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COMPUTING FIXED CLOSURES IN FREE GROUPS

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To Paul Schupp, colleague and friend.

ABSTRACT. Let F be a finitely generated free group. We present an algorithm such that, given a subgroup $H \leq F$, decides whether H is the fixed subgroup of some family of automorphisms, or family of endomorphisms of F and, in the affirmative case, finds such a family. The algorithm combines both combinatorial and geometric methods.

1. Introduction

For all the paper, let $A = \{a_1, \ldots, a_n\}$ be an alphabet with n different letters, and F be the free group (of rank $r(F) = n$) with basis A.

Let $\mathrm{End}(F)$ denote the endomorphism monoid of F, and $\mathrm{Aut}(F)$ the automorphism group of F, so $\mathrm{Aut}(F)$ is the group of units of $\mathrm{End}(F)$. Throughout, we let elements of $\mathrm{End}(F)$ act on the right on F, so $x \mapsto (x)\phi$. Accordingly, compositions are like $(x)\phi\psi = (x\phi)\psi$.

In the last decade, a lot of literature has appeared studying the fixed subgroup of a single, or a family, of automorphisms, or endomorphisms, of F (see the survey [19] for details). But very few algorithmic results are known in this direction. Only few years ago, O. Maslakova (see [13]) has found an algorithm to compute a set of generators for the fixed subgroup of an automorphism of F (which is quite complicated, and whose complexity is quite high). One can easily extend this to compute generators for the fixed subgroup of a finite family of automorphisms, while the related questions on endomorphisms are still open.

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In this note, we shall address the dual problem. We present an algorithm such that, given a subgroup $H \leq F$, decides whether H is the fixed subgroup of some finite family of automorphisms, or finite family of endomorphisms of F and, in the affirmative case, finds such a family. We advice the reader that the provided algorithms are theoretical and far from effective, in the sense that their complexities will be way too high to think about possible effective implementations. A natural open question is whether there exist more natural and efficient algorithms, say polynomial time, for solving such problems.

Recognizing whether H is the fixed subgroup of a family of automorphisms is not difficult, because a classical result by McCool already established that the stabilizer of a finitely generated subgroup of F is finitely generated and computable (as subgroup of $\mathrm{Aut}(F)$), see Theorem 8 below. However, recognizing whether H is the fixed subgroup of a family of endomorphisms is much trickier, because the stabilizer of H, in general, need not be finitely generated as a submonoid of $\mathrm{End}(F)$ (see Example 1 below).

More precisely, the two algorithms given in this paper take a finitely generated subgroup $H \leq F$ as the input, and compute a basis of its automorphism (resp. endomorphism) closure, that is, the smallest subgroup K such that $H \leq K \leq F$ and $K = \mathrm{Fix}(S)$ for some $S \subseteq \mathrm{Aut}\, F$ (resp. $S \subseteq \mathrm{End}\, F$), see the precise definitions below. The main technique used to deal with these problems is the graphical tool called "fringe of a subgroup", which allows to compute the collection of algebraic extensions of a given subgroup H.

In Section 2, we define the concepts, and state the results that will be used, with the corresponding references. Along Section 3, we prove the main result and give the two announced algorithms. Finally, in Section 4, we collect a list of related questions and open problems.

2. Needed tools

Definition 1. For any $S \subseteq \mathrm{End}(F)$, let $\mathrm{Fix}(S)$ denote the subset consisting of those elements of F which are fixed by every element of S (read $\mathrm{Fix}(S) = F$ for the case where S is empty). Then $\mathrm{Fix}(S)$ is a subgroup of F, called the *fixed subgroup of S*.

A subgroup H of F is called an *endo-fixed* subgroup of F if $H = \mathrm{Fix}(S)$ for some $S \subseteq \mathrm{End}(F)$. If S can be chosen to lie in $\mathrm{Aut}(F)$, we further say that H is an *auto-fixed* subgroup of F.

A subgroup H of F is called a 1-*endo-fixed* subgroup of F if $H = \mathrm{Fix}(\phi)$ for some $\phi \in \mathrm{End}(F)$ (here, and throughout, to simplify notation we write $\mathrm{Fix}(\phi)$ rather than $\mathrm{Fix}(\{\phi\})$). If ϕ can be chosen to lie in $\mathrm{Aut}(F)$, we further say that H is a 1-*auto-fixed* subgroup of F. For example, any maximal cyclic subgroup of F is 1-auto-fixed, since it is the subgroup fixed by a suitable inner automorphism. And nonmaximal cyclic subgroups of F are not even endo-fixed, because every endomorphism fixing a power of an element must fix the element itself.

Notice that, since $\mathrm{Fix}(S) = \bigcap_{\alpha \in S} \mathrm{Fix}(\alpha)$, an auto-fixed (resp. endo-fixed) subgroup is an intersection of 1-auto-fixed (resp. 1-endo-fixed) subgroups, and vice-versa. And, clearly, the families of auto-fixed and endo-fixed subgroups of F are closed under arbitrary intersections.

A natural question that arises in this context asks about the relation between the four mentioned families of subgroups of F, namely 1-auto-fixed, 1-endo-fixed, auto-fixed and endo-fixed subgroups. Apart from the obvious inclusions, the relationship among these families is partially known, though not completely since there still are interesting questions in this direction that remain open. For example, it is not known (and conjectured) whether the families of 1-auto-fixed and auto-fixed subgroups (resp. 1-endo-fixed and endo-fixed subgroups) do coincide; in other words, it is not known whether the family of 1-auto-fixed (resp. 1-endo-fixed) subgroups is closed under intersections. As far as we are aware, this is only known to be true when the ambient rank is $n = 2$, and when the involved fixed subgroups have maximal rank, see [10]. In this direction, A. Martino and E. Ventura showed in [10] that every auto-fixed (resp. endo-fixed) subgroup of F is a free factor of a 1-auto-fixed (resp. 1-endo-fixed) subgroup of F. However, they also gave an example of a free factor of a 1-auto-fixed subgroup which is not even endo-fixed.

We point out that, in the definitions of auto-fixed and endo-fixed subgroups, one can always assume that the involved families of morphisms are finite. This was proven by A. Martino and E. Ventura in [12, Corollary 4.2], answering a question previously posed by G. Levitt. So, from now on, a "family" of endomorphisms will always mean a "finite family".

Proposition 2 (Martino–Ventura, [12]). *Let F be a finitely generated free group. For every $S \subseteq \mathrm{End}(F)$ there exists a finite subset $S_0 \subseteq S$ with $|S_0| \leq 2r(F)$, and such that* $\mathrm{Fix}(S_0) = \mathrm{Fix}(S)$.

We do not include in this discussion the families of 1-mono-fixed and mono-fixed subgroups, because they are known to coincide with the families of 1-auto-fixed and auto-fixed subgroups, respectively (see [11, Theorem 11]).

On the other hand, it is known that the families of 1-endo-fixed and 1-auto-fixed subgroups (and the families of endo-fixed and auto-fixed subgroups, as well) *do not* coincide: in [11], the authors exhibited the first known examples of 1-endo-fixed subgroups which are not 1-auto-fixed; see also [3] for more interesting calculations about this phenomena. Hence, determining whether a given subgroup H is an auto-fixed or an endo-fixed subgroup are two different algorithmic problems (the first being much simpler than the second, as will be seen below).

The deepest and most important result about 1-auto-fixed subgroups in the literature was obtained by M. Bestvina and M. Handel in [2], where they developed the theory of train tracks for graphs, and showed that every 1-auto-fixed subgroup of F has rank at most $r(F)$, which had previously been conjectured by G. P. Scott. Soon after the announcement of this result, and using it, W. Imrich and E. Turner showed, in [5], that any 1-endo-fixed subgroup of F also has rank at most $r(F)$. Later, W. Dicks and E. Ventura in [4], using the techniques of [2], showed that any auto-fixed subgroup of F has rank at most $r(F)$; in fact, they proved a stronger result, namely that any mono-fixed subgroup of F is F-inert (a subgroup $H \leq F$ is *F-inert* if $r(H \cap K) \leq r(K)$ for every $K \leq F$). And after this, G. M. Bergman [1], using the result of [4], showed that any endo-fixed subgroup of F also has rank at most $r(F)$ (however, it is not known whether endo-fixed subgroups of F are necessarily F-inert; it is conjectured to be so in the *inertia conjecture*, see [12] and [19]). This brief history is appropriate for our purposes, but is far from complete; for example, it does not mention the ground-breaking work of S. M. Gersten, who first showed that 1-auto-fixed subgroups are finitely generated.

As we mentioned in the Introduction, few algorithmic results are known about fixed subgroups of free groups. The main one is the computability of fixed subgroups of automorphisms which, by the moment, it has only theoretical interest because no precise bound on the complexity is known, and one expects it to be quite high. The corresponding fact for endomorphisms is still an open problem.

Theorem 3 (Maslakova, [13]). *Let $\varphi : F \to F$ be an automorphism of a finitely generated free group F. Then, a basis for* $\mathrm{Fix}(\varphi)$ *is computable.*

An interesting notion to study these questions is the notion of "closure" of a subgroup.

Definition 4. Let $H \leq F$. We denote by $\mathrm{Aut}_H(F)$ the subgroup of $\mathrm{Aut}(F)$ consisting of all automorphisms of F which fix H pointwise,

$$\mathrm{Aut}_H(F) = \{\varphi \in \mathrm{Aut}(F) \mid H \leq \mathrm{Fix}(\varphi)\},$$

usually called the *stabilizer* of H. Analogously, we denote by $\mathrm{End}_H(F)$ the submonoid of $\mathrm{End}(F)$ consisting of all endomorphisms of F which fix every element of H. Clearly, $\mathrm{Aut}_H(F) \leq \mathrm{End}_H(F)$.

Now, $\mathrm{Aut}_{(-)}(F)$ is a function from the set of subgroups of F to the set of subsets of $\mathrm{Aut}(F)$, and $\mathrm{Fix}(-)$ is a function in the reverse direction. This pair of functions form a Galois connection, and their images are called *closed* subsets (in $\mathrm{Aut}(F)$ and F, respectively). Clearly, $\mathrm{Aut}(F)$-closed subgroups of F are precisely the auto-fixed subgroups. Mimicking the classical Galois notions, we define the *auto-closure* of H in F, denoted $a\text{-}Cl_F(H)$, as $\mathrm{Fix}(\mathrm{Aut}_H(F))$, that is, the smallest auto-fixed subgroup of F containing H.

Replacing Aut to End everywhere in the previous paragraph, we obtain another Galois connection, and we similarly define the *endo-closure* of H in F, denoted $e\text{-}Cl_F(H)$, as $\mathrm{Fix}(\mathrm{End}_H(F))$, i.e., the smallest endo-fixed subgroup of F containing H. Since $\mathrm{Aut}_H(F) \leq \mathrm{End}_H(F)$, an obvious relation between closures is

$$e\text{-}Cl_F(H) = \mathrm{Fix}(\mathrm{End}_H(F)) \leq \mathrm{Fix}(\mathrm{Aut}_H(F)) = a\text{-}Cl_F(H).$$

However, the equality does not hold in general, because of the existence of 1-endo-fixed subgroups which are not auto-fixed.

Note that, by the results mentioned above, the ranks of $a\text{-}Cl_F(H)$ and $e\text{-}Cl_F(H)$ are always less than or equal to $r(F)$, even if that of H is not.

The main goal of this note is to show that, for any finitely generated $H \leq F$ (given by a set of generators), one can algorithmically compute a basis for both $a\text{-}Cl_F(H)$ and $e\text{-}Cl_F(H)$. Using this algorithm, one can immediately decide whether the given H is auto-fixed (resp. endo-fixed) or not: H is auto-fixed (resp. endo-fixed) if and only if $a\text{-}Cl_F(H) = H$ (resp. $e\text{-}Cl_F(H) = H$).

To do this, we need to use the concepts of retract and stable image, and the graphical technique called "fringe of a subgroup" to compute the set of algebraic extensions of H. We briefly review now on these two topics.

A subgroup $H \leq F$ is called a *retract of F* (just *retract* if there is no risk of confusion) if there exists a homomorphism $\rho\colon F \to H$ which fixes the elements of H (i.e., such that $\rho^2 = \rho$); such a morphism is called a *retraction*. The obvious examples of retracts are the free factors of F, but there are retracts which are not free factors. Recognizing retracts is algorithmically possible, as showed in [14, Proposition 4.6] following an argument indicated by E. Turner, though quite complicated in practice, because it makes use of Makanin's algorithm to solve systems of equations in free groups.

PROPOSITION 5 (Proposition 4.6 in [14]). *Let $H \leq F$ be a finitely generated subgroup of F, given by a finite set of generators. It is algorithmically decidable whether H is a retract of F and, in the affirmative case, find a retraction $\rho\colon F \to H$.*

For a given endomorphism $\varphi\colon F \to F$, define the *stable image* of φ as $F\varphi^\infty = \bigcap_{m=1}^{\infty} F\varphi^m$. With a simple argument, W. Imrich and E. Turner showed in [5] that: (1) $F\varphi^\infty$ is a φ-invariant subgroup of F; (2) the restriction of φ to its stable image is always an automorphism; and (3) $F\varphi^\infty$ is a retract of F. This will be used later in order to reduce a certain computation with endomorphisms, to a similar one with automorphisms.

Let $H \leq K \leq F$. We say that the extension $H \leq K$ is *algebraic*, denoted $H \leq_{\mathrm{alg}} K$, if H is not contained in any proper free factor of K. The antagonistic situation consists of H being a *free factor* of K, denoted $H \leq_{\mathrm{ff}} K$. It is not difficult to see (see [14]) that every extension $H \leq K$ of finitely generated (free) subgroups of F can be decomposed, in a unique way, as an algebraic extension followed by a free extension, namely $H \leq_{\mathrm{alg}} L \leq_{\mathrm{ff}} K$ (just take L to be the smallest free factor of K containing H, or the biggest algebraic extension of H contained in K). The uniqueness refers to the fact that L

is completely determined by the original extension $H \leq K$; again, mimicking the classical Galois theory, L is called the *algebraic closure of* H *in* K. We refer the reader to [14] for a detailed development of these ideas, including an analysis of the similarities and the significant differences with respect to classical field theory.

The important fact in this story is an old result by M. Takahasi, originally proven by combinatorial methods in [17] (and reproduced in Section 2.4, Exercise 8 of [8]). However, the modern graphical techniques developed by Stalling's in the 1980s (see [16]) lead to a new, clear, concise and very natural proof of Takahasi's theorem, which was discovered independently by E. Ventura in [18], and by I. Kapovich and A. Miasnikov in [6]. S. Margolis, M. Sapir and P. Weil, also independently, considered the same construction in [9] for a slightly different purposes. See [14] for a unification of these three points of view, written in the language of algebraic extensions. In this setting, Takahasi's theorem says the following.

THEOREM 6 (Takahasi). *Let $H \leq F$ be a subgroup of a free group F. If H is finitely generated, then it has finitely many algebraic extensions, that is,*

$$\mathcal{AE}(H) = \{K \leq F \mid H \leq_{\text{alg}} K\}$$

is finite. Furthermore, the elements in $\mathcal{AE}(H)$ are finitely generated, and bases of all of them are computable from any given set of generators for H.

Sketch of proof (*See* [14, *Proposition* 3.7] *for details*). Think $F = \langle A \mid \rangle$ as the fundamental group of a bouquet of n circles, and then H as the corresponding covering $X(H)$, which can be though of as a graph with labels from A on the edges (this graph is easily computable from any given set of generators of H). When H is of infinite index in F, the graph $X(H)$ is infinite but, if H is finitely generated, $X(H)$ consists on a finite core $\Gamma(H)$ with attached infinite trees (each isomorphic to a subtree of the Cayley graph of F with respect to A). Now, consider an arbitrary extension $H \leq K \leq F$. It corresponds to another covering $X(K)$, which is in turn covered by $X(H)$. That is, $X(K)$ is a quotient of $X(H)$ and so, can be obtained from $X(H)$ by performing several identifications of vertices and edges. These identifications may destroy $\Gamma(H)$, but some quotient of $\Gamma(H)$ always remains as a subgraph of $X(K)$ (in fact, of $\Gamma(K)$). If H is finitely generated then $\Gamma(H)$ is finite, and so has finitely many quotients, which are computable from $\Gamma(H)$ (i.e., from any given set of generators of H). This gives a computable finite list of extensions of H, called the *fringe of* H, $\mathcal{O}(H) = \{H_1, \ldots, H_p\}$, $p \geq 1$. And, by construction, it is clear that, for every $H \leq K$, there exists $i = 1, \ldots, p$ such that $H \leq H_i \leq_{\text{ff}} K$. This implies that $\mathcal{AE}(H) \subseteq \mathcal{O}(H)$ and so, we already have a proof of the finiteness part of Takahasi's theorem. Unfortunately, the equality between these two sets is not true in general, as one can possibly find free factor relations between the H_i's. But, after a cleaning process (checking for every pair (i,j) whether $H_i \leq_{\rm ff} H_j$ and, in this case, deleting H_j from the list) one can algorithmically compute $\mathcal{AE}(H) = \{H_1, \ldots, H_q\}$, $1 \leq q \leq p$. See [15] for a polynomial time algorithm to check free factorness. □

Note that the smallest and the biggest of the H_i's in $\mathcal{O}(H)$ correspond, respectively, to the quotient identifying nothing, which gives H itself, and to the quotient identifying all vertices down to a single one, which gives $\langle A' \rangle \leq_{\rm ff} F$, where $A' \subseteq A$ is the set of all letters involved in the generators of H. Note also that the first one belongs to $\mathcal{AE}(H)$ (since $H \leq_{\rm alg} H$) and the same happens for either the second one or a free factor of it. In particular, $\mathcal{AE}(H)$ contains at least H, and a free factor of F (which may coincide). This fact will be used later.

Finally, we mention one of the results in [16]. Given two finitely generated subgroups $H, K \leq F$ (by sets of generators, say), one can algorithmically compute a basis for $H \cap K$ using the technique of pull-backs of graphs.

Proposition 7 (Stallings, [16]). *Let $H, K \leq F$ be two finitely generated subgroups of a free group F, given by finite sets of generators. Then, a basis for $H \cap K$ is algorithmically computable.*

3. The algorithms

Let $H \leq F$ be a finitely generated subgroup of F, given by a set of generators. We shall give two algorithms to compute a basis for $a\text{-}Cl_F(H)$ and $e\text{-}Cl_F(H)$, respectively. The basic fact that we use is a classical result due to J. McCool (see Proposition 5.7 in Chapter I of [7], and the subsequent paragraph).

Theorem 8 (McCool, [7]). *Let $H \leq F$ be a finitely generated subgroup of a (finitely generated) free group, given by a finite set of generators. Then the stabilizer, $\mathrm{Aut}_H(F)$, of H is also finitely generated (in fact finitely presented), and a finite set of generators (and relations) is algorithmically computable from H.*

3.1. The automorphism case. By Theorem 8, $\mathrm{Aut}_H(F)$ is finitely generated; furthermore, a list of generators, say $\mathrm{Aut}_H(F) = \langle \varphi_1, \ldots, \varphi_m \rangle \leq \mathrm{Aut}(F)$, can be algorithmically found from a set of generators of H. Now it is clear that

$$a\text{-}Cl_F(H) = \mathrm{Fix}(\mathrm{Aut}_H(F)) = \bigcap_{\varphi \in \mathrm{Aut}_H(F)} \mathrm{Fix}(\varphi) = \mathrm{Fix}(\varphi_1) \cap \cdots \cap \mathrm{Fix}(\varphi_m).$$

By Maslakova's Theorem 3, we can then compute generators for each of the $\mathrm{Fix}(\varphi_i)$'s and, using Proposition 7, find a basis for their intersection that is $a\text{-}Cl_F(H)$. Finally, Proposition 2 ensures us that a certain subset of at most $2r(F)$ of those φ_i's also makes the job; it only remains to recurrently compute intersections until finding such a set. Thus, we have proven the following.

THEOREM 9. *Let $H \leq F$ be a finitely generated subgroup of a free group, given by a finite set of generators. Then, a basis for the auto-closure a-$Cl_F(H)$ of H is algorithmically computable, together with a set of $m \leq 2r(F)$ automorphisms $\varphi_1, \ldots, \varphi_m \in \mathrm{Aut}(F)$, such that a-$Cl_F(H) = \mathrm{Fix}(\varphi_1) \cap \cdots \cap \mathrm{Fix}(\varphi_m)$.*

COROLLARY 10. *Let $H \leq F$ be a finitely generated subgroup of a free group, given by a finite set of generators. Then, it is algorithmically decidable whether H is auto-fixed and, in the affirmative case, find a set of $m \leq 2r(F)$ automorphisms $\varphi_1, \ldots, \varphi_m \in \mathrm{Aut}(F)$, such that $H = \mathrm{Fix}(\varphi_1) \cap \cdots \cap \mathrm{Fix}(\varphi_m)$.*

Proof. Apply Theorem 9 to H. If a-$Cl_F(H)$ is strictly bigger than H, then H is not auto-fixed (there are elements outside H which are fixed by *every* automorphism of F fixing H). Otherwise, a-$Cl_F(H) = H$ and the algorithm in Theorem 9 also outputs a list of $m \leq 2r(F)$ automorphisms $\varphi_1, \ldots, \varphi_m \in \mathrm{Aut}(F)$, such that $\mathrm{Fix}(\varphi_1) \cap \cdots \cap \mathrm{Fix}(\varphi_m) = $ a-$Cl_F(H) = H$. □

We don't play much attention to the complexity of this algorithm because it seems far from fast. McCool's algorithm is a brute force search which is not conceptually complicated, but has strongly exponential complexity. Maslakova's algorithm is conceptually much more sophisticated, and its complexity also seems to be quite high. Finally, the algorithm to compute intersections is both easy and fast.

3.2. The endomorphism case. There is no hope that a similar strategy could work in general for endomorphisms instead of automorphisms. On one hand, Maslakova's theorem makes strong use of train tracks, a machinery that only works for monomorphisms and definitely does not work in presence of nontrivial kernel; in fact, at the time of writing, no algorithm is known to compute the fixed subgroup of an arbitrary endomorphism of F. On the other hand, as the following example shows, $\mathrm{End}_H(F)$ is not always finitely generated as submonoid of $\mathrm{End}(F)$ and so, there is no hope of having a variation of McCool's result for endomorphisms.

EXAMPLE 1 (*Ciobanu–Dicks*, [3]). We reproduce here Example 1.4 of [3] to show that $\mathrm{End}_H(F)$ is not always finitely generated as a submonoid of $\mathrm{End}(F)$, even with H being finitely generated as a subgroup of F.

Let $F = \langle a, b, c\rangle$ be the free group of rank 3, let $d = ba[c^2, b]a^{-1}$ (where $[x, y] = xyx^{-1}y^{-1}$), and consider the subgroup $H = \langle a, d\rangle \leq F$. Consider the endomorphism $\psi : F \to F$ given by $a \mapsto a$, $b \mapsto d$, $c \mapsto 1$, and the automorphism $\phi : F \to F$ given by $a \mapsto a$, $b \mapsto b$, $c \mapsto cb$. Straightforward computations show that $d\psi = d$ hence, $\psi \in \mathrm{End}_H(F)$. Moreover, $\phi^n\psi$ acts as $a \mapsto a$, $b \mapsto d$, $c \mapsto d^n$ and so, we also have $\phi^n\psi \in \mathrm{End}_H(F)$ for every $n \in \mathbb{Z}$. Now, Corollary 3.4 from [3] shows that this is precisely the whole stabilizer of H,

$$\mathrm{End}_H(F) = \{1, \phi^n\psi \mid n \in \mathbb{Z}\} = \{1\} \cup \langle\phi\rangle\psi.$$

Again, an easy calculation shows that $(\phi^n\psi)\cdot(\phi^m\psi)=\phi^n\psi$, for every $n,m\in\mathbb{Z}$. Thus, the monoid $\mathrm{End}_H(F)$ is *not* finitely generated.

Being convinced that the above algorithm for the automorphism case cannot be adapted to the endomorphism case, a different strategy is needed. We shall use algebraic extensions, Takahasi's theorem and retractions to reduce the computation of the endo-closure $e\text{-}Cl_F(H)$ to finitely many computations of auto-closures.

THEOREM 11. *Let $H\leq F$ be a finitely generated subgroup of a free group, given by a finite set of generators. Then, a basis for the endo-closure $e\text{-}Cl_F(H)$ of H is algorithmically computable, together with a set of $m\leq 2r(F)$ endomorphisms $\varphi_1,\dots,\varphi_m\in\mathrm{End}(F)$, such that $e\text{-}Cl_F(H)=\mathrm{Fix}(\varphi_1)\cap\cdots\cap\mathrm{Fix}(\varphi_m)$.*

Proof. Consider the set $\mathcal{AE}(H)=\{H_1,H_2,\dots,H_q\}$ of algebraic extensions of H, and the subset of those which are retracts of F, say $\mathcal{AE}_{\mathrm{ret}}(H)=\{H_1,\dots,H_r\}$, $1\leq r\leq q$ (note that $\mathcal{AE}_{\mathrm{ret}}(H)$ is not empty because, as we noted above, $\mathcal{AE}(H)$ contains at least a free factor (and so a retract) of F). By Theorem 6, we can algorithmically compute $q\geq 1$, and a basis for each $H_1,\dots,H_q$. Now, using Theorem 5, we can algorithmically decide which of these H_i's are retracts of F, and so compute $r\geq 1$, $\mathcal{AE}_{\mathrm{ret}}(H)=\{H_1,\dots,H_r\}$, and retractions $\rho_i: F\to H_i$, for $i=1,\dots,r$. Then, write the generators of H in terms of the computed bases of each one of these H_i's, and apply r times Theorem 9 to compute, for every $i=1,\dots,r$, a basis for $a\text{-}Cl_{H_i}(H)$ together with a collection of (at most $2r(H_i)$) automorphisms $\alpha_{i,j}\in\mathrm{Aut}(H_i)$ such that $\bigcap_j \mathrm{Fix}(\alpha_{i,j})=a\text{-}Cl_{H_i}(H)$, and bases for all these fixed subgroups $\mathrm{Fix}(\alpha_{i,j})$. Finally, use Proposition 7 to find a basis for $\bigcap_{i=1}^r a\text{-}Cl_{H_i}(H)$.

Now, we claim that $\bigcap_{i=1}^r a\text{-}Cl_{H_i}(H)=e\text{-}Cl_F(H)$. In fact, we shall prove this equality under the form

$$\bigcap_{i=1}^{r}\ \bigcap_{\substack{\alpha\in\mathrm{Aut}(H_i)\\ H\leq\mathrm{Fix}(\alpha)}}\mathrm{Fix}(\alpha)=\bigcap_{\substack{\beta\in\mathrm{End}(F)\\ H\leq\mathrm{Fix}(\beta)}}\mathrm{Fix}(\beta), \tag{1}$$

by showing that every intersecting subgroup in one side is also present in the opposite side (or otherwise a subgroup of it).

Let $\beta\in\mathrm{End}(F)$ be such that $H\leq\mathrm{Fix}(\beta)$. Consider the stable image of β, which is a retract of F, and contains $\mathrm{Fix}(\beta)$ and so H; then, look at the algebraic closure of H in it,

$$H\leq_{\mathrm{alg}} H_i\leq_{\mathrm{ff}} F\beta^{\infty}\leq_{\mathrm{ret}} F.$$

Since free factors of retracts of F are retracts of F, this H_i is an element of $\mathcal{AE}_{\mathrm{ret}}(H)$. Furthermore, the endomorphism β restricts to an automorphism of $F\beta^{\infty}$ which, in turn, restricts to an automorphism $\alpha=\beta_{|H_i}$ of H_i: the automorphism $\beta_{|F\beta^{\infty}}$ leaves H invariant (in fact, pointwise fixed) and sends free factors of $F\beta^\infty$ to free factors of $F\beta^\infty$; hence, the smallest one containing H, which is H_i, must be sent to itself. And, clearly, $H \leq \mathrm{Fix}(\alpha) \leq \mathrm{Fix}(\beta)$. This shows inclusion "$\leq$" in equation (1).

Now, let $H_i \in \mathcal{AE}_{\mathrm{ret}}(H)$, and let $\alpha \in \mathrm{Aut}(H_i)$ with $H \leq \mathrm{Fix}(\alpha)$. Consider $\beta = \rho_i \alpha \iota_i \in \mathrm{End}(F)$, where $\rho_i : F \to H_i$ is a retraction, and $\iota_i : H_i \to F$ is the inclusion map. It is clear that $H \leq \mathrm{Fix}(\alpha) = \mathrm{Fix}(\beta)$ and so, $\mathrm{Fix}(\alpha)$ is also one of the subgroups appearing in the intersection on the right-hand side of (1). This shows inclusion "$\geq$" in equation (1) and completes the proof of the claim.

Thus, the algorithm described in the first paragraph of this proof, certainly computes a basis for $e\text{-}Cl_F(H)$ (together with some side information, namely the retractions ρ_i, the collection of automorphisms $\alpha_{i,j} \in \mathrm{Aut}(H_i)$, and bases for their fixed subgroups $\mathrm{Fix}(\alpha_{i,j})$). To conclude, it only remains to explicitly construct a list of at most $2r(F)$ endomorphisms of F whose fixed set is exactly $e\text{-}Cl_F(H)$. This is easy from the previous paragraph: the collection of endomorphisms $\beta_{i,j} = \rho_i \alpha_{i,j} \iota_i \in \mathrm{End}(F)$ satisfy $\mathrm{Fix}(\beta_{i,j}) = \mathrm{Fix}(\alpha_{i,j})$ and so,

$$\bigcap_{i,j} \mathrm{Fix}(\beta_{i,j}) = \bigcap_i \left(\bigcap_j \mathrm{Fix}(\alpha_{i,j}) \right) = \bigcap_i a\text{-}Cl_{H_i}(H) = e\text{-}Cl_F(H).$$

It could happen that the computed set, $\{\beta_{i,j} \mid i,j\}$, of endomorphisms of F exceeded in number the maximum wanted quantity of $2r(F)$. In this case, Proposition 2 ensures us that a certain subset of cardinal at most $2r(F)$ makes the job, too. Since, as a side product of our computation, we also have a basis of each $\mathrm{Fix}(\beta_{i,j}) = \mathrm{Fix}(\alpha_{i,j})$, it only remains to recurrently compute intersections until finding the desired set (knowing it exists). This concludes the proof. □

Corollary 12. *Let $H \leq F$ be a finitely generated subgroup of a free group, given by a finite set of generators. Then, it is algorithmically decidable whether H is endo-fixed and, in the affirmative case, find a set of $m \leq 2r(F)$ endomorphisms $\varphi_1, \ldots, \varphi_m \in \mathrm{End}(F)$ such that $H = \mathrm{Fix}(\varphi_1) \cap \cdots \cap \mathrm{Fix}(\varphi_m)$.*

4. Open problems

In this last section, we collect a list of interesting questions and open problems in this subject.

Problem 1. Find an algorithm to compute $\mathrm{Fix}(\varphi)$ for a given $\varphi \in \mathrm{End}(F)$.

Problem 2. Find an algorithm to determine whether a given finitely generated subgroup $H \leq F$ is 1-auto-fixed, or 1-endo-fixed.

Problem 3. Do the families of auto-fixed and 1-auto-fixed subgroups of F coincide? And those of 1-endo-fixed and endo-fixed subgroups?

PROBLEM 4. Find effective, say polynomial time, algorithms to compute the fix subgroup of a given endomorphism, and to determine whether a given finitely generated subgroup H of F is 1-auto-fixed, or 1-endo-fixed, or auto-fixed, or endo-fixed.

PROBLEM 5. Are endo-fixed subgroups of F F-inert?

PROBLEM 6. Find an algorithm to decide whether a given subgroup $H \leq F$ is F-inert.

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