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TRANSPORTATION COST INEQUALITIES ON PATH SPACES OVER RIEMANNIAN MANIFOLDS

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Abstract. Some transportation cost inequalities are established on the path space over a connected complete Riemannian manifold with Ricci curvature bounded from below. The reference distance on the path space is the L^2-norm of the Riemannian distance along paths.

1. Introduction

Let M be a connected complete Riemannian manifold either with convex boundary ∂M or without boundary. Assume that there is a nonnegative constant K such that

$$\mathrm{Ric}(X,X) \geq -K|X|^2, \quad X \in TM. \tag{1.1}$$

Then it is well-known that the (reflecting if $\partial M \neq \emptyset$) Brownian motion on M is nonexplosive.

For fixed $p \in M$ and $T > 0$, let μ^T denote the distribution of the (reflecting) Brownian motion starting from p before time T. Then μ^T is a probability measure on $M^{[0,T]} := \{x_\cdot : [0,T] \to M\}$ with σ-field $\mathcal{A}^T$ induced by cylindrically measurable functions. Since the diffusion process is continuous, μ^T has full measure on the path space

$$M_p^T := \{x_\cdot \in C([0,T]; M) : x_0 = p\}$$

with σ-algebra $\mathcal{A}_p^T := M_p^T \cap \mathcal{A}^T$. Our aim is to establish Talagrand's transportation cost inequality for the measure μ^T. This inequality was first introduced in [13] for the standard Gaussian measure on $\mathbb{R}^d$.

Before we state our main results, let us recall the known results in finite dimensions. Let $\rho(x,y)$ be the Riemannian distance between x and y for $x, y \in M$. Let $\mu := e^{V(x)}dx$ be a probability measure on M, where dx denotes

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the Riemannian volume element. For any probability measure ν on M, let $W_2(\nu,\mu)$ be the L^2-Wasserstein distance of ν and μ induced by ρ, i.e.,

$$W_2(\nu,\mu)^2 := \inf_{\pi\in\mathcal{C}(\nu,\mu)} \int_{M\times M} \rho(x,y)^2 \pi(dx,dy),$$

where $\mathcal{C}(\nu,\mu)$ stands for the set of probability measures on $M\times M$ with marginal distributions ν and μ. In 1996, Talagrand [13] proved for $M=\mathbb{R}^d$ and μ the standard Gaussian measure that

$$W_2(f\mu,\mu)^2 \le 2\mu(f\log f), \quad f\ge 0,\ \mu(f)=1.$$

This inequality was subsequently established by Otto and Villani [11] for general M under a curvature condition: If $\mathrm{Ric}-\mathrm{Hess}_V$ is bounded below, then the log-Sobolev inequality implies the transportation cost inequality. Recently, Otto and Villani's result was proved by Bobkov, Gentil and Ledoux [4] for measurable V without any curvature condition; see Section 2.3 and the equivalence of (1.12) and (1.13) therein. This result is a starting point of our present work, and we therefore state it explicitly:

THEOREM 1.1 ([11] and [4]). *Let $\mu := e^V dx$ be a probability measure on M. If there is a constant $C>0$ such that*

$$\mu(f^2\log f^2) \le 2C\mu(|\nabla f|^2), \quad f\in C_b^1(M),\ \mu(f^2)=1, \tag{1.2}$$

then

$$W_2(f\mu,\mu)^2 \le 2C\mu(f\log f), \quad f\ge 0,\ \mu(f)=1. \tag{1.3}$$

In view of Theorem 1.1 we may ask for transportation cost inequalities on the path space as the log-Sobolev inequality holds for the O-U Dirichlet form on M_p^T provided Ric is bounded; see, e.g., [1], [8], [5]. For this purpose one may take the intrinsic distance of this Dirichlet form. Indeed, such a transportation cost inequality has been established recently by Gentil [6] for $M=\mathbb{R}^d$ and by the author [16] for compact M. In this paper, we work with the following simple but natural distance and establish a transport cost inequality depending only on the lower bound of the curvature.

For any $T>0$, let

$$\rho^T(x_\cdot,y_\cdot) := \left\{\int_0^T \rho(x_s,y_s)^2 ds\right\}^{1/2}, \quad x_\cdot,y_\cdot\in M_p^T.$$

Let W_2^T be the corresponding L^2-Wasserstein distance. Moreover, for $I=\{s_1,\cdots,s_n\}$ with $0<s_1<\cdots<s_n<T$ define the distance on $M^I := \{x_I=(x_{s_1},\cdots,x_{s_n}) : x_{s_i}\in M, 1\le i\le n\}$ by

$$\rho^I(x_I,y_I) := \left\{\sum_{i=1}^n (s_{i+1}-s_i)\rho(x_{s_i},y_{s_i})^2\right\}^{1/2}, \quad x_I,y_I\in M^I,\ s_{n+1}:=T.$$

Let W_2^I be the corresponding probability distance. For a probability measure ν on M_p^T, let ν^I denote its projection onto M^I. For two probability measures μ_1, μ_2 on M_p^T, define

$$\widetilde{W}_2^T(\mu_1,\mu_2) := \sup\Big\{W_2^I(\mu_1^I,\mu_2^I) : I \subset (0,T) \text{ is finite}\Big\}.$$

We have the following result where only the lower bound of Ric is involved.

THEOREM 1.2. *Assume* (1.1). *For any nonnegative measurable function f on M_p^T with $\mu^T(f)=1$, we have*

$$W_2^T(f\mu^T,\mu^T)^2 \le \widetilde{W}_2^T(f\mu^T,\mu^T)^2 \le \frac{2}{K^2}(e^{KT}-1-KT)\mu^T(f\log f). \tag{1.4}$$

Among other applications, the transportation cost inequality can be applied to obtain exponential convergence of a Markov semigroup in the Wasserstein distance. For instance, let $\widetilde{P}_t$ be a symmetric Markov semigroup on $L^2(\mu^T)$ whose Dirichlet form satisfies a log-Sobolev inequality. Then it is well-known that for nonnegative f with $\mu^T(f)=1$ and $\mu^T(f\log f)<\infty$, $\mu^T(\widetilde{P}_t f\log \widetilde{P}_t f)$ converges to zero exponentially fast as $t\to\infty$. Thus, by Theorem 1.2, so does $W_2^T((\widetilde{P}_t f)\mu^T,\mu^T)^2$.

Note that (1.4) does not make sense when $T\to\infty$. To establish a transportation cost inequality which holds also for $T=\infty$, we introduce below a modified distance. For $K\ge 0$, $T>0$ and $h\in C[0,\infty)$ with $h(r)>0$ for $r>0$ such that $\int_0^1 s^{-1}h(s)ds<\infty$, define

$$\rho_h^T(x_\cdot,y_\cdot) := \bigg\{\int_0^T \frac{h(s)\rho(x_s,y_s)^2}{\int_0^s dr\int_r^T h(t)e^{K(t-r)}dt}ds\bigg\}^{1/2}.$$

Let $W_2^{T,h}$ be the corresponding L^2-Wasserstein distance. Let $\widetilde{W}_2^{T,h}$ be defined in the same way as $\widetilde{W}_2^T$ with ρ^I replaced by

$$\rho_h^I(x_I,y_I) := \bigg\{\sum_{j=1}^n \frac{\rho(x_{s_j},y_{s_j})^2\int_{s_j}^{s_{j+1}}h(s)ds}{\int_0^{s_j}ds\int_s^T e^{K(t-s)}h(t)dt}\bigg\}^{1/2}, \quad s_{n+1}:=T.$$

THEOREM 1.3. *Assume* (1.1). *For any $T>0$ and any $h\in C(0,\infty)$ with $h(r)>0$ for $r>0$ such that $\int_0^1 s^{-1}h(s)ds<\infty$, we have*

$$W_2^{T,h}(f\mu^T,\mu^T)^2 \le \widetilde{W}_2^{T,h}(f\mu^T,\mu^T)^2 \le 2\mu^T(f\log f), \quad f\ge 0,\ \mu^T(f)=1.$$

In particular, if $\int_0^\infty h(t)e^{tK}dt<\infty$, then

$$W_2^{\infty,h}(f\mu^\infty,\mu^\infty)^2 \le \widetilde{W}_2^{\infty,h}(f\mu^\infty,\mu^\infty)^2 \le 2\mu^\infty(f\log f), \quad f\ge 0,\ \mu^\infty(f)=1.$$

REMARK. Theorems 1.2 and 1.3 can be extended to diffusion processes with time-dependent drifts. Consider, for instance, the process generated by $L(\cdot,t) := \frac{1}{2}(\Delta+Z_t)$, where Z_t is a C^1-vector field for each $t\in[0,T)$. In particular, let $p_t(x,y)$ be the transition density of the Brownian motion and let

$$Z_t := 2\nabla \log p_{T-t}(\cdot, q), \quad t \in [0,T)$$

for a fixed point q. Then the distribution of the diffusion process starting from p is the Brownian bridge measure on the pinned path space $\{x_\centerdot \in M_p^T : x_T = q\}$.

Assume that $K_\centerdot \in C([0,T);[0,\infty))$ is such that

$$\mathrm{Ric}(X,X) - \langle \nabla_X Z_t, X\rangle \geq -K_t|X|^2, \quad t\in[0,T),\ X\in TM.$$

Then

$$W_2^T(f\mu^T,\mu^T)^2 \leq \widetilde{W}_2^T(f\mu^T,\mu^T)^2 \leq 2\mu^T(f\log f)\int_0^T ds \int_s^T e^{K_t(t-s)}dt$$

for all $f\geq 0$ with $\mu^T(f)=1$. Moreover, Theorem 1.3 remains true with K replaced by K_t in the definitions of ρ_h^T and ρ_h^I.

2. Proofs of Theorem 1.2 and 1.3

To apply Theorem 1.1, we first prove a log-Sobolev inequality for cylindrical functions.

LEMMA 2.1. *Assume* (1.1). *Let f be a cylindrically smooth function with $f(x_\centerdot) = f(x_{s_1},\cdots,x_{s_n})$, $0<s_1<\cdots s_n\leq T$. If $\mu^T(f^2)=1$ then*

(2.1)
$$\mu^T(f^2\log f^2) \leq 2\sum_{i=1}^n \int \bigg(\sum_{j=i}^n |\nabla_j f|\Big(\frac{e^{K(s_j-s_{i-1})}-e^{K(s_j-s_i)}}{K}\Big)^{1/2}\bigg)^2 d\mu^T,$$

where $s_0:=0$ and ∇_j denotes the gradient w.r.t. x_{s_j}.

Proof. Let P_t be the semigroup of the (reflecting) Brownian motion. By (1.1) we have (see, e.g., [12], [9], [15])

$$|\nabla P_t\xi(x)| \leq e^{Kt/2}P_t|\nabla\xi|(x), \quad t\geq 0,\ \xi\in C_b^1(M),\ x\in M. \tag{2.2}$$

By Bakry's semigroup argument, (2.2) implies that (see, e.g., [3], [8])

$$P_t(\xi^2\log\xi^2) \leq \frac{2(e^{Kt}-1)}{K}P_t|\nabla\xi|^2 + (P_t\xi^2)\log P_t\xi^2 \tag{2.3}$$

for any $t\geq 0, \xi\in C_b^1(M)$. Hence (2.1) holds for $n=1$ since in this case $\mu^T(f^2\log f^2) = P_{s_1}(f^2\log f^2)(p)$. Assume that (2.1) holds for $n\leq k$ for some $k\geq 1$. It remains to prove (2.1) for $n=k+1$. Let

$$\mu^{\{s_1,\cdots,s_n\}}(dx_{s_1},\ldots,dx_{s_n}) = P(s_1,p,dx_{s_1})P(s_2-s_1,x_{s_1},dx_{s_2}) \cdots P(s_k-s_{k-1},x_{s_{k-1}},dx_{s_k}),$$

where $P(t,x,\ dy)$ is the transition kernel of the (reflecting) Brownian motion. Note that for fixed $y \in M^k$, it follows from (2.2) with $t = s_{k+1} - s_k$ that

$$\begin{aligned}(2.4)\quad &\left|\nabla \int_M f^2(y, x_{s_{k+1}}) P(s_{k+1}-s_k, \cdot, dx_{s_{k+1}})\right| \\ &\qquad \le 2e^{K(s_{k+1}-s_k)/2} \int_M (|f|\cdot|\nabla_{k+1} f|)(y, x_{s_{k+1}}) P(s_{k+1}-s_k, \cdot, dx_{s_{k+1}}).\end{aligned}$$

Applying (2.3) with $t = s_{k+1} - s_k$, (2.1) with $n = k$, and taking (2.4) into account, we obtain

$$\begin{aligned}\mu^T(f^2\log f^2) &= \int_{M^k} d\mu^{\{s_1,\cdots,s_k\}} \int_M (f^2 \log f^2) P(s_{k+1}-s_k, x_{s_k}, dx_{s_{k+1}}) \\ &\le \frac{2(e^{K(s_{k+1}-s_k)}-1)}{K}\mu^T(|\nabla_{k+1} f|^2) \\ &\quad + 2\int_{M^k}\sum_{i=1}^k \frac{\mu^{\{s_1,\cdots,s_k\}}(dx_{s_1},\cdots,dx_{s_k})}{\int_M f^2 P(s_{k+1}-s_k, x_{s_k}, dx_{s_{k+1}})}\cdot \\ &\qquad \cdot\Big\{\int_M |f|\Big(\sum_{j=i}^{k+1} |\nabla_j f|\Big(\frac{e^{K(s_j-s_{i-1})}-e^{K(s_j-s_i)}}{K}\Big)^{1/2}\Big)\cdot \\ &\qquad\qquad \cdot P(s_{k+1}-s_k, x_{s_k}, dx_{s_{k+1}})\Big\}^2 \\ &\le 2\sum_{i=1}^{k+1}\int \Big(\sum_{j=i}^{k+1}|\nabla_j f|\Big(\frac{e^{K(s_j-s_{i-1})}-e^{K(s_j-s_i)}}{K}\Big)^{1/2}\Big)^2 d\mu^T. \qquad \square\end{aligned}$$

COROLLARY 2.2. *In the situation of Lemma* 2.1, *let* $I = \{s_1,\cdots,s_n\}$ *with* $0 < s_1 < \cdots < s_n \le T$ *and let* μ^I *denote the projection of* μ^T *onto* M^I. *For any* $s_{n+1} > s_n$ *and any function* $h : (0,T] \to (0,\infty)$, *we have*

$$\mu^I(f^2\log f^2) \le 2\sum_{j=1}^n \frac{\mu^I(|\nabla_j f|^2)}{\int_{s_j}^{s_{j+1}} h(s)ds}\int_0^{s_j} ds \int_s^{s_{n+1}} e^{K(t-s)} h(t)dt.$$

Proof. Note that

$$\begin{aligned}&\Big\{\sum_{j=i}^n |\nabla_j f|\Big(\int_{s_{i-1}}^{s_i} e^{K(s_j-s)}ds\Big)^{1/2}\Big\}^2 \\ &\qquad \le \Big(\sum_{j=i}^n \frac{|\nabla_j f|^2}{\int_{s_j}^{s_{j+1}} h(s)ds}\Big)\sum_{k=i}^n \int_{s_{i-1}}^{s_i} e^{K(s_k-s)}ds \int_{s_k}^{s_{k+1}} h(t)dt \\ &\qquad \le \Big(\sum_{j=i}^n \frac{|\nabla_j f|^2}{\int_{s_j}^{s_{j+1}} h(s)ds}\Big)\int_{s_{i-1}}^{s_i} ds \int_s^{s_{n+1}} e^{K(t-s)}h(t)dt.\end{aligned}$$

Then the desired result follows from Lemma 2.1. $\square$

LEMMA 2.3. *Let $\rho_t(x_\cdot, y_\cdot) := \rho(x_t, y_t)$. We have*

$$(\mu^T \times \mu^T)(\rho_t^2) \leq \frac{1}{K}(e^{Kt} - 1), \quad t \in [0, T].$$

Proof. Let $(x_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ be two independent (reflecting) Brownian motions with $x_0 = y_0 = p$. Since ∂M is either empty or convex, we have (see [10], [14])

$$d\rho(x_t, y_t) = \sqrt{2}db_t + \frac{1}{2}(\Delta\rho(x_t, \cdot)(y_t) + \Delta\rho(\cdot, y_t)(x_t))dt - dL_t,$$

where b_t is the one-dimensional Brownian motion and L_t is an increasing process. By (1.1) and the Laplacian comparison theorem we have

$$\begin{aligned}\frac{1}{2}(\Delta\rho(x, \cdot)(y) + \Delta\rho(\cdot, y)(x)) &\leq \sqrt{K(d-1)}\coth\big(\sqrt{K(d-1)}\rho(x,y)\big)\\ &\leq \frac{d-1}{\rho(x,y)} + \sqrt{K(d-1)}.\end{aligned}$$

Therefore, by Ito's formula we obtain

$$\begin{aligned}d\rho(x_t, y_t)^2 &\leq 2\sqrt{2}\rho(x_t, y_t)db_t + (2d + 2\sqrt{K(d-1)}\rho(x_t, y_t))dt\\ &\leq 2\sqrt{2}\rho(x_t, y_t)db_t + (3d - 1 + K\rho(x_t, y_t)^2)dt.\end{aligned}$$

Since $\rho(x_0, y_0) = 0$, this implies that

$$E\rho(x_t, y_t)^2 \leq \frac{1}{K}(3d-1)(e^{Kt} - 1), \quad t > 0.$$

Hence the proof is finished. □

LEMMA 2.4. *Assume* (1.1). *Let $c_t = (e^{tK_t} - 1)/K$. We have*

$$[\mu^T \times \mu^T](e^{\alpha\rho(x_t, y_t)^2}) \leq \frac{\exp[\alpha(3d-1)c_t/(1 - 4\alpha c_t)]}{\sqrt{1 - 4\alpha c_t}}, \ t \in [0, T], \ \alpha \in (0, 1/4c_t).$$

Proof. By (2.3) and the additivity of the log-Sobolev inequality (see [7]) we have

$$(P_t \times P_t)(\xi^2 \log \xi^2) \leq 2c_t(P_t \times P_t)(|\nabla_{M\times M}\xi|^2) + (P_t \times P_t)(\xi^2)\log(P_t \times P_t)(\xi^2)$$

for any $t > 0, \xi \in C_b^1(M \times M)$. Since $|\nabla_{M\times M}\rho|^2 = 2$, according to [2] this implies that

$$(P_t \times P_t)(e^{\alpha\rho^2}) \leq \frac{\exp[\alpha((P_t \times P_t)(\rho))^2/(1 - 4\alpha c_t)]}{\sqrt{1 - 4\alpha c_t}}, \quad t > 0. \tag{2.5}$$

Applying Lemma 2.3 completes the proof. □

Proof of Theorem 1.2. For $I = \{s_i : 1 \leq i \leq n\}$ with $0 < s_1 < \cdots < s_n < T$, let $f^I(x_{s_1}, \cdots, x_{s_n}) = \mu^T(f|x_{s_1}, \cdots, x_{s_n})$ and let μ^I be the projection of

μ^T onto M^I. It is easy to check that ρ^I is the Riemannian distance on M^I with metric

$$\langle X, Y\rangle_I := \sum_i (s_{i+1} - s_i)\langle X_{s_i}, Y_{s_i}\rangle_M,$$

where X_{s_i} (resp. Y_{s_i}) is the i-th component of X (resp. Y) which is tangent to $M^{\{s_i\}}$. Moreover, let ∇_I denote the corresponding gradient operator. For $g \in C^\infty(M^I)$ one has

$$\langle \nabla_I g, \nabla_I g\rangle_I = \sum_{j=1}^n (s_{j+1} - s_j)^{-1}|\nabla_j g|^2.$$

Thus, by Theorem 1.1 and Corollary 2.2 with $h \equiv 1$, we obtain

$$\begin{aligned} W_2^I(f^I\mu_p^I, \mu^I)^2 &\le 2\mu^I(f^I \log f^I)\int_0^{s_n} ds \int_s^{s_{n+1}} e^{K(t-s)}dt \\ &\le 2\mu^T(f\log f)\int_0^T ds\int_s^T e^{K(t-s)}dt. \end{aligned} \tag{2.6}$$

It remains to prove the first inequality in (1.4). Since (M_p^T, ρ_∞^T) is a Polish space with Borel σ-algebra $\mathcal{A}_p^T$, where $\rho_\infty^T(x_\cdot, y_\cdot) := \sup_{t\in[0,T]}\rho(x_t, y_t)$, $\{\mu^T, f\mu^T\}$ is tight. Moreover, for any compact set $D \subset M_p^T$ and any $\pi \in \mathcal{C}(f\mu^T, \mu^T)$ one has

$$\pi((D\times D)^c) \le \mu^T(D^c) + (f\mu^T)(D^c).$$

Thus $\mathcal{C}(f\mu^T, \mu^T)$ is tight too. Let $\{I_n\}$ be increasing such that $\delta(I_n) \downarrow 0$ as $n \uparrow \infty$, where $\delta(I_n) := \max_{1\le i\le k_n+1}(s_i - s_{i-1})$ for $I_n := \{0 = s_0 < s_1 < \cdots < s_{k_n} < T = s_{k_n+1}\}$. For each $n \ge 1$ let $\pi^{I_n} \in \mathcal{C}(f^{I_n}\mu^{I_n}, \mu^{I_n})$ be such that

$$\pi^{I_n}((\rho^{I_n})^2) \le W_2^{I_n}(f^{I_n}\mu^{I_n}, \mu^{I_n})^2 + \frac{1}{n}.$$

Let

$$\pi_n(\cdot) := \int \pi^{I_n}(dx_{I_n}, dy_{I_n})[(f\mu^T)\times\mu^T](\cdot|x_{I_n}, y_{I_n});$$

i.e., for any set $A \subset \mathcal{A}_p^T \times \mathcal{A}_p^T$,

$$\pi_n(A) := \int_{M^{I_n}\times M^{I_n}} [(f\mu^T)\times\mu^T](A|x_{I_n}, y_{I_n})\pi^{I_n}(dx_{I_n}, dy_{I_n}).$$

Then $\{\pi_n\} \subset \mathcal{C}(f\mu^T, \mu^T)$. Let $\{\pi_{n'}\}$ be a subsequence such that $\pi_{n'} \to \pi$ weakly for a probability measure π on $M_p^T \times M_p^T$. Then $\pi \in \mathcal{C}(f\mu^T, \mu^T)$. Thus for any $n \ge 1$ and any $N > 0$, if we let $\rho_N^{I_n}$ be defined in the same way as ρ^{I_n}, but with ρ replaced by $\rho\wedge N$, we have

$$\begin{aligned} \pi((\rho_N^{I_n})^2) &= \lim_{n'\to\infty}\pi^{I_{n'}}((\rho_N^{I_n})^2) \\ &\le (1+\varepsilon)\widetilde{W}_2^T(f\mu^T, \mu^T)^2 + (1+\varepsilon^{-1})\sup_{n'>n}\pi^{I_{n'}}(|\rho_N^{I_n} - \rho_N^{I_{n'}}|^2) \end{aligned} \tag{2.7}$$

for any $\varepsilon > 0$. Noting that $|\rho(x_s, y_s) - \rho(x_t, y_t)| \le \rho(x_s, x_t) + \rho(y_s, y_t)$, we have

$$\sup_{n'>n} \pi^{I_{n'}}(|\rho_N^{I_n} - \rho_N^{I_{n'}}|^2)$$
$$\le 2\int_{M_p^T} \left\{ N \wedge \sup_{0<s<t<T, t-s\le\delta(I_n)} \rho(x_s, x_t) \right\}^2 (f\mu^T + \mu^T)(dx_\bullet),$$

which converges to zero as $n \to \infty$ according to the dominated convergence theorem. Letting first $n \uparrow \infty$, then $N \uparrow \infty$, and finally $\varepsilon \downarrow 0$ in (2.7), we complete the proof. □

Proof of Theorem 1.3. We simply note that the argument in the proof of Theorem 1.2 yields

$$W_2^{I,h}(f^I\mu^T, \mu^T)^2 \le 2\mu^T(f \log f);$$

hence the first assertion follows. It remains to prove the second assertion, where $\int_0^\infty e^{tK_t} h(t)dt < \infty$. To this end, it suffices to show

$$W_2^{\infty,h}(f\mu^\infty, \mu^\infty) \le \limsup_{T\to\infty} W_2^{T,h}(f\mu^T, \mu^T). \tag{2.8}$$

For nonnegative f with $\mu^\infty(f) = 1$ and $\mu^\infty(f \log f) < \infty$, by Lemma 2.4 with $\alpha_t = 1/8c_t$ for each $t > 0$ we obtain

$$\begin{aligned}[(f\mu^\infty) \times \mu^\infty]((\rho_h^\infty)^2) &= \int_0^\infty \frac{h(t)[(f\mu^\infty) \times \mu^\infty](\rho(x_t, y_t)^2)dt}{\int_0^t ds \int_s^\infty e^{K(r-s)}h(r)dr} \\ &\le \int_0^\infty \frac{h(t)\mu^\infty(f \log f)dt}{\alpha_t \int_0^t ds \int_s^\infty e^{K(r-s)}h(r)dr} \\ &\quad + \int_0^\infty \frac{h(t)[\mu^\infty \times \mu^\infty](\exp[\alpha_t \rho(x_t, y_t)^2])dt}{\int_0^t ds \int_s^\infty e^{K(r-s)}h(r)dr} < \infty.\end{aligned}$$

Therefore

$$\mu^\infty\big((1+f)(\rho_h^\infty(\cdot, z_\bullet)^2\big) < \infty \tag{2.9}$$

for μ^∞-a.s. $z_\bullet \in M_p^\infty$. Let us fix $z_\bullet \in M_p^\infty$ such that (2.9) holds. For any coupling π^T for $f^T\mu^T$ and μ^T, where $f^T(x_{[0,T]}) := \mu^\infty(f|x_{[0,T]})$, we have

$$\pi(\cdot) := \int_{M_p^T \times M_p^T} \pi^T(dx_{[0,T]}, dy_{[0,T]})[(f\mu^\infty) \times \mu^\infty](\cdot|x_{[0,T]}, y_{[0,T]})$$
$$\in \mathcal{C}(f\mu^\infty, \mu^\infty).$$

Then

$$\begin{aligned} W_2^{\infty,h}(f\mu^\infty,\mu^\infty)^2 &\le \int_{M_p^T\times M_p^T}(\rho_h^T)^2 d\pi^T \\ &\qquad + 2\int_T^\infty \frac{h(s)[\rho(x_s,z_s)^2+\rho(y_s,z_s)^2]\pi(dx_\cdot,dy_\cdot)}{\int_0^s dr\int_r^\infty e^{K(t-r)}h(t)dt}ds \\ &= \int_{M_p^T\times M_p^T}(\rho_h^T)^2 d\pi^T + 2\int_T^\infty \frac{h(s)\int\rho(x_s,z_s)^2[(1+f)\mu^\infty](dx_\cdot)}{\int_0^s dr\int_r^\infty e^{K(t-r)}h(t)dt}ds \\ &=: \int_{M_p^T\times M_p^T}(\rho_h^T)^2 d\pi^T + \varepsilon(T). \end{aligned}$$

Combining this with the first assertion, we arrive at

$$W_2^{\infty,h}(f\mu^\infty,\mu^\infty)^2 \le W_2^{T,h}(f^T\mu^T,\mu^T)^2 + \varepsilon(T).$$

Then (2.8) follows by noting that $\lim_{T\to\infty}\varepsilon(T)=0$ according to (2.9). □

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