν are finite measures, μ is saturated and ν is absolutely continuous with respect to μ, then ν is also saturated. This is because the measure algebra induced by ν contains the measure algebra induced by μ.

2.2. Saturation and vector measures. Let $(\Omega, \mathcal{F})$ be a measurable space and X be a Banach space. A vector-valued function $m : \mathcal{F} \to X$ is *countably additive* if for every pairwise disjoint sequence $\{A_n\}$ in $\mathcal{F}$, we have $m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$, where the series is unconditionally convergent under the norm of X. Thanks to the Orlicz–Pettis theorem (see [6, Corollary I.4.4] or [7, Theorem IV.10.1]), m is countably additive if and only if x^*m is a finite signed measure on $\mathcal{F}$ for every $x^* \in X^*$. Throughout this paper, a countably additive vector-valued function $m : \mathcal{F} \to X$ is called a *vector measure*.

A set $A \in \mathcal{F}$ is an *atom* of m if $m(A) \neq \mathbf{0}$ and for every $E \in \mathcal{F}$ with $E \subset A$, either $m(E) = \mathbf{0}$ or $m(A \setminus E) = \mathbf{0}$. If m has no atom, it is said to be *nonatomic*. A set $N \in \mathcal{F}$ is *m-null* if $m(A \cap N) = \mathbf{0}$ for every $A \in \mathcal{F}$.

A vector measure $m : \mathcal{F} \to X$ is *absolutely continuous* (or *μ-continuous*) with respect to a (scalar) measure μ if $\mu(A) = 0$ implies that $m(A \cap E) = \mathbf{0}$ for every $E \in \mathcal{F}$. A finite measure μ is a *control measure* of a vector measure m whenever $\mu(A) = 0$ if and only if $m(A \cap E) = \mathbf{0}$ for every $E \in \mathcal{F}$. The Bartle–Dunford–Schwartz theorem guarantees that every vector measure in a Banach space possesses a control measure (see [3, Corollary 2.4], [6, Corollary I.2.6] or [7, Lemma IV.10.5]).

The significance of a control measure is exemplified by the observation that a set in $\mathcal{F}$ is an atom of m if and only if it is an atom of a control measure for m. This leads to the following characterization of the nonatomicity of m.

PROPOSITION 2.2. *Let μ be a control measure of a vector measure $m : \mathcal{F} \to X$. Then m is nonatomic if and only if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is infinite.*

DEFINITION 2.2. A vector measure is *saturated* if it has a control measure that is saturated.

It follows from the definition that m is saturated if and only if it is absolutely continuous with respect to a saturated finite measure. In view of Proposition 2.2, the above definition obviously reinforces the nonatomicity of vector measures. Saturated vector measures are nonatomic. Since any two control measures for m are equivalent, they generate the same measure algebra on $\mathcal{F}$. Thus, the above definition is independent of the particular choice of the control measures for m. For the role of control measures in the analysis of the range of a vector measure, see [6], [17], [42].

Unless otherwise noted, for the remainder of this paper, $(\Omega, \mathcal{F}, \mu)$ is a finite measure space, X is a Banach space and $m : \mathcal{F} \to X$ is assumed to be a μ-continuous vector measure.

3. An operator-theoretic approach

In this section, we present a systematic treatment of the range of vector measures in terms of the linear operators from $L^\infty(\mu)$ to X. The dependence of the validity of Lyapunov's theorem for infinite-dimensional Banach spaces on the noninjectivity of the suitable linear operator, and hence, on the dimensionality condition for $L^\infty(\mu)$ and X, is an important observation that can be traced to Rudin's [38] original reworking of Lindenstrauss' [30] finite-dimensional proof.

3.1. Lyapunov measures and Lyapunov operators. If $f:\Omega\to\mathbb{R}$ is a simple function of the form $f=\sum_{i=1}^n \alpha_i\chi_{A_i}$ with $\alpha_i\in\mathbb{R}$ and $A_i\cap A_j=\emptyset$ for $i\neq j$, define as usual the integral of f with respect to m by $\int f\,dm=\sum_{i=1}^n \alpha_i m(A_i)$. For any $f\in L^\infty(\mu)$, taking a sequence of simple functions $\{f_n\}$ converging to $f\in L^\infty(\mu)$ in the essential sup norm allows us to define $\int f\,dm=\lim_n\int f_n\,dm$ (see [6, Definition I.1.12]). Thus, we may define the *integration operator* $T_m:L^\infty(\mu)\to X$ of m by $T_m f=\int f\,dm$ for $f\in L^\infty(\mu)$. The integration operator T_m is a continuous linear operator (see [6, Theorem I.1.13]). The following continuity property of integration operators is well known (see [6, Lemma IX.1.3]).

Lemma 3.1. *The integration operator $T_m:L^\infty(\mu)\to X$ is continuous for the weak* topology of $L^\infty(\mu)$ and the weak topology of X.*

We first characterize nonatomic vector measures in terms of integration operators. To this end, we introduce the notion of nonatomic operators.

Definition 3.1. The integration operator $T_m:L^\infty(\mu)\to X$ is an *nonatomic operator* if for every $E\in\mathcal{F}$ with $\mu(E)>0$ and $\varepsilon>0$ there exists $f\in L_E^\infty(\mu)\setminus\{0\}$ with signed values $\{-1,0,1\}$ such that $\|T_m f\|<\varepsilon$.

Theorem 3.1. *A vector measure $m:\mathcal{F}\to X$ is nonatomic if and only if $T_m:L^\infty(\mu)\to X$ is a nonatomic operator.*

Proof. Suppose that T_m is a nonatomic operator. Then for every $\varepsilon>0$ there exists $f\in L_E^\infty(\mu)\setminus\{0\}$ with signed values $\{-1,0,1\}$ such that $\|T_m f\|<\varepsilon$. If m has an atom $E\in\mathcal{F}$, then $m(E)\neq\mathbf{0}$ and E is an atom of μ by the absolute continuity of m with respect to μ. Since measurable functions are constant on atoms, either $f=\chi_E$ or $f=-\chi_E$. We thus obtain $\|m(E)\|=\|T_m f\|<\varepsilon$ for every $\varepsilon>0$, and hence, $m(E)=\mathbf{0}$, a contradiction.

Conversely, suppose that T_m is not a nonatomic operator. Then there exists $E\in\mathcal{F}$ with $\mu(E)>0$ and $\varepsilon>0$ such that $\|T_m f\|\geq\varepsilon$ for every $f\in L_E^\infty(\mu)\setminus\{0\}$ with signed values $\{-1,0,1\}$. Thus, for every $A\in\mathcal{F}_E$ with $\mu(A)>0$, we have $\|T_m\chi_A\|=\|m(A)\|\geq\varepsilon$. If E is not an atom of m, then there exists $A\in\mathcal{F}_E$ such that $m(A)\neq m(E)$ and $m(A)\neq\mathbf{0}$. By the μ-continuity of m, we have $\mu(A)>0$. Hence, there exists $\delta>0$ such that for every $B\in\mathcal{F}$ with

$\mu(A \cap B) < \delta$, we have $\|m(A \cap B)\| < \varepsilon$, a contradiction. Therefore, E is an atom of m. $\square$

DEFINITION 3.2.

(i) A vector measure $m : \mathcal{F} \to X$ is a *Lyapunov measure* if for every $E \in \mathcal{F}$ the set $m(\mathcal{F}_E)$ is weakly compact and convex in X.
(ii) The integration operator $T_m : L^\infty(\mu) \to X$ is a *Lyapunov operator* of m if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the restriction $T_m : L^\infty_E(\mu) \to X$ is not injective.

The range of a Lyapunov measure of m is weakly compact and convex in X in view of $m(\mathcal{F}) = m(\mathcal{F}_\Omega)$ with $\Omega \in \mathcal{F}$. If m has an atom $E \in \mathcal{F}$, then evidently, $m(\mathcal{F}_E)$ is not convex in X. Therefore, every Lyapunov measure is nonatomic. As the next result demonstrates, the nonatomicity of vector measures is reinforced as well by the notion of Lyapunov operators.

THEOREM 3.2. *If $T_m : L^\infty(\mu) \to X$ is a Lyapunov operator, then it is a nonatomic operator.*

Proof. If m is not a nonatomic operator, then m has an atom $E \in \mathcal{F}$ by Theorem 3.1. Thus, $m(E) \neq \mathbf{0}$ and E is an atom of μ. Since every $f \in L^\infty_E(\mu)$ is constant on the atom E of μ, we have $T_m f = \int_E f\,dm = \alpha m(E)$ with $\alpha \in \mathbb{R}$, which implies that $T_m f = \mathbf{0}$ if and only if $f = 0$. Therefore, $T_m : L^\infty_E(\mu) \to X$ is an injection. $\square$

Instead of using $T_m f$, denote by $m(f)$ the integral of f with respect to m, that is, $m(f) = \int f\,dm$. We next characterize Lyapunov measures in terms of Lyapunov operators. The proof of the following proposition was found in [6, Theorem IX.1.4].

PROPOSITION 3.1. *A vector measure $m : \mathcal{F} \to X$ is a Lyapunov measure if and only if $T_m : L^\infty(\mu) \to X$ is a Lyapunov operator of m. The range of a Lyapunov measure m is given by*

$$m(\mathcal{F}) = \left\{ m(f) \in X \mid 0 \leq f \leq 1, f \in L^\infty(\mu) \right\}.$$

Next, we rework a celebrated example of Uhl [46].

EXAMPLE 3.1. Let $(I, \mathcal{L}, \lambda)$ be the Lebesgue unit interval with $I = [0,1]$ and define $m : \mathcal{L} \to L^1(\lambda)$ by $m(A) = \chi_A$ for $A \in \mathcal{L}$. Then m is a nonatomic vector measure with a control measure of λ. The closure of the range of m is neither compact nor convex in $L^1(\lambda)$ (see [46]). Hence, by Proposition 3.1, the integration operator $T_m : L^\infty(\lambda) \to L^1(\lambda)$ is not a Lyapunov operator. This fact can be verified directly as follows.

Let f be a simple function of the form $f = \sum_{i=1}^k \alpha_i \chi_{A_i}$, where $A_1, \dots, A_k$ are mutually disjoint sets in $\mathcal{L}$ and $\alpha_1, \dots, \alpha_k$ are real numbers. It is easy to see that $T_m f = f$. For an arbitrarily given $f \in L^\infty(\lambda)$, choose a sequence of simple functions $\{f_k\}$ that converges to f in the norm topology of $L^\infty(\lambda)$. Then $\{f_k\}$ converges to f as well in the weak* topology of $L^\infty(\lambda)$. Since $T_m : L^\infty(\lambda) \to X$ is continuous for the weak* topology of $L^\infty(\lambda)$ and the weak topology of X by Lemma 3.1, we have $T_m f_k \to Tf$ weakly in $L^1(\lambda)$. It follows from $L^\infty(\lambda) \subset L^1(\lambda)$ that $\{f_k\}$ converges to f both in the norm and weak topologies of $L^1(\lambda)$. Hence, $\langle T_m f - f, \varphi\rangle = \lim_k \langle T_m f_k - f_k, \varphi\rangle = 0$ for every $\varphi \in L^1(\lambda)$ in view of $T_m f_k = f_k$ for each k. Therefore, we obtain $T_m f = f$ for every $f \in L^\infty(\lambda)$. This means that T_m is not a Lyapunov operator, but a nonatomic operator.

Remark 3.1. The equivalence between Lyapunov measures and Lyapunov operators were established first by [26] for the case where X is a quasicomplete locally convex space and then elaborated in [6] in the current simpler form when X is a Banach space. An alternative notion of Lyapunov measures in which the closure of the range $m(\mathcal{F})$ is convex was proposed in [17] and characterized in terms of the "sign-embedding" operators from $L^\infty(\mu)$ to X. For another characterization of Lyapunov measures in terms of the sign-embedding operators from $L^1(\mu)$ to X, see [16].

3.2. The dimensionality condition. Denote by $\mathrm{ca}(\mathcal{F}, \mu, X)$ the space of μ-continuous, X-valued vector measures on $\mathcal{F}$ and by $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ the space of linear operators from $L^\infty(\mu)$ to X which are continuous for the weak* topology of $L^\infty(\mu)$ and the weak topology of X.

Definition 3.3. A linear operator $T : L^\infty(\mu) \to X$ is a *local injection* if there exist $E \in \mathcal{F}$ with $\mu(E) > 0$ such that the restriction $T : L^\infty_E(\mu) \to X$ is an injection.

Theorem 3.3. *There exists a linear bijection from* $\mathrm{ca}(\mathcal{F}, \mu, X)$ *onto* $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$.

Proof. By Lemma 3.1, for every $m \in \mathrm{ca}(\mathcal{F}, \mu, X)$ the integration operator $T_m : L^\infty(\mu) \to X$ is continuous for the weak* topology of $L^\infty(\mu)$ and the weak topology of X. Hence, the mapping $\Psi :\ \mathrm{ca}(\mathcal{F}, \mu, X) \to \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ defined by $\Psi(m) = T_m$ for $m \in \mathrm{ca}(\mathcal{F}, \mu, X)$ is a linear injection. Conversely, for given $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$, define $m \in \mathrm{ca}(\mathcal{F}, \mu, X)$ by $m(A) = T\chi_A$ for $A \in \mathcal{F}$. To demonstrate the countable additivity of m, let $\{A_i\}$ be a mutually disjoint sequence in $\mathcal{F}$ and $A = \bigcup_{i=1}^\infty A_i$. Since $\sum_{i=1}^k \chi_{A_i}$ converges weakly* to χ_A in $L^\infty(\mu)$ as $k \to \infty$, we have

$$\sum_{i=1}^k x^* m(A_i) = x^* T\left(\sum_{i=1}^k \chi_{A_i}\right) \to x^* T\chi_A = x^* m(A)$$

for every $x^* \in X^*$, where the weak convergence in X follows from the continuity of T with respect to the weak* and weak topologies. Therefore, $x^* m$ is a finite signed measure for every $x^* \in X^*$, and hence, m is countably additive by the Orlicz–Pettis theorem. It is obvious that m is absolutely continuous with respect to μ.

Define the linear mapping by $\Phi : \mathcal{L}(L^\infty_{w^*}(\mu), X_w) \to \mathrm{ca}(\mathcal{F}, \mu, X)$ by

$$\Phi(T)(A) = T\chi_A, \quad A \in \mathcal{F}.$$

Let $\Phi(T) = m$ and consider the integration operator T_m. Then T_m coincides with T on the set of all characteristic functions. Since the linear span of the set of all characteristic functions is norm dense in $L^\infty(\mu)$, it is also weakly* dense in $L^\infty(\mu)$. Let $f \in L^\infty(\mu)$ be given arbitrary and $\{f_k\}$ be a sequence of simple functions in $L^\infty(\mu)$ which converges weakly* to f. We then have $x^*T_mf_k = x^*Tf_k \to x^*Tf$ for every $x^* \in X^*$ as $k \to \infty$ because T is continuous for the weak* and weak topologies. Therefore, $T_mf = Tf$ for every $f \in L^\infty(\mu)$. This means that $\Psi \circ \Phi(T) = T$ for every $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$. Similarly, let $\Psi(m) = T$ and consider $m' \in \mathrm{ca}(\mathcal{F}, \mu, X)$ given by $m'(A) = T\chi_A$ for $A \in \mathcal{F}$. Then $m'(A) = T_m\chi_A = m(A)$ for every $A \in \mathcal{F}$. This implies that $\Phi \circ \Psi(m) = m$ for every $m \in \mathrm{ca}(\mathcal{F}, \mu, X)$. Therefore, $\Psi = \Phi^{-1}$ is a bijection. □

COROLLARY 3.1. *Every vector measure in* $\mathrm{ca}(\mathcal{F}, \mu, X)$ *is a Lyapunov measure if and only if every linear operator in* $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ *is not a local injection.*

We turn to a canonical situation in the literature where $(\Omega, \mathcal{F}, \mu)$ is saturated and X is separable.

PROPOSITION 3.2. *If* $(\Omega, \mathcal{F}, \mu)$ *is saturated and* X *is separable, then every linear operator in* $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ *is not a local injection.*

Proof. As shown by [35, Lemma 1], if X is separable and $L^1_E(\mu)$ is nonseparable for $E \in \mathcal{F}$ with $\mu(E) > 0$, then for every $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$, the restriction $T : L^\infty_E(\mu) \to X$ is not an injection. By Proposition 2.1(ii), a saturated measure space $(\Omega, \mathcal{F}, \mu)$ has the property such that $L^1_E(\mu)$ is nonseparable for every $E \in \mathcal{F}$ with $\mu(E) > 0$. Therefore, every $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ is not a local injection. □

REMARK 3.2. The *algebraic dimension* of a Banach space X is the cardinality of a Hamel basis in X. We denote by $\dim_{\mathrm{alg}} X$ the algebraic dimension of X. Consider the algebraic dimensionality condition

$$\dim_{\mathrm{alg}} L^\infty_E(\mu) > \dim_{\mathrm{alg}} X \quad \text{for every } E \in \mathcal{F} \text{ with } \mu(E) > 0. \tag{3.1}$$

This is obviously a sufficient condition for every $T \in \mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ not to be a local injection. This algebraic dimensionality condition was introduced first by Rudin [38, Theorem 5.5] in the proof of the Lyapunov theorem for the case where X is finite dimensional so as to replace the induction argument of the proof by Lindenstrauss [30]. The significance of condition (3.1) was also recognized by Kluvánek and Knowles [26, Theorem V.2.1] when X is a locally convex space, and as emphasized in [39, (A1)]), it plays a crucial role for demonstrating the existence of equilibria for economies with a measure space of agents and an infinite-dimensional commodity space.

By the *topological dimension* of a Banach space X, we mean the smallest cardinal corresponding to a subset of X with linear span norm dense in X, which is denoted by $\dim_{\mathrm{top}} X$ to discriminate it from $\dim_{\mathrm{alg}} X$. Note that $\dim_{\mathrm{alg}} X \geq \dim_{\mathrm{top}} X$ for every Banach space X (see [29, p. 110]). As observed by [44, p. 140] and [35, p. 840], imposing the algebraic dimensionality condition (3.1) rules out natural measure spaces. In particular, discrepancies between the topological and algebraic dimensions lead to the existence of a saturated measure space that violates (3.1) even if X is separable. To illustrate this point, let $[0,1]^{\mathfrak{c}}$ be the product space of the Lebesgue unit interval $[0,1]$. We then have $\dim_{\mathrm{top}} L^1[0,1]^{\mathfrak{c}} = \mathfrak{c}$ (see [36, Remark, p. 221]) and $\mathrm{card} L^\infty[0,1]^{\mathfrak{c}} = \dim_{\mathrm{top}} L^\infty[0,1]^{\mathfrak{c}} = \mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$ (see [36, Remark, p. 222]). Thus, $\dim_{\mathrm{alg}} L^\infty[0,1]^{\mathfrak{c}} = \dim_{\mathrm{top}} L^\infty[0,1]^{\mathfrak{c}} = \mathfrak{c}$. On the other hand, if X is separable and infinite-dimensional, then $\dim_{\mathrm{alg}} X = \mathfrak{c}$ (see [28]).

4. Lyapunov measures in separable Banach spaces

In this section, we focus on separable Banach spaces, with the first subsection devoted to a sufficiency result, and the second to a necessity result. Our results testify to the relevance of the saturation property for the conclusion of Lyapunov's theorem.

4.1. Saturation and Lyapunov measures: A sufficiency theorem.

The next theorem states that the saturation of m is a sufficient condition for m to be a Lyapunov measure, or equivalently, for T_m to be a Lyapunov operator, whenever X is separable.

THEOREM 4.1. *Let X be a separable Banach space and $m : \mathcal{F} \to X$ be a μ-continuous vector measure. If $(\Omega, \mathcal{F}, \mu)$ is saturated, then it is a Lyapunov measure with its range $m(\mathcal{F})$ given by*

$$m(\mathcal{F}) = \big\{ m(f) \in X \mid 0 \leq f \leq 1, f \in L^\infty(\mu) \big\}.$$

Proof. Since $T_m : L^\infty(\mu) \to X$ is not a local injection by Lemma 3.1 and Proposition 3.2, it is a Lyapunov operator of m and the result follows from Proposition 3.1. □

The separability of X is indispensable for Theorem 4.1. This can be demonstrated easily by an examination of the counterexample of Uhl [46] as shown below. (See also Example 3.1.)

EXAMPLE 4.1. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and define the vector measure $m : \mathcal{F} \to L^1(\mu)$ by $m(A) = \chi_A$. As shown in [6, p. 261] (see also [40, Example 1.2]), m is of bounded variation with the total variation $\mu(\Omega)$ and the range $m(\mathcal{F})$ is closed in $L^1(\mu)$ by the dominated convergence theorem. It is evident that μ is a control measure of m.

Suppose that $(\Omega,\mathcal{F},\mu)$ is nonatomic and $\mathcal{F}$ is countably generated. One can choose $A\in\mathcal{F}$ with $0<\mu(A)<\mu(\Omega)$ by the nonatomicity of μ. Then, $\chi_A,\chi_{\Omega\setminus A}\in m(\mathcal{F})$, and hence $\frac{1}{2}\chi_A+\frac{1}{2}\chi_{\Omega\setminus A}=\frac{1}{2}\chi_\Omega\in\mathrm{co}\,m(\mathcal{F})$. On the other hand, for every $A\in\mathcal{F}$, we have

$$\left\|m(A)-\frac{1}{2}\chi_\Omega\right\|_1=\left\|\chi_A-\left(\frac{1}{2}\chi_A+\frac{1}{2}\chi_{\Omega\setminus A}\right)\right\|_1=\frac{1}{2}\|\chi_\Omega\|_1=\frac{1}{2}\mu(\Omega),$$

where $\|\cdot\|_1$ is the L^1-norm. Therefore, the range $m(\mathcal{F})$ is not convex in X. It should be noted that $L^1(\mu)$ is separable since $\mathcal{F}$ is countably generated.

What happens if the measure space $(\Omega,\mathcal{F},\mu)$ is saturated? The range nonconvexity of m is still valid as it stands, but $L^1(\mu)$ is nonseparable here. Moreover, we have a following characterization: *A vector measure $m:\mathcal{F}\to L^1(\mu)$ defined above is saturated if and only if for every $E\in\mathcal{F}$ with $\mu(E)>0$, $L^1_E(\mu)$ is nonseparable.* We have a saturated vector measure taking values in a nonseparable Banach space which is *not* a Lyapunov measure. Indeed, this nonseparable Banach space can be taken to be a most natural one—the one generated by the saturated control measure of the saturated vector measure—generated in the L^1-sense.

Remark 4.1. By Theorem 4.1, every vector saturated vector measure has a convex range in a separable Banach space. A celebrated example of Lyapunov's exhibits the existence of an l^2-valued vector measure on the Lebesgue unit interval with a nonconvex range (see [6, Example IX.1.1]). The Lebesgue unit interval is a typical example of a nonatomic probability space that is not saturated. This example then suggests that the saturation property is indispensable in guaranteeing the convexity of the range of a vector measure in infinite-dimensional Banach spaces. It is worth noting that Lyapunov's counterexample is no longer a counterexample for a countably-generated Lebesgue extension of the unit interval; see [24]. Such an extension is of course not saturated, and yet furnishes the convexity of the range of the "specific" l^2-valued vector measure.

4.2. Saturation and Lyapunov measures: A characterization.

Lemma 4.1. *Let X be an infinite-dimensional Banach space. If a nonatomic finite measure space $(\Omega,\mathcal{F},\mu)$ is not saturated, then there exists $E\in\mathcal{F}$ with $\mu(E)>0$ and $m\in\mathrm{ca}(\mathcal{F}_E,\mu_E,X)$ such that m is not a Lyapunov measure.*

Proof. Let $(\Omega,\mathcal{F},\mu)$ be a nonatomic finite measure space that is not saturated. Then by Proposition 2.1(ii), there exists $E\in\mathcal{F}$ with $\mu(E)>0$ such that $L^1_E(\mu)$ is separable. Without loss of generality, we may assume that μ is a nonatomic probability measure because it is finite. Since the measure algebra $(\widehat{\mathcal{F}_E},\widehat{\mu_E})$ is separable, by the isomorphism theorem, it is isomorphic to the measure algebra $(\widehat{\mathcal{L}},\hat{\lambda})$ of the Lebesgue measure space $(I,\mathcal{L},\lambda)$ on the unit interval $I=[0,1]$. Denote by $\Phi:\widehat{\mathcal{F}_E}\to\widehat{\mathcal{L}}$ the isomorphism from $(\widehat{\mathcal{F}_E},\widehat{\mu_E})$

onto $(\widehat{\mathcal{L}}, \hat{\lambda})$. Then Φ is a σ-isomorphism, that is, it satisfies $\Phi(\bigvee_{n=1}^{\infty} \widehat{A_n}) = \bigvee_{n=1}^{\infty} \Phi(\widehat{A_n})$ for every sequence $\{\widehat{A_n}\}$ in $\widehat{\mathcal{F}_E}$ (see [37, Problem 15.3.11]).

By virtue of [6, Corollary IX.1.6], there exists a vector measure $G : \mathcal{L} \to X$ of bounded variation and a set $J \subset I$ with $J \in \mathcal{L}$ such that the set $\{G(S \cap J) \mid S \in \mathcal{L}\}$ is not a weakly compact convex set in X and λ is a control measure for G. Define $\widehat{G} : \widehat{\mathcal{L}} \to X$ by $\widehat{G}(\widehat{S}) = G(S)$ for $\widehat{S} \in \widehat{\mathcal{L}}$. If $\widehat{S} = \widehat{S}'$, then $\lambda(S \triangle S') = 0$, and hence, $S \triangle S' \in \mathcal{L}$ is a G-null set. We then have $\widehat{G}(\widehat{S}) = G(S) = G(S') = \widehat{G}(\widehat{S}')$. Thus, $\widehat{G}$ is well defined. Define the vector measure $m : \mathcal{F}_E \to X$ by $m(A) = \widehat{G}(\Phi(\widehat{A}))$ for $A \in \mathcal{F}_E$. We claim that μ_E is a control measure for m, and hence, $m \in \mathrm{ca}(\mathcal{F}_E, \mu_E, X)$. To this end, recall the construction of m and note the following equivalence for $A \in \mathcal{F}_E$.

$$\begin{aligned}
\mu_E(A) = 0 \iff \widehat{\mu_E}(\widehat{A}) = 0 &\iff \hat{\lambda}\big(\Phi(\widehat{A})\big) = 0 \\
&\iff \lambda(S) = 0 \ \forall S \in \Phi(\widehat{A}) \\
&\iff G\big(S \cap J'\big) = \mathbf{0} \ \forall S \in \Phi(\widehat{A}) \ \forall J' \in \mathcal{L} \\
&\iff \widehat{G}\big(\Phi\big(\widehat{A} \wedge \widehat{E'}\big)\big) = \mathbf{0} \ \forall \widehat{E'} \in \widehat{\mathcal{F}_E} \\
&\iff m\big(A \cap E'\big) = \mathbf{0} \ \forall E' \in \mathcal{F}_E,
\end{aligned}$$

where the second equivalence employs the fact that Φ is measure-preserving and the forth equivalence follows from the fact that λ is a control measure for $G \in \mathrm{ca}(\mathcal{L}, \lambda, X)$. □

As illustrated by [18], the power of saturation arises prominently in necessity results in various applications. We present here a necessity result on saturation for vector measures that provides a characterization of saturated measure spaces in terms of Lyapunov measures and Lyapunov operators.

Theorem 4.2. *Let X be an infinite-dimensional separable Banach space. Then the following conditions are equivalent.*

(i) $(\Omega, \mathcal{F}, \mu)$ *is saturated.*
(ii) *Every vector measure in* $\mathrm{ca}(\mathcal{F}, \mu, X)$ *is a Lyapunov measure.*
(iii) *The integration operator $T_m : L^\infty(\mu) \to X$ is a Lyapunov operator for every $m \in \mathrm{ca}(\mathcal{F}, \mu, X)$.*

Proof. (i) ⇒ (ii) ⇔ (iii): See Theorem 4.1 and Proposition 3.1.

(ii) ⇒ (i): Take any $g \in L^1(\mu, X)$ with $g(\omega) \neq \mathbf{0}$ a.e. $\omega \in \Omega$. Then the vector measure $m_g : \mathcal{F} \to X$ defined by $m_g(A) = \int_A g\,d\mu$ for $A \in \mathcal{F}$ is a Lyapunov measure in $\mathrm{ca}(\mathcal{F}, \mu, X)$ with control measure μ. Since m_g is nonatomic, μ is also nonatomic. If μ is not saturated, then by Lemma 4.1, there exists a non-Lyapunov measure $m_E \in \mathrm{ca}(\mathcal{F}_E, \mu_E, X)$ for some $E \in \mathcal{F}$ with $\mu(E) > 0$. Extend m_E from $\mathcal{F}_E$ to $\mathcal{F}$ by $m(A) = m_E(A \cap E)$ for $A \in \mathcal{F}$. Then m is a non-Lyapunov measure in $\mathrm{ca}(\mathcal{F}, \mu, X)$, a contradiction. □

Remark 4.2. One cannot remove the infinite-dimensionality of Banach spaces from Theorem 4.2. Suppose that X is a finite-dimensional Banach space. If $(\Omega, \mathcal{F}, \mu)$ is a nonatomic finite measure space with $\mathcal{F}$ countably generated, which is satisfied for the Lebesgue unit interval, then the Maharam type of $(\Omega, \mathcal{F}, \mu)$ is countable by Proposition 2.1(i), and consequently, μ is not saturated. If $m \in \mathrm{ca}(\mathcal{F}, \mu, X)$, then the range $m(\mathcal{F})$ is compact and convex by the classical Lyapunov theorem. This leads to a counterexample of Theorem 4.2 whenever the infinite-dimensionality requirement is violated.

5. Lyapunov measures in nonseparable Banach spaces

The aim this section is to provide a classification of Banach spaces for which Lyapunov's theorem holds in terms of the Maharam-types. It is now well-understood that the Maharam theorem identifies the basic "building blocks" for measure spaces, and our investigation relies on them to go beyond Lyapunov's theorem for separable Banach spaces established in Theorem 4.1 and opens the door to Lyapunov's theorem for their nonseparable counterparts. We also present an exact version of the Lyapunov theorem for Banach spaces with the Radon–Nikodym property along the lines of [42], [46].

5.1. Maharam-types and the Lyapunov property.

Definition 5.1. A Banach space X has the *Lyapunov property* with respect to a finite measure space $(\Omega, \mathcal{F}, \mu)$ if every vector measure in $\mathrm{ca}(\mathcal{F}, \mu, X)$ is a Lyapunov measure.

The next result reveals an ambivalence of the Lyapunov property in infinite-dimensional Banach spaces.

Proposition 5.1. *No infinite-dimensional Banach space has the Lyapunov property with respect to a countably-generated, nonatomic probability space. However, for every Banach space there exists a nonatomic probability space with respect to which it has the Lyapunov property.*

Proof. It follows from the isomorphism theorem that the measure algebra of a countably-generated, nonatomic probability space is isomorphic to the measure algebra of the Lebesgue unit interval. The first part of the proposition draws from Uhl's celebrated example (already referred to above) whereby, for every infinite-dimensional Banach space X, there exists an X-valued vector measure defined on the Lebesgue unit interval such that its range is not convex in X.

For the second part, in view of Proposition 3.1, Corollary 3.1 and Theorem 3.2, it suffices to show that for every infinite-dimensional Banach space X, there exists a saturated probability space such that condition (3.1) is satisfied. Let $\mathfrak{m} = \dim_{\mathrm{alg}} X$ and take any infinite cardinal $\mathfrak{n} > \mathfrak{m}$. Consider the probability space $(\Omega, \mathcal{F}, \mu)$ with the product space $\Omega = [0,1]^{\mathfrak{n}}$ of the Lebesgue unit interval. Since $[0,1]^{\mathfrak{n}}$ is isomorphic to $\{0,1\}^{\mathfrak{n}}$, by the Maharam theorem, $[0,1]^{\mathfrak{n}}$ is a Maharam-type-homogenous, of Maharam type $\mathfrak{n}$ (see [12, Theorem 331K]). That is, for every $E \in \mathcal{F}$ with $\mu(E) > 0$, the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is $\mathfrak{n}$. Since $\mathfrak{n}$ is uncountable, $(E, \mathcal{F}_E, \mu_E)$ is saturated. Without loss of generality, we may assume that μ_E is a probability measure on $\mathcal{F}_E$. Again by the Maharam theorem (see [12, Theorem 331I]), the measure algebra $(\widehat{\mathcal{F}}, \hat{\mu})$ of $(\Omega, \mathcal{F}, \mu)$ is isomorphic to the measure algebra $(\widehat{\mathcal{F}_E}, \widehat{\mu_E})$ of $(E, \mathcal{F}_E, \mu_E)$. This means that there is a linear isometry from $L^\infty(\mu)$ onto $L^\infty_E(\mu)$ for every $E \in \mathcal{F}$ with $\mu(E) > 0$ (see [12, Theorem 363F]). Since $\operatorname{card} L^\infty(\mu) = \dim_{\mathrm{top}} L^\infty(\mu) = \mathfrak{n}^{\aleph_0}$ (see [36, Remark, p. 222]), we obtain $\dim_{\mathrm{alg}} L^\infty_E(\mu) \geq \dim_{\mathrm{top}} L^\infty_E(\mu) = \mathfrak{n}^{\aleph_0} > \mathfrak{m} = \dim_{\mathrm{alg}} X$. $\square$

Let $(\Omega, \mathcal{F}, \mu)$ be a Maharam-type-homogeneous, finite measure space and let $(\widehat{\mathcal{F}}, \hat{\mu})$ and $(\widehat{\mathcal{F}_E}, \widehat{\mu_E})$ be measure algebras induced by μ and μ_E respectively. Since for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the Maharam type of $(E, \mathcal{F}_E, \mu_E)$ is equal to that of $(\Omega, \mathcal{F}, \mu)$, by the Maharam theorem (see [34, Theorem 1] or [12, Theorem 331.I]), there is an isomorphism $\Phi : \widehat{\mathcal{F}_E} \to \widehat{\mathcal{F}}$ for every $E \in \mathcal{F}$ with $\mu(E) > 0$ with the measure-preserving property $\widehat{\mu_E}(\Phi^{-1}(\widehat{A})) = \hat{\mu}(\widehat{A})$ for every $A \in \mathcal{F}$.

Denote by $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$ a finite measure space that induces a homogeneous measure algebra of Maharam-type $\mathfrak{m}$. Then $\mu_{\mathfrak{m}}$ is nonatomic whenever $\mathfrak{m} \geq \aleph_0$ and moreover $\mathfrak{m} = \aleph_0$ if and only if $\mathcal{F}_{\mathfrak{m}}$ is countably generated modulo the null sets. By definition, $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$ is saturated whenever $\mathfrak{m} \geq \aleph_1$. By Theorem 4.1, if $\mathfrak{m} \geq \aleph_1$, then separable Banach spaces have the Lyapunov property with respect to $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$. Thus, the smallest cardinal satisfying the Lyapunov property for infinite-dimensional separable Banach spaces is $\mathfrak{m} = \aleph_1$ in view of Proposition 5.1.

When X is nonseparable, condition (3.1) guarantees that the Lyapunov property for X is satisfied if $\mathfrak{m}^{\aleph_0} > \dim_{\mathrm{alg}} X$ because $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$ is isomorphic to $[0,1]^{\mathfrak{m}}$ by the Maharam theorem and $\dim_{\mathrm{top}} L^\infty(\mu_{\mathfrak{m}}) = \mathfrak{m}^{\aleph_0}$ (see [36, Remark, p. 222]). Since the class of all cardinals is well-ordered with respect to cardinality, the smallest cardinal satisfying the Lyapunov property for X indeed exists.

For any Banach space X, let $\mathfrak{m}(X)$ be the smallest cardinal $\mathfrak{m}$ such that X has the Lyapunov property with respect to $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$. Then $\mathfrak{m}(X) = \aleph_0$ if X is a finite-dimensional Banach space; $\mathfrak{m}(X) = \aleph_1$ if X is an infinite-dimensional separable Banach space; $\mathfrak{m}(X) \geq \aleph_1$ if X is a nonseparable Banach space.

We summarize the above observations through the following characterization of the Lyapunov property for Banach spaces.

PROPOSITION 5.2. *A Banach space X has the Lyapunov property with respect to a Maharam-type-homogenous, finite measure space if and only if its Maharam type is greater than or equal to* $\mathfrak{m}(X)$.

Denote by $\mathfrak{m}^+$ the smallest cardinal strictly greater than $\mathfrak{m}$. Under the generalized continuum hypothesis, $\mathfrak{m}^+ = 2^{\mathfrak{m}}$, in particular, $\mathfrak{c} = \aleph_1$. (However, we do not assume it here.) For nonseparable Banach spaces, we have the following estimate of $\mathfrak{m}(X)$.

THEOREM 5.1. *If X is a nonseparable Banach space with* $\dim_{\mathrm{alg}} X = \mathfrak{m}$, *then* $\aleph_1 \le \mathfrak{m}(X) \le \mathfrak{m}^+$.

Proof. Let $(\Omega, \mathcal{F}, \mu)$ be a homogeneous finite measure space of Maharam type $\mathfrak{m}^+$. By the Maharam theorem, every measure space $(E, \mathcal{F}_E, \mu_E)$ with $\mu(E) > 0$ is isomorphic to the product space $[0,1]^{\mathfrak{m}^+}$ of the Lebesgue unit interval $[0,1]$. We thus have:

$$\dim_{\mathrm{alg}} L^\infty_E(\mu) \ge \dim_{\mathrm{top}} L^\infty_E(\mu) = \left(\mathfrak{m}^+\right)^{\aleph_0} \ge \mathfrak{m}^+ > \mathfrak{m},$$

where the second equality follows from [36, Remark, p. 222]. Take any $m \in \mathrm{ca}(\mathcal{F}, \mu, X)$. By Corollary 3.1 and the dimensionality condition (3.1), T_m is a Lyapunov operator. Therefore, $\mathfrak{m}(X) \le \mathfrak{m}^+$. □

COROLLARY 5.1. *If $(\Omega, \mathcal{F}, \mu)$ is Maharam-type-homogeneous, then the following conditions are equivalent.*

(i) *X has the Lyapunov property with respect to $(\Omega, \mathcal{F}, \mu)$.*
(ii) *Every linear operator in $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ is not a local injection.*
(iii) *The Maharam type of $(\Omega, \mathcal{F}, \mu)$ is greater than or equal to $\mathfrak{m}(X)$.*

The significance of the Maharam type homogeneity in measure theory is emphasized by Fremlin [12, p. 131] as follows.

> Maharam's theorem belongs with the Radon–Nikodym theorem, Fubini's theorem and the strong law of large numbers as one of the theorems which make measure theory what it is. Once you have this theorem and its consequences [...] properly absorbed, you will never again look at a measure space without classifying its measure algebra in terms of the types of its homogeneous principal ideals. As one might expect, a very large proportion of the important measure spaces of analysis are homogeneous, and indeed a great many are homogeneous with Maharam type $\aleph_0$.

5.2. Saturation and the Radon–Nikodym property. By $L^1(\mu, X)$ we denote the space of X-valued Bochner integrable functions on Ω. Uhl's [46] "approximate" version of the Lyapunov theorem for Banach spaces with the RNP has already been referred to, and for the reader's convenience we note that Banach space X has the *Radon–Nikodym property* (*RNP*) with respect to $(\Omega, \mathcal{F}, \mu)$ if for every μ-continuous vector measure $m : \mathcal{F} \to X$ of bounded variation, there exists $g \in L^1(\mu, X)$ such that $m(A) = \int_A g\, d\mu$ for every $A \in \mathcal{F}$.

A Banach space has the *RNP* if it has the RNP with respect to every finite measure space.

We can now recall Uhl's theorem (see also [6, Theorem IV.1.10]): *Let X be a Banach space with the RNP. If $m:\mathcal{F}\to X$ is a nonatomic vector measure of bounded variation, then the norm closure of the range of m is norm compact and convex in X.* When the underlying vector measures are saturated, we obtain an "exact" version of the Lyapunov theorem for Banach spaces with the RNP, in which the closure operation is removed.

THEOREM 5.2. *Let X be a Banach space with the RNP. If $m:\mathcal{F}\to X$ is a saturated vector measure of bounded variation, then the range of m is norm compact and convex in X.*

Proof. Since m is of bounded variation, its control measure μ is given by its variation (see [6, Proposition I.1.9]). By the RNP of X, there exists a function $g\in L^1(\mu,X)$ such that $m(A)=\int_A g\,d\mu$ for every $A\in\mathcal{F}$. Since Bochner integrable functions are μ-essentially separably valued (see [6, Theorem II.1.2]), we may assume without loss of generality that g takes values in a separable Banach space X. (Consider the linear span generated by the essential range of g.) By Theorem 4.1, the range $m(\mathcal{F})$ is weakly compact and convex in X. Since m is nonatomic, the norm closure of $m(\mathcal{F})$ is norm compact by Uhl's Theorem. Therefore, $m(\mathcal{F})$ is norm compact and convex in X because on convex sets in X norm closedness is equivalent to weak closedness. □

REMARK 5.1. Theorem 5.2 is an extension of [44, Remark to Theorem 2], who obtained the same exact result for nonatomic Loeb probability spaces. For a discussion of the importance of the bounded variation assumption on vector measures in Uhl's theorem, see the discussion in [40] and in [16, Introduction and Theorem 1]. The bounded variation condition is needed in Theorem 5.2 to obtain a Radon–Nikodym derivative from the RNP, but the proof does not require the control measure to be the variation of the given vector measure. In Section 6, we also present an alternative proof of Theorem 5.2 that revolves on these considerations.

6. Two applications

In this final section, we turn to two applications that bring out the power of the saturation property. The first applies our principal result to show the convexity of the integral of a multifunction under the saturation assumption and to present an alternative proof of Theorem 5.2; for details as to the importance of this result in mathematical economics and in control theory, see [5], [26] and their references. The second applies our principal result to an application of Lyapunov's theorem in mathematical economics that has proved to be rather influential and leads us to richer notions of restricted cores allocations of saturated economies; see [41], and subsequent work referenced in [10], [19]. It bears emphasis that our primary motivation is to argue for the relevance Lyapunov's theorem on saturated spaces in applied work, and that comprehensive investigations of both applications in their substantive registers will be presented elsewhere.

6.1. On convexity of the set-valued integral. Let $\mathcal{B}_X$ be the Borel σ-algebra of a Banach space $(X, \|\cdot\|)$. A set-valued mapping from Ω to X with nonempty values is called a *multifunction*. A multifunction $\Gamma : \Omega \twoheadrightarrow X$ is *measurable* if its graph is $\mathcal{F} \otimes \mathcal{B}_X$-measurable. It is *integrably bounded* if there exists $\varphi \in L^1(\mu)$ such that $\Gamma(\omega) \subset \varphi(\omega)B$ a.e. $\omega \in \Omega$, where B is a closed unit ball in X. If X is a measurable multifunction from a finite measure space $(\Omega, \mathcal{F}, \mu)$ to a separable Banach space X, then by the measurable selection theorem (see [1, Corollary 18.27]), there exists a measurable function $g : \Omega \to X$ such that $g(\omega) \in \Gamma(\omega)$ a.e. $\omega \in \Omega$. Denote by $\mathcal{S}^1_\Gamma$ the set of all Bochner integrable selections of Γ, that is,

$$\mathcal{S}^1_\Gamma = \left\{ g \in L^1(\mu, X) \mid g(\omega) \in \Gamma(\omega) \text{ a.e. } \omega \in \Omega \right\}.$$

The *integral* of Γ with respect to μ is defined by

$$\int \Gamma \, d\mu = \left\{ \int g \, d\mu \in X \;\middle|\; g \in \mathcal{S}^1_\Gamma \right\}.$$

We denote by $\mathcal{G}(\Omega, X)$ the space of multifunctions from Ω to X. If $(\Omega, \mathcal{F}, \mu)$ is a finite measure space and $\Gamma \in \mathcal{G}(\Omega, X)$ is measurable and integrably bounded, then $\mathcal{S}^1_\Gamma$ is nonempty.

In [35, Theorem 1], the following result is proved: *If X is an infinite-dimensional Banach space, then $(\Omega, \mathcal{F}, \mu)$ is saturated if and only if the integral $\int \Gamma \, d\mu$ is convex in X for every $\Gamma \in \mathcal{G}(\Omega, X)$.* We demonstrate the sufficiency of saturation for any Banach space with a simpler proof. We invite to the reader to compare our convexity result with that presented in the literature: with the availability of Theorem 4.1, the sufficiency proof simply mimics that for the case of the multifunction taking values in a finite-dimensional Euclidean space as in [5]; also see [35], [44], [45].

THEOREM 6.1. *If $(\Omega, \mathcal{F}, \mu)$ is saturated, then the integral $\int \Gamma \, d\mu$ is convex in X for every $\Gamma \in \mathcal{G}(\Omega, X)$.*

Proof. Choose any $g_0, g_1 \in \mathcal{S}^1_\Gamma$ and $\alpha \in [0,1]$. Since Bochner integrable functions are μ-essentially separably valued (see [6, Theorem II.1.2]), we may assume without loss of generality that g_0 and g_1 take values in a separable Banach space X. It suffices to show that there exists $g \in \mathcal{S}^1_\Gamma$ such that $\int g \, d\mu = \int (\alpha g_0 + (1-\alpha) g_1) \, d\mu$. To this end, define the vector measure $m : \mathcal{F} \to X \times X$ by:

$$m(A) = \left(\int_A g_0 \, d\mu, \int_A g_1 \, d\mu \right), \quad A \in \mathcal{F}.$$

Since m is saturated by its μ-continuity, the range of m is convex by Theorem 4.1. Then there exists $A \in \mathcal{F}$ such that $m(A) = \alpha m(\Omega)$, which is equivalent to $\int_A g_i \, d\mu = \alpha \int g_i \, d\mu$ for each $i = 0, 1$. Define $g = g_0 \chi_A + g_1 \chi_{\Omega \setminus A}$. We then have $g \in \mathcal{S}^1_\Gamma$ and:

$$\int g \, d\mu = \int_A g_0 \, d\mu + \int_{\Omega \setminus A} g_1 \, d\mu = \alpha \int g_0 \, d\mu + (1 - \alpha) \int g_1 \, d\mu. \qquad \square$$

To present an alternative proof of Theorem 5.2, we cite the following result from [45, Proposition 1] (see also [44, Theorem 2]).

Proposition 6.1. *If $(\Omega, \mathcal{F}, \mu)$ is saturated, then the integral $\int \Gamma \, d\mu$ is norm compact in X for every integrably bounded, norm compact-valued multifunction $\Gamma \in \mathcal{G}(\Omega, X)$.*

An alternative proof of Theorem 5.2. By the RNP of X, there is a function $g \in L^1(\mu, X)$ such that $m(A) = \int_A g \, d\mu$ for every $A \in \mathcal{F}$, where μ is a control measure of m. Define the multifunction $\Gamma : \Omega \twoheadrightarrow X$ by $\Gamma(\omega) = \{\mathbf{0}, g(\omega)\}$ for $\omega \in \Omega$. Then, Γ is obviously integrably bounded and norm-compact valued. Moreover, any measurable selection of Γ is of the form $g\chi_A$ with $A \in \mathcal{F}$, and hence, $\int \Gamma \, d\mu = m(\mathcal{F})$. An appeal to Theorem 6.1 and Proposition 6.1 yields the norm compactness and convexity of $m(\mathcal{F})$. $\square$

This alternative proof leads us to record the validity of the following corollary.

Corollary 6.1. *Let $m : \mathcal{F} \to X$ a vector measure such that $m(A) = \int_A g \, d\mu$ for every $A \in \mathcal{F}$ with $g \in L^1(\mu, X)$. If $(\Omega, \mathcal{F}, \mu)$ saturated, then the range of m is norm compact and convex in X.*

Corollary 6.1 is an exact version of the Lyapunov theorem for vector measures with a Radon–Nikodym derivative in $L^1(\mu, X)$. In this context, Uhl's evaluation is worth keeping in mind. (We quote from [46, p. 162], but use the notation of this paper.)

> If X is allowed to be a general Banach space and m is an X-valued measure of bounded variation, then one can assert that the range of m is precompact and that, in the nonatomic case the closure of the range of m is convex if [...] m has the representation $m(A) = \int_A g \, d\mu$, $A \in \mathcal{F}$ for some measure μ and some measurable g with $\int \|g\| \, d\mu < \infty$. However, this restriction appears, to the author, to be too severe for a general result.

6.2. On restricted cores of a saturated economy. We present an application of Theorem 6.1 to a characterization of the restricted core allocations of an economy with a measure space of agents and an infinite-dimensional commodity space; we follow the formulations of [10], [23]. The following preliminary result clarifies the power of the saturation assumption.

LEMMA 6.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a nonatomic finite measure space, X be a Banach space, Y be a Banach space with the RNP with respect to $(\Omega, \mathcal{F}, \mu)$ and $m : \mathcal{F} \to Y$ be a μ-continuous vector measure of bounded variation. If $\int_E f\, d\mu = \mathbf{0}$ for $f \in L^1(\mu, X)$ and $E \in \mathcal{F}$ with $\mu(E) > 0$, then for every $\alpha \in [0,1]$ there exists a sequence $\{F_n\}$ in $\mathcal{F}$ with $F_n \subset E$ for each n such that*

$$\lim_{n\to\infty}\left(\int_{F_n} f\, d\mu, m(F_n), \mu(F_n)\right) = \big(\mathbf{0}, \alpha m(E), \alpha\mu(E)\big).$$

Proof. Since Y has the RNP regarding to $(\Omega, \mathcal{F}, \mu)$, there exists $g \in L^1(\mu, Y)$ such that $m(A) = \int_A g\, d\mu$ for every $A \in \mathcal{F}$. Define $\varphi \in L^1(\mu, X \times Y \times \mathbb{R})$ by $\varphi(\omega) = (f(\omega), g(\omega), 1)$ for $\omega \in \Omega$ and the nonatomic vector measure $\tilde{m} : \mathcal{F}_E \to X \times Y \times \mathbb{R}$ by $\tilde{m}(A) = \int_A \varphi\, d\mu$ for $A \in \mathcal{F}_E$. An appeal to Uhl's theorem [46, p. 162] guarantees that the closure of $\tilde{m}(\mathcal{F}_E)$ is convex in $X \times Y \times \mathbb{R}$. Take any $\alpha \in [0,1]$. Since $(\mathbf{0}, tm(E), \alpha\mu(E)) = \alpha\tilde{m}(E) + (1-\alpha)\tilde{m}(\emptyset)$ is in the closure of $\tilde{m}(\mathcal{F}_E)$, there exists a sequence $\{F_n\}$ in $\mathcal{F}_E$ such that $\tilde{m}(F_n) \to (\mathbf{0}, \alpha m(E), \alpha\mu(E))$. □

LEMMA 6.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a saturated finite measure space, X be a Banach space, Y be a Banach space with the RNP with respect to $(\Omega, \mathcal{F}, \mu)$ and $m : \mathcal{F} \to Y$ be a μ-continuous vector measure of bounded variation. If $\int_E f\, d\mu = \mathbf{0}$ for $f \in L^1(\mu, X)$ and $E \in \mathcal{F}$ with $\mu(E) > 0$, then for every $\alpha \in [0,1]$ there exists $F \in \mathcal{F}$ with $F \subset E$ such that*

$$\left(\int_F f\, d\mu, m(F), \mu(F)\right) = \big(\mathbf{0}, \alpha m(E), \alpha\mu(E)\big).$$

Proof. Let φ be the function and $\tilde{m}$ be the vector measure respectively given in the proof of Lemma 6.1. Define $\Gamma \in \mathcal{G}(E, X \times Y \times \mathbb{R})$ by $\Gamma(\omega) = \{\tilde{\mathbf{0}}, \varphi(\omega)\}$ for $\omega \in E$, where $\tilde{\mathbf{0}}$ is the origin of $X \times Y \times \mathbb{R}$. Since any measurable selection of Γ is of the form $\varphi\chi_A$ with $A \in \mathcal{F}_E$ and $\tilde{m}(A) = \int_A \varphi\, d\mu$ for every $A \in \mathcal{F}_E$, we have $\int_E \Gamma\, d\mu = \tilde{m}(\mathcal{F}_E)$. In view of the saturation of $(E, \mu_E, \mathcal{F}_E)$, an appeal to Theorem 6.1 guarantees that $\tilde{m}(\mathcal{F}_E)$ is convex in $X \times Y \times \mathbb{R}$. Take any $\alpha \in [0,1]$. Since $(\mathbf{0}, \alpha m(E), \alpha\mu(E)) = \alpha\tilde{m}(E) + (1-\alpha)\tilde{m}(\emptyset)$, there exists $F \in \mathcal{F}_E$ such that $(\mathbf{0}, \alpha m(E), \alpha\mu(E)) = \tilde{m}(F)$ by the convexity of $\tilde{m}(\mathcal{F}_E)$. □

REMARK 6.1. The above lemmas are significant extensions of [41] to the case where the underlying space is an arbitrary Banach space. When $X = \mathbb{R}^n$ and $Y = \mathbb{R}$ in Lemma 6.1, $\{F_n\}$ is shown to be a constant sequence equal to some $F \in \mathcal{F}$ with $F \subset E$ by applying the classical Lyapunov's theorem, which is the case precisely covered by [41]. Lemma 6.1 is also a generalization of [10, p. 1188] who treat the case that X is a Banach space and $Y = \mathbb{R}$. Under the saturation assumption, the approximation result in Lemma 6.1 can be strengthened to the exact result in Lemma 6.2.

We can now turn to the substantive formulation of an economy. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space of *agents* with its generic element denoted by $a \in \Omega$. A *commodity space* X is a Banach space. A *consumption set* $\mathcal{X}(a)$ of each agent is described by a multifunction $\mathcal{X} : \Omega \twoheadrightarrow X$ with $\mathcal{X}(a) \subset X$ for every $a \in \Omega$. A Bochner integrable function $f \in L^1(\mu, X)$ is an *allocation* if $f(a) \in \mathcal{X}(a)$ a.e. $a \in \Omega$. A *preference relation* on $\mathcal{X}(a)$ is a multifunction $P_a : \mathcal{X}(a) \twoheadrightarrow \mathcal{X}(a)$ such that $f(a) \notin P_a(f(a))$ for every allocation f. An *initial endowment* $e(a)$ is given by a function $e \in L^1(\mu, X)$ with $e(a) \in \mathcal{X}(a)$. An *economy* $\mathcal{E}$ is a quadruple $\mathcal{E} = [(\Omega, \mathcal{F}, \mu), \mathcal{X}, (P_a)_{a\in\Omega}, e]$. When $(\Omega, \mathcal{F}, \mu)$ is nonatomic, $\mathcal{E}$ is called an *nonatomic economy* and $(\Omega, \mathcal{F}, \mu)$ is saturated, it is called a *saturated economy*.

An allocation f for an economy $\mathcal{E}$ is *feasible* if $\int f\, d\mu = \int e\, d\mu$. A *coalition* is a set A in $\mathcal{F}$ with $\mu(A) > 0$. A coalition $A \in \mathcal{F}$ *blocks* a feasible allocation f if there exists a feasible allocation g such that $g(a) \in P_a(f(a))$ a.e. $a \in A$ and $\int_A g\, d\mu = \int_A e\, d\mu$. Such a coalition A is called a *blocking coalition* to f. The set of all feasible allocations that no coalition in $\mathcal{F}$ can block is the *core* of the economy $\mathcal{E}$, denoted by $\mathcal{C}(\mathcal{E})$.

The following definition of the core with restricted coalitions is due to [19]. Let $\varepsilon \in (0,1)$ be given arbitrarily. A coalition $A \in \mathcal{F}$ is an *upper ε-coalition* (resp. a *lower ε-coalition*) if $\mu(A) \ge \varepsilon\mu(\Omega)$ (resp. $\mu(A) \le \varepsilon\mu(\Omega)$). The set of all feasible allocations that no upper ε-coalition (resp. lower ε-coalition) in $\mathcal{F}$ can block is the *upper ε-core* (resp. *lower ε-core*) of the economy $\mathcal{E}$, denoted by $\mathcal{C}^{\varepsilon}(\mathcal{E})$ (resp. $\mathcal{C}_{\varepsilon}(\mathcal{E})$). It follows from the definitions that $\mathcal{C}(\mathcal{E}) \subset \mathcal{C}^{\varepsilon}(\mathcal{E})$ and $\mathcal{C}(\mathcal{E}) \subset \mathcal{C}_{\varepsilon}(\mathcal{E})$ for every $\varepsilon \in (0,1)$ and $\varepsilon_1 < \varepsilon_2$ with $\varepsilon_1, \varepsilon_2 \in (0,1)$ implies $\mathcal{C}^{\varepsilon_1}(\mathcal{E}) \subset \mathcal{C}^{\varepsilon_2}(\mathcal{E})$ and $\mathcal{C}_{\varepsilon_2}(\mathcal{E}) \subset \mathcal{C}_{\varepsilon_1}(\mathcal{E})$.

THEOREM 6.2. *Let $\mathcal{E} = [(\Omega, \mathcal{F}, \mu), \mathcal{X}, (P_a)_{a\in\Omega}, e]$ be a saturated economy.*

(i) *If f is a feasible allocation that is blocked by a coalition $A \in \mathcal{F}$ via a feasible allocation g, then for every $\varepsilon \in (0,1)$ there exists a coalition $E \in \mathcal{F}$ with $E \subset A$ and $\mu(E) = \varepsilon\mu(A)$ such that E blocks f via g.*

(ii) *$\mathcal{C}(\mathcal{E}) = \mathcal{C}_{\varepsilon}(\mathcal{E})$ for every $\varepsilon \in (0,1)$.*

(iii) *$\mathcal{C}(\mathcal{E}) = \bigcap_{\varepsilon\in(0,1)} \mathcal{C}^{\varepsilon}(\mathcal{E})$.*

Proof. (i) Since $A \in \mathcal{F}$ is a blocking coalition to f, there exists a feasible allocation g such that $\int_A (g - e)\, d\mu = \mathbf{0}$. By Lemma 6.2, for every $\varepsilon \in (0,1)$ there exists a coalition $E \in \mathcal{F}$ with $E \subset A$ satisfying $\int_E (g - e)\, d\mu = \mathbf{0}$. Hence, f is blocked by the coalition E via the feasible allocation g.

(ii) Take any $\varepsilon \in (0,1)$. If $f \notin \mathcal{C}(\mathcal{E})$, then there exist a coalition $A \in \mathcal{F}$ and a feasible allocation g such that $g(a) \in P_a(f(a))$ a.e. $a \in A$ and $\int_A (g - e)\, d\mu = \mathbf{0}$. By the condition (i) above, there exists a subcoalition $E \subset A$ with $\mu(E) = \varepsilon\mu(A) \le \varepsilon\mu(\Omega)$ satisfying $\int_E (g - e)\, d\mu = \mathbf{0}$. Hence, f is blocked by the lower ε-coalition E via g, and thus $f \notin \mathcal{C}_{\varepsilon}(\mathcal{E})$. Therefore, $\mathcal{C}_{\varepsilon}(\mathcal{E}) \subset \mathcal{C}(\mathcal{E})$.

(iii) If $f \notin \mathcal{C}(\mathcal{E})$, then there exist a coalition $A \in \mathcal{F}$ and a feasible allocation g such that $g(a) \in P_a(f(a))$ a.e. $a \in A$ and $\int_A (g - e)\, d\mu = \mathbf{0}$. When $\mu(A) <$

$\mu(\Omega)$, choose $\varepsilon \in (0,1)$ sufficiently small such that $\mu(A) \geq \varepsilon\mu(\Omega)$. Then f is blocked by the upper ε-coalition A via g, and hence $f \notin \mathcal{C}^{\varepsilon}(\mathcal{E})$. When $\mu(A) = \mu(\Omega)$, take any $\varepsilon \in (0,1)$. By the condition (i) above, there exists a subcoalition $E \subset A$ with $\mu(E) = \varepsilon\mu(A)$ satisfying $\int_E (g - e)\, d\mu = \mathbf{0}$. Hence, f is blocked by the upper ε-coalition E via g, and thus $f \notin \mathcal{C}^{\varepsilon}(\mathcal{E})$. Therefore, $\bigcap_{\varepsilon\in(0,1)} \mathcal{C}^{\varepsilon}(\mathcal{E}) \subset \mathcal{C}(\mathcal{E})$. □

Condition (i) implies that if A is a blocking coalition to a feasible allocation f via a feasible allocation g, then there is an arbitrary small blocking coalition $E \subset A$ with $\mu(E) < \varepsilon$ to the same feasible allocation f also via g. Condition (ii) means that for every positive number ε, the core coincides with the set of all feasible allocations that are not blocked by any coalition of measure less than ε. Condition (iii) is the continuity property of the upper ε-core in the sense that $\mathcal{C}^{\varepsilon}(\mathcal{E})$ is monotone decreasing as $\varepsilon \uparrow 1$ and converges to $\mathcal{C}(\mathcal{E})$.

Remark 6.2. We stress that Theorem 6.2 retains the original spirit of [41], and that, unlike the treatment in [10] which involves an approximation argument based on the special case of Lemma 6.1, there is no additional assumption on the underlying data of the economy: the commodity space, consumption sets, preferences and initial endowments. However, note the asymmetry between the upper and lower cases of the ε-core, and one needs further assumptions to strengthen the "shrinkage" result reported here to one asserting the coincidence of the upper and exact cases, as in the finite-dimensional setting.

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References

[1] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis: A hitchhiker's guide*, 3rd ed., Springer, Berlin, 2006. MR 2378491

[2] N. Azarnia and J. D. M. Wright, *On the Lyapunoff–Knowles theorem*, Q. J. Math. **33** (1982), 257–261. MR 0668170

[3] R. G. Bartle, N. Dunford and J. T. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. **7** (1955), 289–305. MR 0070050

[4] C. Carathéodory, *Die homomorphieen von Somen und die Multiplikation von Inhalts-funionen*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) **8** (1939), 8–130. MR 1556820

[5] G. Debreu, *Integration of correspondences*, Proc. fifth Berkeley sympos. math. statist. probability, vol. II: Contributions to probability theory, part 1, Univ. California Press, Berkeley, 1967, pp. 351–372. MR 0228252
[6] J. Diestel and J. J. Uhl Jr., *Vector measures*, Amer. Math. Soc., Providence, 1977. MR 0453964
[7] N. Dunford and J. T. Schwartz, *Linear operators, part I: General theory*, Wiley, New York, 1958. MR 1009162
[8] A. Dvoretzky, *On Liapunov's convexity theorem*, Proc. Natl. Acad. Sci. USA **91** (1994), 2145. MR 1264123
[9] M. Džamonja and K. Kunen, *Properties of the class of measure separable compact spaces*, Fund. Math. **147** (1995), 261–277. MR 1348722
[10] Ö. Evren and F. Hüsseinov, *Theorems on the core of an economy with infinitely many commodities and consumers*, J. Math. Econom. **44** (2008), 1180–1196. MR 2456475
[11] S. Fajardo and H. J. Keisler, *Model theory of stochastic processes*, AK Peters, Natick, 2002. MR 1939107
[12] D. H. Fremlin, *Measure theory, volume 3: Measure algebras*, Torres Fremlin, Colchester, 2002. MR 2459668
[13] P. R. Halmos and J. von Neumann, *Operator methods in classical mechanics, II*, Ann. of Math. (2) **43** (1942), 332–350. MR 0006617
[14] D. Hoover and H. J. Keisler, *Adapted probability distributions*, Trans. Amer. Math. Soc. **286** (1984), 159–201. MR 0756035
[15] V. M. Kadets, *Remark on the Lyapunov theorem on vector measures*, Funct. Anal. Appl. **25** (1991), 295–297. MR 1167727
[16] V. M. Kadets and M. M. Popov, *On the Lyapunov convexity theorem with applications to sign-embeddings*, Ukrainian Math. J. **44** (1992), 1091–1098. MR 1211141
[17] V. M. Kadets and G. Shekhtman, *The Lyapunov theorem for ℓ_p-valued measures*, St. Petersburg Math. J. **4** (1993), 961–966. MR 1202728
[18] H. J. Keisler and Y. N. Sun, *Why saturated probability spaces are necessary*, Adv. Math. **221** (2009), 1584–1607. MR 2522428
[19] M. A. Khan, *Some remarks on the core of a "large" economy*, Econometrica **42** (1974), 633–642. MR 0452542
[20] M. A. Khan, *Approximately convex average sums of sets in normed spaces*, Appl. Math. Comput. **9** (1981), 27–34. MR 0650675
[21] M. A. Khan, *On the integration of set-valued mappings in a non-reflexive Banach space, II*, Simon Stevin **59** (1985), 257–267. MR 0833482
[22] M. A. Khan and L. P. Rath, *The Shapley–Folkman theorem and the range of a bounded measure: An elementary and unified treatment*, Positivity **17** (2013), 381–394. MR 3090669
[23] M. A. Khan and N. C. Yannelis, *Equilibria in markets with a continuum of agents and commodities*, Equilibrium theory in infinite dimensional spaces (M. A. Khan and N. C. Yannelis, eds.), Springer, Berlin, 1991, pp. 233–248.
[24] M. A. Khan and Y. C. Zhang, *Set-valued functions, Lebesgue extensions and saturated probability spaces*, Adv. Math. **229** (2012), 1080–1103. MR 2855087
[25] J. F. C. Kingman and A. P. Robertson, *On a theorem of Lyapunov*, J. London Math. Soc. **43** (1968), 347–351. MR 0224768
[26] I. Kluvánek and G. Knowles, *Vector measures and control systems*, North-Holland, Amsterdam, 1975. MR 0499068
[27] G. Knowles, *Lyapunov vector measures*, SIAM J. Control Optim. **13** (1975), 294–303. MR 0388216
[28] H. E. Lacey, *The Hamel dimension of any infinite dimensional separable Banach space is c*, Amer. Math. Monthly **80** (1973), 298. MR 0320717

[29] H. E. Lacey, *The isometric theory of classical Banach spaces*, Springer, Berlin, 1974. MR 0493279

[30] J. Lindenstrauss, *A short proof of Liapounoff's convexity theorem*, J. Math. Mech. **15** (1966), 971–972. MR 0207941

[31] P. A. Loeb, *Conversion from nonstandard to standard measure spaces and applications in probability theory*, Trans. Amer. Math. Soc. **211** (1975), 113–122. MR 0390154

[32] P. A. Loeb and Y. N. Sun, *Purification and saturation*, Proc. Amer. Math. Soc. **137** (2009), 2719–2724. MR 2497484

[33] A. Liapounoff, *Sur les fonctions-vecteurs complètement additives*, Bull. Acad. Sci. URSS. Sér. Math. **4** (1940), 465–478 (in Russian). MR 0004080

[34] D. Maharam, *On homogeneous measure algebras*, Proc. Natl. Acad. Sci. USA **28** (1942), 108–111. MR 0006595

[35] K. Podczeck, *On the convexity and compactness of the integral of a Banach space valued correspondence*, J. Math. Econom. **44** (2008), 836–852. MR 2414904

[36] H. P. Rosenthal, *On the injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measures μ*, Acta Math. **124** (1970), 205–248. MR 0257721

[37] H. L. Royden, *Real analysis*, 3rd ed., Macmillan, New York, 1988. MR 0151555

[38] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973. MR 0365062

[39] A. Rustichini and N. C. Yannelis, *What is perfect competition?* Equilibrium theory in infinite dimensional spaces (M. A. Khan and N. C. Yannelis, eds.), Springer, Berlin, 1991, pp. 249–265.

[40] E. Saab and P. Saab, *Lyapunov convexity type theorems for non-atomic vector measures*, Quaest. Math. **26** (2003), 371–383. MR 2018918

[41] D. Schmeidler, *A remark on the core of an atomless economy*, Econometrica **40** (1972), 579–580. MR 0342144

[42] Y. N. Sun, *On the theory of vector valued Loeb measures and integration*, J. Funct. Anal. **104** (1992), 327–362. MR 1153991

[43] Y. N. Sun, *Distributional properties of correspondences on Loeb spaces*, J. Funct. Anal. **139** (1996), 68–93. MR 1399686

[44] Y. N. Sun, *Integration of correspondences on Loeb spaces*, Trans. Amer. Math. Soc. **349** (1997), 129–153. MR 1401529

[45] Y. N. Sun and N. C. Yannelis, *Saturation and the integration of Banach valued correspondences*, J. Math. Econom. **44** (2008), 861–865. MR 2414906

[46] J. J. Uhl Jr., *The range of a vector-valued measure*, Proc. Amer. Math. Soc. **23** (1969), 158–163. MR 0264029

[47] R. G. Ventner, *Liapounoff convexity-type theorems*, Vector measures, integration and related topics (G. P. Curbera, G. Mockenhaupt and W. J. Ricker, eds.), Birkhäuser, Berlin, 2010, pp. 371–380. MR 2744004

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