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SUBALGEBRAS OF $C(\Omega, M_n)$ AND THEIR MODULES

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Abstract. We give an operator space characterization of subalgebras of $C(\Omega, M_n)$. We also describe injective subspaces of $C(\Omega, M_n)$ and then give applications to sub-TROs of $C(\Omega, M_n)$. Finally, we prove an 'n-minimal version' of the Christensen-Effros-Sinclair representation theorem.

1. Introduction and preliminaries

Let $n \in \mathbb{N}^*$. An operator space X is called *n-minimal* if there exists a compact Hausdorff space Ω and a completely isometric map $i : X \to C(\Omega, M_n)$. The readers are referred to [13] and [7] for details on operator space theory. Recall that the C^*-algebra $C(\Omega, M_n)$ can be identified $*$-isomorphically with $C(\Omega) \otimes_{\min} M_n$ or $M_n(C(\Omega))$ (see [12, Proposition 12.5] for details). Obviously, in the case $n = 1$ we just deal with the well-known class of minimal operator spaces. Smith noticed that any linear map into M_n is completely bounded and its cb norm is achieved at the nth amplification, i.e., $\|u\|_{cb} = \|\operatorname{id}_{M_n} \otimes u\|$ (see [12, Proposition 8.11]). Clearly, this property remains true for maps into $C(\Omega, M_n)$. In fact, Pisier showed that this property characterizes n-minimal operator spaces. More precisely, if X is an operator space such that any linear map u into X is necessarily completely bounded and $\|u\|_{cb} = \|\operatorname{id}_{M_n} \otimes u\|$, then X is n-minimal (see [14, Theorem 18]).

We now recall a few facts about injectivity (see [7], [12] or [2] for details). A Banach space X is *injective* if for any Banach spaces $Y \subset Z$ each contractive map $u : Y \to X$ has a contractive extension $\tilde{u} : Z \to X$. It has been known since the 1950s that a Banach space is injective if and only if it is isometric to a $C(K)$-space with K a Stonean space, and dual injective Banach spaces are exactly L^∞-spaces (see [6] for more details). More recently, injectivity has also been studied in the operator spaces category. Analogously, an operator space X is said to be *injective* if for any operator spaces $Y \subset Z$ each completely contractive map $u : Y \to X$ has a completely contractive extension $\tilde{u} : Z \to X$.

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Note that a Banach space is injective if and only if it is injective as a minimal operator space.

Let X be an operator space. (Y, i) is an *injective envelope of* X if Y is an injective operator space, $i : X \to Y$ is a complete isometry and for any injective operator space Z with $i(X) \subset Z \subset Y$ we have $Z = Y$. Sometimes we may ignore the completely isometric embedding. In fact, any operator space admits a unique injective envelope (up to complete isometry) and we write $I(X)$ the injective envelope of X. See [7, Chapter 6] for a proof of this construction.

Obviously, an ℓ^∞-direct sum of n-minimal operator spaces is again n-minimal. In the following proposition, we give some other easy properties of n-minimal operator spaces:

Proposition 1.1. *Let X be an n-minimal operator space. Then we have:*

(i) *Its bidual X^{**} and its injective envelope $I(X)$ are n-minimal too.*
(ii) *If, moreover, X is a dual operator space, then there is a set I and a w^*-continuous complete isometry $i : X \to \ell^\infty_I(M_n)$.*

Proof. The first assertion of (i) follows from $C(\Omega, M_n)^{**} = M_n(C(\Omega))^{**} = M_n(C(\Omega)^{**})$ $*$-isomorphically. For the second, suppose $X \subset C(\Omega, M_n)$ completely isometrically. By the description of injective Banach spaces, $I(C(\Omega)) = C(\Omega')$ with Ω' Stonean. Then $X \subset C(\Omega', M_n)$ and the latter C^*-algebra is injective, so $I(X) \subset C(\Omega', M_n)$ completely isometrically.

Suppose that W is an operator space predual of X. Then $X = CB(W, \mathbb{C})$, and if $I = \bigcup_n \mathrm{Ball}(M_n(W))$ we have a w^*-continuous complete isometry $\psi : X \longrightarrow \oplus^\infty_{w\in I} M_{n_w}$ (where $n_w = m$ if $w \in M_m(W)$) defined by $\psi(x) = ([x(w_{ij})])_{w\in I}$. Let $x \in M_k(X) = CB(W, M_k)$. As X is n-minimal, by [12, Proposition 8.11], $\|x^*\|_{cb} = \|\mathrm{id}_{M_n} \otimes x^*\|$, where $x^* : M_k^* \to X$ denotes the adjoint map. However, for any l, $\|\mathrm{id}_{M_l} \otimes x\| = \|\mathrm{id}_{M_l} \otimes x^*\|$. Hence, $\|x\|_{cb} = \|\mathrm{id}_{M_n} \otimes x\|$ and so, in the definition of ψ, we can majorize the n_w's by n and obtain a complete isometry. $\square$

We mentioned above that an injective minimal operator space is a C^*-algebra. However, this property does not hold for n-minimal operator spaces (as soon as $n \geq 2$). Generally, an injective operator space only admits a structure of a ternary ring of operators. We recall that a closed subspace X of a C^*-algebra is a *ternary ring of operators* (TRO in short) if $XX^\star X \subset X$, where $X^\star$ denotes the adjoint space of X. A *W^*-TRO* is a w^*-closed subspace of a von Neumann algebra that is stable under the above "triple product". TROs and W^*-TROs can be regarded as generalizations of C^*-algebras and W^*-algebras. For instance, the Kaplansky Density Theorem and the Sakai Theorem remain valid for TROs (see, e.g., [6]). A *triple morphism* between TROs is a linear map which preserves their "triple products". This category enjoys some "rigidity properties" similar to those of the category of C^*-algebras (see, e.g., [6] or [2, Section 8.3] for details).

So far we have seen that certain properties of the minimal case extend to the n-minimal situation. Therefore, the basic idea of this paper is to extend valid results in the commutative case to the more general n-minimal case.

A first commutative result that can be extended to the n-minimal case is a theorem on operator algebras due to Blecher. We recall that an *operator algebra* is a closed subalgebra of $B(H)$; see [2] or [12] for some background and recent developments. An operator algebra is said to be *approximately unital* if it possesses a contractive approximate identity. In [1], Blecher showed that an approximately unital operator algebra which is minimal is in fact a uniform algebra (i.e., a subalgebra of a commutative C^*-algebra). Let A be an approximately unital operator algebra and assume that A is n-minimal. We obtain a completely isometric homomorphism from A into a certain space $C(\Omega, M_n)$ (see Corollary 2.3). Of course, we can ask this type of question in various categories of operator spaces. More precisely, let $\mathcal{C}$ denote a certain subcategory of the category of operator spaces with completely contractive maps. Let X be an object of $\mathcal{C}$ which is n-minimal (as an operator space). Can we obtain a completely isometric morphism of $\mathcal{C}$ from X into a C^*-algebra of the form $C(\Omega, M_n)$? For example, in Proposition 1.1 we answer this question in the category of dual operator spaces and w^*-continuous completely contractive maps. We will also give a positive answer in the following categories:

- C^*-algebras and $*$-homomorphisms (see Theorem 2.2);
- von Neumann algebras and w^*-continuous $*$-homomorphisms (see Remark 2.4);
- approximately unital operator algebras and completely contractive homomorphisms (see Corollary 2.3);
- operator systems and completely positive unital maps (see Corollary 3.3);
- TRO and triple morphisms (see Proposition 4.1);
- W^*-TRO and w^*-continuous triple morphisms (see Corollary 4.5).

In other words, in any of the above categories, the n-minimal operator space structure encodes the additional structure. Since the injective envelope of an n-minimal operator space is n-minimal too (see Proposition 1.1), passing to the injective envelope will be a useful technique to answer these questions. In any case, the description of n-minimal injective operator spaces (established in Theorem 3.5) will be of major importance.

The Christensen-Effros-Sinclair Theorem (CES-Theorem in short) is a second example of a theorem that can be treated in the n-minimal case. Let A be an operator algebra (or more generally a Banach algebra endowed with an operator space structure) and let X be an operator space which is a left A-module. Then, following [2, Chapter 3], we say that X is a left *h-module over*

A if the action of A on X induces a completely contractive map from $A \otimes_h X$ in X (where $\otimes_h$ denotes the Haagerup tensor product). The CES-Theorem states that if X is a non-degenerate h-module over an approximately unital operator algebra A (i.e., AX is dense in X), then there exists a C^*-algebra C, a complete isometry $i : X \to C$ and a completely contractive homomorphism $\pi : A \to C$ such that $i(a \cdot x) = \pi(a)i(x)$ for any $a \in A,\ x \in X$. We will prove that if X is n-minimal, then we can choose C to be n-minimal too. This leads to an 'n-minimal version' of the CES-Theorem. The case $n = 1$ has been treated (see [3]) in a Banach space framework; here we will use an operator space approach based on the multiplier algebra of an operator space.

2. Subalgebras of $C(\Omega, M_n)$

Recall that a C^*-algebra is *subhomogeneous of degree* $\leq n$ if it is contained $*$-isomorphically in a C^*-algebra of the form $C(\Omega, M_n)$, where Ω is a compact Hausdorff space. Hence n-minimality can be seen as an operator space analog of subhomogeneity of degree $\leq n$. We also recall the well-known characterization of subhomogeneous C^*-algebras in terms of representations. Indeed, a C^*-algebra A is subhomogeneous of degree $\leq n$ if and only if every irreducible representation of A has dimension no greater than n. The "if part" is easily obtained by taking a separating family of irreducible representations. Conversely, if A is contained $*$-isomorphically in $C(\Omega, M_n)$, then every irreducible representation of A extends to one on $C(\Omega, M_n)$ (because irreducible representations correspond to pure states). As any irreducible representation of $C(\Omega, M_n)$ has dimension no greater than n, we obtain the result. (The author thanks Roger Smith for these explanations.)

LEMMA 2.1. *Let $k \in \mathbb{N}^*$, Ω a compact Hausdorff space and t_k the transpose mapping*

$$\begin{array}{rccc} t_k : & C(\Omega, M_k) & \to & C(\Omega, M_k), \\ & [f_{ij}] & \mapsto & [f_{ji}]. \end{array}$$

Then for any $l \in \mathbb{N}^*$, $\| \mathrm{id}_{M_l} \otimes t_k \| = \inf(k, l)$. Thus t_k is completely bounded and $\| \mathrm{id}_{M_k} \otimes t_k \| = \|t_k\|_{cb} = k$.

Proof. The equality $\|t_k\|_{cb} = k$ is obtained by adapting the proof of [7, Proposition 2.2.7]. Hence in the case $k \leq l$, by [12, Proposition 8.11] we obtain $\| \mathrm{id}_{M_l} \otimes t_k \| = \inf(k, l)$. Next, we prove $\| \mathrm{id}_{M_l} \otimes t_k \| \leq l$. Let π be the cyclical permutation matrix

$$\pi = \begin{pmatrix} 0 & 0 & \cdots & 0 & I_k \\ I_k & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & I_k & 0 \end{pmatrix} \in M_l(C(\Omega, M_k)).$$

Let $D_l : M_l(C(\Omega, M_k)) \to M_l(C(\Omega, M_k))$ be the diagonal truncation of M_l, i.e., $D_l(\epsilon_{ij} \otimes y) = \delta_{ij}\epsilon_{ij} \otimes y$, where ϵ_{ij} $(i, j \leq l)$ denotes the matrix elements of M_l and $y \in C(\Omega, M_k)$. Let $x = [x_{ij}]_{i,j\leq l} \in M_l(C(\Omega, M_k))$ and for simplicity of notation write $t(x) = \mathrm{id}_{M_l} \otimes t_k(x) \in M_l(C(\Omega, M_k))$. Then $t(x) = \sum_{i=0}^{l-1} D_l(t(x)\pi^i)\pi^{-i}$, and so $\|t(x)\| \leq \sum_{i=0}^{l-1} \|D_l(t(x)\pi^i)\|$ (because π is unitary). To obtain the result, it suffices to majorize each term of the above sum by the norm of x. However, for any i, $D_l(t(x)\pi^i)$ is of the form $\sum_{j=1}^{l} \epsilon_{jj} \otimes t_k(x_{p_j q_j})$ and we can majorize its norm

$$\left\| \sum_{j=1}^{l} \epsilon_{jj} \otimes t_k(x_{p_j q_j}) \right\|^2 = \left\| \sum_{j=1}^{l} \epsilon_{jj} \otimes t_k(x_{p_j q_j} x^*_{p_j q_j}) \right\| = \max_j \left\{ \left\| t_k(x_{p_j q_j} x^*_{p_j q_j}) \right\| \right\}.$$

Now, $x_{p_j q_j} x^*_{p_j q_j}$ is a selfadjoint element of $C(\Omega, M_k)$, so its norm is unchanged by t_k and $\|t_k(x_{p_j q_j} x^*_{p_j q_j})\| = \|x_{p_j q_j}\|^2 \leq \|x\|^2$. Finally, for any i, $\|D_l(t(x)\pi^i)\| \leq \|x\|$, which gives the result.

Moreover, by adapting [7, Proposition 2.2.7], we get easily $\|\mathrm{id}_{M_l} \otimes t_k\| = l$, if $l \leq k$. □

In the next theorem, we denote by A^{op} the opposite structure of a C^*-algebra A (see, e.g., [13, Paragraph 2.10] or [2, Paragraph 1.2.25] for details). More generally, if X is an operator space, X^{op} is the same vector space but with the new matrix norm defined by

$$\|[x_{ij}]\|_{M_n(X^{\mathrm{op}})} = \|[x_{ji}]\|_{M_n(X)} \quad \text{for any } [x_{ij}] \in M_n(X).$$

Hence the assumption (iii) in the following theorem is equivalent to

$$\|\mathrm{id}_A \otimes t_k\| \leq n \quad \text{for any } k \in \mathbb{N}^*,$$

where t_k denotes the transpose mapping from M_k to M_k discussed above.

THEOREM 2.2. *Let A be a C^*-algebra. Then the following are equivalent:*

(i) *A is subhomogeneous of degree $\leq n$.*
(ii) *A is n-minimal.*
(iii) *$\|\mathrm{id} : A \to A^{\mathrm{op}}\|_{cb} \leq n$.*

Proof. (i)⇒(ii) is obvious and (ii)⇒(iii) follows from the first equality in the above lemma.

Suppose (iii) holds. Let $\pi : A \to B(H)$ be an irreducible representation and $k \in \mathbb{N}^*$ such that $M_k \subset B(H)$; from the first paragraph of this section, we see that we must prove that $k \leq n$. By the above lemma (with a singleton as Ω), there is $x \in M_k(M_k) \subset M_k(B(H))$ satisfying

$$k = \|\mathrm{id}_{M_k} \otimes t_k(x)\| \quad \text{and} \quad \|x\| \leq 1.$$

The representation $\pi_k = \mathrm{id}_{M_k} \otimes \pi$ is also irreducible, so $\pi_k(M_k(A))' = \mathbb{C}I_{H^k}$. Thus, by von Neumann's Double Commutant Theorem,

$$\overline{M_k(\pi(A))}^{so} = M_k(B(H)).$$

By the Kaplansky Density Theorem, there exists a net $(x_\lambda)_{\lambda\in\Lambda} \subset M_k(\pi(A))$ converging to x in the σ-strong operator topology and such that $\|x_\lambda\| \leq 1$. Therefore $\mathrm{id}_{B(H)} \otimes t_k(x_\lambda)$ tends to $\mathrm{id}_{M_k} \otimes t_k(x)$ in the w^*-topology, and by the semicontinuity of the norm in the w^*-topology we have

$$k = \|\,\mathrm{id}_{M_k} \otimes t_k(x)\| \leq \limsup_{\lambda} \|\,\mathrm{id}_{B(H)} \otimes t_k(x_\lambda)\|.$$

Let $\epsilon > 0$. For any λ, there exists $y_\lambda \in M_k(A)$ such that $x_\lambda = \pi_k(y_\lambda)$ and $\|y_\lambda\| \leq 1 + \epsilon$. By assumption,

$$\|\,\mathrm{id}_A \otimes t_k\| \leq n.$$

Moreover, $(\mathrm{id}_{B(H)} \otimes t_k) \circ \pi_k = \pi_k \circ (\mathrm{id}_A \otimes t_k)$. Combining these arguments we finally obtain

$$\begin{aligned} k = \|\,\mathrm{id}_{M_k} \otimes t_k(x)\| &\leq \limsup_{\lambda} \|\,\mathrm{id}_{B(H)} \otimes t_k(\pi_k(y_\lambda))\| \\ &\leq \limsup_{\lambda} \|\pi_k(\mathrm{id}_A \otimes t_k(y_\lambda))\| \\ &\leq \|\,\mathrm{id}_A \otimes t_k\|(1+\epsilon) \\ &\leq n(1+\epsilon). \end{aligned}$$

Hence $k \leq n$. □

Now we extend the equivalence (i)⇔(ii) of the above theorem, which concerns C^*-algebras, to the larger category of operator algebras and completely contractive homomorphisms.

Corollary 2.3. *Let A be an approximately unital operator algebra. Then the following are equivalent:*

(i) *There exists a compact Hausdorf space Ω and a completely isometric homomorphism $\pi : A \to C(\Omega, M_n)$.*
(ii) *A is n-minimal.*

Proof. (i)⇒(ii) is obvious. Suppose (ii). We know that the injective envelope $I(A)$ is a C^*-algebra and there is a completely isometric homomorphism from A into $I(A)$ (see [2, Corollary 4.2.8]). Since A is n-minimal, $I(A)$ is n-minimal too, by Proposition 1.1. Applying Theorem 2.2 to $I(A)$, we obtain the result. □

Remark 2.4. Using the well-known description of subhomogeneous W^*-algebras, we easily obtain the result that, if M is a W^*-algebra and M is n-minimal, then

$$M = \oplus_{i\in I}^{\infty} L^{\infty}(\Omega_i, M_{n_i})$$

via a normal $*$-isomorphism. Here Ω_i is a measure space and $n_i \leq n$, for any $i \in I$. This result will be extended to the category of W^*-TROs (see Corollary 4.5).

3. Injective n-minimal operator spaces

Before describing injective n-minimal operator spaces, we treat the more "rigid" case of injective n-minimal C^*-algebras as an easy consequence of [16].

PROPOSITION 3.1. *Let A be an n-minimal C^*-algebra. Then the following are equivalent:*

(i) *A is injective.*
(ii) *There exists a finite family of Stonean compact Hausdorff spaces $(\Omega_i)_{i\in I}$ such that $A = \oplus^\infty_{i\in I} C(\Omega_i, M_{n_i})$ $*$-isomorphically with $n_i \leq n$, for any $i \in I$.*

Proof. As A is injective, A is monotone complete (see [7, Theorem 6.1.3]). Thus A is an AW^*-algebra. Moreover, by [16, Proposition 6.6], A either contains $M_\infty = \oplus^\infty_k M_k$ or is of the desired form. The first alternative is impossible because A is n-minimal. This proves the "only if" part. The converse is clear, since each Ω_i is Stonean. $\square$

REMARK 3.2. This theorem enables us to give a short proof of the implication (ii)$\Rightarrow$(i) in Theorem 2.2. If A is an n-minimal C^*-algebra, its injective envelope $I(A)$ is n-minimal too (by Proposition 1.1). $I(A)$ is a C^*-algebra and contains A $*$-isomorphically (see [7, Theorem 6.2.4]). Applying the above proposition to $I(A)$, we obtain that

$$I(A) = \oplus^\infty_{i\in I} C(\Omega_i, M_{m_i}) \quad *\text{-isomorphically}$$

with $n_i \leq n$, for any $i \in I$. Now it is not difficult to construct a $*$-isomorphism from A into $C(\Omega, M_n)$, where Ω denotes the (finite) disjoint union of the Ω_i's.

We recall that an operator space X is *unital* if there exists $e \in X$ and a complete isometry from X into a certain $B(H)$ which sends e on I_H. By the following result, an n-minimal operator system can be embedded into a C^*-algebra of the form $C(\Omega, M_n)$ via a unital complete order isomorphism.

COROLLARY 3.3. *Let X be a unital operator space. Then the following are equivalent:*

(i) *There exists a compact Hausdorff space Ω and a completely isometric unital map $\pi : X \to C(\Omega, M_n)$.*
(ii) *X is n-minimal.*

Proof. (i)$\Rightarrow$(ii) is obvious. Suppose (ii). We know that the injective envelope $I(X)$ is a C^*-algebra and there is a unital complete isometry from X

into $I(X)$ (see [2, Corollary 4.2.8]). As X is n-minimal, $I(X)$ is n-minimal too (by Proposition 1.1). By the above theorem

$$I(X) = \oplus_{i\in I}^{\infty} C(\Omega_i, M_{n_i}) \ \ast\text{-isomorphically.}$$

Next, we show that for any i there exists a unital complete isometry $\varphi_i : M_{n_i} \to M_n$. By iteration, we only need to prove that for any $k \in \mathbb{N}^*$ there exists a unital complete isometry from M_k into M_{k+1}. The map

$$\begin{array}{rccl} i_k : & M_k & \to & M_{k+1}, \\ & x & \mapsto & x \oplus \mathrm{tr}_k(x), \end{array}$$

where tr_k denotes the normalized trace on M_k, is a unital complete order isomorphism and thus a unital complete isometry. We can define a unital complete isometry

$$\begin{array}{rccl} \psi : & \oplus_{i\in I}^{\infty} C(\Omega_i, M_{n_i}) & \to & C(\Omega, M_n), \\ & (f_i \otimes x_i)_i & \mapsto & \sum_i \tilde{f}_i \otimes \varphi_i(x_i), \end{array}$$

where Ω denotes the disjoint union of Ω_i's and $\tilde{f}_i$ the continuous extension by 0 of f_i on Ω. Finally, we have

$$X \subset I(X) \subset C(\Omega, M_n)$$

via unital complete isometries. $\square$

REMARK 3.4. This last corollary cannot be extended to the category of operator algebras and completely contractive homomorphisms. In fact, if $\pi : M_p \to C(\Omega, M_q)$ is a unital completely contractive homomorphism, then π is positive, so it is a $\ast$-homomorphism. Therefore (after composing with an evaluation) we obtain a unital $\ast$-homomorphism from M_p in M_q, and thus p divides q (see [12, Exercise 4.11]).

We recall a crucial construction of the injective envelope of an operator space X, which will be useful in this paper (see [2, Paragraph 4.4.2] for more details on this construction). Assume that $X \subset B(H)$. We consider its Paulsen system

$$S(X) = \begin{pmatrix} \mathbb{C} & X \\ X^\star & \mathbb{C} \end{pmatrix} \subset M_2(B(H)),$$

where $X^\star$ denotes the adjoint space of X. The injective envelope of $S(X)$ is the range of a completely contractive projection $\varphi : M_2(B(H)) \to M_2(B(H))$ which leaves $S(X)$ invariant. By [7, Theorem 6.1.3], $I(S(X))$ admits a C^*-algebraic structure, but it is not necessarily a sub-C^*-algebra of $M_2(B(H))$. However,

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 - p$$

(which are invariant by φ) are still orthogonal projections (i.e., selfadjoint idempotents) of the new C^*-algebra $I(S(X))$. Since they satisfy $p+q=1$ and $pq=0$, we can decompose $I(S(X))$ in 2×2 matrices, as follows:

$$I(S(X)) = \begin{pmatrix} I_{11}(X) & I_{12}(X) \\ I_{21}(X) & I_{22}(X) \end{pmatrix},$$

where $I_{11}(X) = pI(S(X))p$ and $I_{22}(X) = qI(S(X))q$ are injective C^*-algebras, $I_{12}(X) = pI(S(X))q$ is in fact the injective envelope of X and $I_{21}(X) = qI(S(X))p$ coincides with $I_{12}(X)^\star$. Therefore we obtain the Hamana-Ruan Theorem, i.e., an injective operator space is an "off-diagonal" corner of an injective C^*-algebra (see [7, Theorem 6.1.6]). This theorem links the study of injective operator spaces to injective C^*-algebras (and, by the way, it proves that an injective operator space is a TRO).

Theorem 3.5. *Let X be an n-minimal operator space. Then the following are equivalent:*

(i) *X is injective.*
(ii) *There exists a finite family of Stonean compact Hausdorff spaces $(\Omega_i)_{i\in I}$ such that $X = \oplus_{i\in I}^\infty C(\Omega_i, M_{r_i,k_i})$ completely isometrically with $r_i, k_i \le n$, for any $i \in I$.*

Proof. (ii)$\Rightarrow$(i) is obvious. Let X be an injective n-minimal operator space. By the above discussion there exists an injective C^*-algebra A and a projection $p \in A$ such that

$$X = pA(1-p) \quad \text{completely isometrically.}$$

In fact, A is the injective envelope of $S(X)$, the Paulsen system of X (see above). As X is n-minimal, $S(X)$ is $2n$-minimal, and so is A (by Proposition 1.1). By Proposition 3.1,

$$A = \oplus_{i\in I}^\infty C(\Omega_i, M_{m_i}) \quad *\text{-isomorphically,}$$

where $m_i \le 2n$. For simplicity of notation, assume for the moment that the cardinality of I is equal to 1, and so

$$X = pC(\Omega, M_m)(1-p) \quad \text{completely isometrically,}$$

for some projection $p \in C(\Omega, M_m)$. By [5, Corollary 3.3] or [8, Theorem 3.2], there is a unitary u of $C(\Omega, M_m)$ such that for any $\omega \in \Omega$, $upu^*(\omega)$ is of the form $\mathrm{diag}(1,\dots,1,0,\dots,0)$. So we may assume that for any $\omega \in \Omega$, $p(\omega)$ is a diagonal matrix of the form given above. For any $k \le m$ we define

$$\Omega_k = \{\omega \in \Omega \;:\; rg(p(\omega)) = k\},$$

which is a closed subset of Ω (because the rank and the trace of a projection coincide), and the family $(\Omega_k)_{k\le m}$ forms a partition of Ω. Hence any Ω_k

is open (and closed) in Ω, so Ω_k is still Stonean. We have the completely isometric identifications

$$X = pC(\Omega, M_m)(1-p) = \oplus^{\infty}_{k\leq m} C(\Omega_k, M_{k,m-k}) = \oplus^{\infty}_{1\leq k\leq m-1} C(\Omega_k, M_{k,m-k}).$$

Moreover, for any $1 \leq k \leq m-1$, we have the completely isometric embeddings

$$M_{k,m-k} \subset C(\Omega_k, M_{k,m-k}) \subset X.$$

As X is n-minimal, this forces $k \leq n$ and $m-k \leq n$, for otherwise at least the row Hilbert space R_{n+1} or the column Hilbert space C_{n+1} would be n-minimal. Thus X has the announced form. In general, I is a finite set and

$$X = p \oplus^{\infty}_{i\in I} C(\Omega_i, M_{m_i})(1-p) = \oplus^{\infty}_{i\in I} p_i C(\Omega_i, M_{m_i})(1-p_i),$$

where p_i is a projection in $C(\Omega_i, M_{m_i})$ and $p = \oplus_i p_i$. Applying the above argument to each term $p_i C(\Omega_i, M_{m_i})(1-p_i)$, we obtain the result. $\square$

Corollary 3.6. *Let X be an n-minimal dual operator space. Then the following are equivalent:*

(i) *X is injective.*
(ii) *There exists a finite family of measure spaces $(\Omega_i)_{i\in I}$ such that $X = \oplus^{\infty}_{i\in I} L^{\infty}(\Omega_i, M_{r_i,k_i})$ via a completely isometric w^*-homeomorphism with $r_i, k_i \leq n$, for any $i \in I$.*

Proof. By the above theorem, $X = \oplus^{\infty}_i C(K_i, M_{r_i,k_i})$ completely isometrically, where K_i is Stonean. Since X is a dual operator space, this forces $C(K_i)$ to be a dual commutative C^*-algebra, i.e., $C(K_i) = L^{\infty}(\Omega_i)$ (via a normal $*$-isomorphism) for some measure space Ω_i. $\square$

4. Application to n-minimal TROs

In this section, we will use the description of injective n-minimal operator spaces to obtain results on n-minimal TROs. First, we will show that the n-minimal operator structure of a TRO determines its whole triple structure. See, e.g., [6] or [2, Section 8.3] for details on TROs.

Proposition 4.1. *Let X be a TRO. The following are equivalent:*

(i) *There exists a compact Hausdorff space Ω and an injective triple morphism $\pi : X \to C(\Omega, M_n)$.*
(ii) *X is n-minimal.*

Proof. (i)$\Rightarrow$(ii) follows from the fact that an injective triple morphism is necessarily completely isometric (see, e.g., [6, Proposition 2.2] or [2, Lemma 8.3.2]).

Suppose (ii). By [2, Remark 4.4.5 (1)], the injective envelope of X admits a TRO structure and X can be viewed as a sub-TRO of $I(X)$. By Theorem

3.5 we can describe this injective envelope as a direct sum,

$$I(X) = \oplus_{i\in I}^{\infty} C(\Omega_i, M_{r_i,k_i}) \quad \text{completely isometrically.}$$

But the right hand side of this equality admits a canonical TRO structure and it is known (see, e.g., [2, Corollary 4.4.6]) that a surjective complete isometry between TROs is automatically a triple morphism. In addition, for any i, the embedding $\varphi_i : M_{r_i,k_i} \to M_n$ into the "up-left" corner of M_n is an injective triple morphism. As at the end of the proof of Corollary 3.3, we finally obtain

$$X \subset I(X) = \oplus_{i\in I}^{\infty} C(\Omega_i, M_{r_i,k_i}) \subset C(\Omega, M_n)$$

as TROs. □

For details on the theory of C^*-modules the reader is referred to [11] or [2, Chapter 8] for an operator space approach. We recall the construction of the *linking C^*-algebra* of a C^*-module. If X is a left C^*-module over a C^*-algebra A, then its conjugate vector space $\overline{X}$ is a right C^*-module over A with the action $\overline{x} \cdot a = \overline{a^*x}$ and inner product $\langle \overline{x}, \overline{y}\rangle = \langle x, y\rangle$, for any $a \in A$, $x, y \in X$. We denote by ${}_A\mathbb{K}(X)$ the C^*-algebra of "compact" adjointable maps of X. Then

$$\mathcal{L}(X) = \begin{pmatrix} A & X \\ \overline{X} & {}_A\mathbb{K}(X) \end{pmatrix}$$

is a C^*-algebra too, called the *linking C^*-algebra of X*. If X is an equivalence bimodule (see [2, Paragraph 8.1.2]) over two C^*-algebras A and B, we define

$$\mathcal{L}(X) = \begin{pmatrix} A & X \\ \overline{X} & B \end{pmatrix} \quad \text{and} \quad \mathcal{L}^1(X) = \begin{pmatrix} A^1 & X \\ \overline{X} & B^1 \end{pmatrix},$$

where A^1 and B^1 denote the unitizations of A and B, which are also C^*-algebras (see [2, Paragraph 8.1.17] for details on linking C^*-algebras). We notice that X is an "off-diagonal" corner of a C^*-algebra, i.e., $X = p\mathcal{L}^1(X)(1-p)$ for some projection $p \in \mathcal{L}^1(X)$. Hence a C^*-module admits a TRO structure. The converse will be proved later, so we have a correspondence between C^*-modules, equivalence bimodules and TROs (see [2, Paragraph 8.1.19, 8.3.1]). The following corollary is a reformulation of the above proposition in the language of C^*-modules; it can be compared with Theorem 5.4.

COROLLARY 4.2. *Let X be a full left C^*-module over a C^*-algebra A. Then the following are equivalent:*

(i) *There exists a compact Hausdorff space Ω, a complete isometry $i : X \to C(\Omega, M_n)$ and a $*$-isomorphism $\sigma : A \to C(\Omega, M_n)$ such that for any $a \in A$, $x, y \in X$,*

$$i(a \cdot x) = \sigma(a)i(x),$$
$$\sigma(\langle x, y\rangle) = i(x)i(y)^*.$$

(ii) *X is n-minimal and A is subhomogeneous of degree $\leq n$.*

(iii) *X is n-minimal.*

Proof. Only (iii)$\Rightarrow$(i) needs a proof. Since X is a C^*-module, it is also a TRO (see above). By Proposition 4.1, there exists a compact Hausdorf space Ω and an injective triple morphism $i : X \to C(\Omega, M_n)$. By [2, Corollary 8.3.5], we can construct a corner preserving $*$-isomorphism $\pi : \mathcal{L}(X) \to M_2(C(\Omega, M_n))$ such that $i = \pi_{12}$. Choosing $\sigma = \pi_{11}$, we obtain the desired relations. $\square$

An equivalence bimodule version of the above corollary can be stated. In the above result we transfer n-minimality from X to A. We can also treat the "converse reverse" question: Let X be an equivalence bimodule over two n-minimal C^*-algebras. We will prove that X is n-minimal. We first translate this proposition into the language of TROs. Let X be a TRO contained in a C^*-algebra B via an injective triple morphism. As in the notation of the second section of [15], we define $C(X)$ (resp. $D(X)$) as the norm closure of $\mathrm{span}\{xy^*,\ x, y \in X\}$ (resp. $\mathrm{span}\{x^*y,\ x, y \in X\}$). As X is a sub-TRO of B, $C(X)$ and $D(X)$ are sub-C^*-algebras of B and

$$A(X) = \begin{pmatrix} C(X) & X \\ X^\star & D(X) \end{pmatrix}$$

is a sub-C^*-algebras of $M_2(B)$. Hence a TRO can be regarded as an "off-diagonal" corner of a C^*-algebra. This establishes the correspondence between C^*-modules, equivalence bimodules and TROs. $A(X)$ is also called the *linking C^*-algebra of X*. Analogously, in the W^*-TROs category, let X be a W^*-TRO contained in a W^*-algebra B via a w^*-continuous injective triple morphism. We define $M(X)$ (resp. $N(X)$) as the w^*-closure of $\mathrm{span}\{xy^*,\ x, y \in X\}$ (resp. $\mathrm{span}\{x^*y,\ x, y \in X\}$). As X is a sub-W^*-TRO of B, $M(X)$ and $N(X)$ are sub-W^*-algebras of B and

$$R(X) = \begin{pmatrix} M(X) & X \\ X^\star & N(X) \end{pmatrix}$$

is a sub-W^*-algebras of $M_2(B)$, called the *linking von Neumann algebra of X*. In fact, the linking algebras do not depend on the embedding of X into a C^*-algebra.

Obviously, if X is an equivalence bimodule over two C^*-algebras A and B, $C(X)$ and $D(X)$ play the roles of A and B in the correspondence between equivalence bimodules and TROs. Hence in the language of TROs we obtain (in the dual case):

Proposition 4.3. *Let X be a W^*-TRO such that $M(X)$ and $N(X)$ are n-minimal von Neumann algebras. Then X is n-minimal and*

$$X = \oplus_i^\infty\ L^\infty(\Omega_i)\overline{\otimes}M_{r_i,k_i},$$

where Ω_i is a measure space, $r_i, k_i \le n$, for any i.

Proof. We write $R(X)$ for the linking von Neumann algebra of X. By [9, Theorem 6.5.2], there exist p_1, p_2 and p_3, three central projections of $R(X)$, such that

$$R(X) = p_1 R(X) \oplus^{\infty} p_2 R(X) \oplus^{\infty} p_3 R(X)$$

and such that, for $i = 1, 2, 3$, $p_i R(X)$ is a von Neumann algebra of type i or $p_i = 0$. Since $M(X)$ is n-minimal, $M(X)$ is of type I. However, $M(X) = pR(X)p$ for some projection p in $R(X)$ and for any i,

$$p_i M(X) = pp_i pM(X)pp_i p.$$

As the type is unchanged by compression (see [9, Exercise 6.9.16]), $p_i M(X)$ is of type I or $p_i M(X) = 0$. On the other hand, for any i,

$$p_i M(X) = p_i pR(X) = pp_i R(X)p_i p,$$

so $p_i M(X)$ has the same type as $p_i R(X)$ or $p_i M(X) = 0$. Thus $p_i M(X) = 0$ for $i = 2, 3$, i.e., $p_i p = 0$ for $i = 2, 3$. Analogously, using our assumption on $N(X)$, we have $p_i(1 - p) = 0$ for $i = 2, 3$. Hence $p_i = 0$ for $i = 2, 3$, i.e., $R(X)$ is of type I. By [15, Theorem 4.1],

$$X = \oplus_k^{\infty} \ L^{\infty}(\Omega_k) \overline{\otimes} M_{I_k, J_k},$$

where Ω_k is a measure space, I_k, J_k are sets and $M_{I_k, J_k} = B(\ell^2_{I_k}, \ell^2_{J_k})$. Since $M(X)$ (resp. $N(X)$) is n-minimal, this forces the cardinality of I_k (resp. J_k) to be no greater than n, for any k. So X is n-minimal and has the desired form. □

REMARK 4.4. In the following two results, we will use the fact that the multiplier algebra of an n-minimal C^*-algebra is also n-minimal. This follows from Proposition 1.1.

The following corollary on W^*-TROs extends Remark 2.4.

COROLLARY 4.5. *Let X be a W^*-TRO. The following are equivalent:*

(i) *X is n-minimal.*
(ii) *There exists a measure space Ω and a w*-continuous injective triple morphism $\pi : X \to L^{\infty}(\Omega, M_n)$.*
(iii) *There exists a finite family of measure spaces $(\Omega_i)_{i \in I}$ such that $X = \oplus^{\infty}_{i \in I} L^{\infty}(\Omega_i, M_{r_i, k_i})$ with $r_i, k_i \leq n$, for any $i \in I$.*

Proof. Only (i)⇒(iii) needs a proof. Suppose (i). By Proposition 4.1 we can regard X as a sub-TRO of $C(\Omega, M_n)$. Hence, by construction, $C(X)$ and $D(X)$ are n-minimal C^*-algebras. By [10], $M(X)$ (resp. $N(X)$) is the multiplier algebra of $C(X)$ (resp. $D(X)$), so $M(X)$ and $N(X)$ are n-minimal W^*-algebras (by Remark 4.4). The result follows from the above proposition. □

Finally, we can generalize the implications (ii)⇔(iv)⇔(v) of [2, Proposition 8.6.5] on minimal TROs to the n-minimal case.

Theorem 4.6. *Let X be a TRO. The following are equivalent:*
(i) *X is n-minimal.*
(ii) *X^{**} is an injective n-minimal operator space (see Corollary 3.6).*
(iii) *$C(X)$ and $D(X)$ are n-minimal C^*-algebras.*

Proof. (ii)⇒(i) and (i)⇒(iii) are obvious. Suppose (iii). By [10, Proposition 2.4] the multiplier algebra of $C(X^{**})$ is $C(X)^{**}$, and this C^*-algebra is n-minimal by our assumption on $C(X)$ and Remark 4.4. Moreover, by [15], $M(X^{**})$ is also the multiplier algebra of $C(X^{**})$, so $M(X^{**})$ is n-minimal too. The same argument works for $N(X^{**})$ and we can apply Proposition 4.3 to X^{**}. □

5. An n-minimal version of the CES-theorem

To prove the "n-minimal" version the CES-Theorem we need the notion of *left multiplier algebra* of an operator space X. A left multiplier of an operator space X is a map $u : X \to X$ such that there exists a C^*-algebra A containing X via a complete isometry i and $a \in A$ satisfying $i(u(x)) = ai(x)$ for any $x \in X$. Let $\mathcal{M}_l(X)$ denote the set of left multipliers of X. The *multiplier norm of u* is the infimum of $\|a\|$ over all possible A, i, a as above. In fact, Blecher and Paulsen proved that any left multiplier can be represented in the embedding of X into the C^*-algebra (discussed in Section 3)

$$I(S(X)) = \begin{pmatrix} I_{11}(X) & I(X) \\ I(X)^\star & I_{22}(X) \end{pmatrix}.$$

More precisely, for any left multiplier u of norm no greater than 1 there exists a unique $a \in I_{11}(X)$ of norm no greater than 1 such that $u(x) = ax$ for any $x \in X$ (see [2, Theorem 4.5.2]). This result enables us to consider $\mathcal{M}_l(X)$ as an operator subalgebra of $I_{11}(X)$ (see the proof of [2, Proposition 4.5.5] and [2, Paragraph 4.5.3] for more details) and

$$\mathcal{M}_l(X) = \{a \in I_{11}(X),\ aX \subset X\}$$

as operator algebras. The product used in this formula is that on the C^*-algebra $I(S(X))$. The operator algebra $\mathcal{M}_l(X)$ is called the *multiplier algebra of* X. We let $\mathcal{A}_l(X) = \Delta(\mathcal{M}_l(X))$ denote the diagonal (see [2, Paragraph 2.1.2]) of $\mathcal{M}_l(X)$. This C^*-algebra is called the *left adjointable multiplier algebra of* X and we have

$$\mathcal{A}_l(X) = \{a \in I_{11}(X),\ aX \subset X \text{ and } a^*X \subset X\}$$

$*$-isomorphically. In fact, if X is originally a C^*-algebra, then $\mathcal{A}_l(X)$ is just its multiplier algebra, and we recover Remark 4.4.

Analogously, the *right multiplier algebra of* X is given by

$$\mathcal{M}_r(X) = \{b \in I_{22},\ Xb \subset X\},$$

and its diagonal $\mathcal{A}_r(X) = \{b \in I_{22},\ Xb \subset X \text{ and } Xb^* \subset X\}$ is the *right adjointable multiplier algebra* of X.

LEMMA 5.1. *Let X be an operator space and $I(X)$ its injective envelope. Then there exists a completely contractive unital homomorphism $\theta : \mathcal{M}_l(X) \to \mathcal{M}_l(I(X))$ such that $\theta(u)_{|X} = u$, for any $u \in \mathcal{M}_l(X)$. Thus, $\theta_{|\mathcal{A}_l(X)} : \mathcal{A}_l(X) \to \mathcal{A}_l(I(X))$ is a $*$-isomorphism. Moreover, the same results hold for right multipliers.*

Proof. Let $u \in \mathcal{M}_l(X)$. Then u can be represented by an element a in $\{a \in I_{11}(X),\ aX \subset X\}$. Using the multiplication inside $I(S(X))$, $aI(X) \subset I(X)$, so a can be regarded as an element of $\mathcal{M}_l(I(X))$, which we write as $\theta(u)$. Therefore, θ is an injective unital completely contractive homomorphism. The rest of the proof follows from [2, Paragraph 2.1.2]. □

In the following lemma, we use the C^**-envelope* of a unital operator space; see [2, Theorem 4.3.1] for details. We write R_n (resp. C_n) for the row (resp. column) Hilbert space of dimension n. If X is an operator space, we let $C_n(X)$ be the minimal tensor product of C_n and X, or equivalently,

$$C_n(X) = \left\{ \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \dots & \vdots \\ x_n & 0 & \cdots & 0 \end{pmatrix},\ x_i \in X \right\} \subset M_n(X).$$

The definition of $R_n(X)$ is similar using a row instead of a column. Adapting the proof of the first example of the third section of [17], we obtain:

LEMMA 5.2. *Let A be an injective C^*-algebra and $k \in \mathbb{N}^*$. Then:*

(1) *$\mathcal{M}_l(R_k(A)) = A$ $*$-isomorphically and the action is given by*

$$a \cdot (x_1, \dots, x_k) = (ax_1, \dots, ax_k), \quad \text{for any } a, x_i \in A.$$

(2) *$\mathcal{M}_r(C_k(A)) = A$ $*$-isomorphically and the action is given by*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \cdot a = \begin{pmatrix} x_1 a \\ \vdots \\ x_k a \end{pmatrix}, \quad \text{for any } a, x_i \in A.$$

Proof. We only prove (1); the proof of (2) is similar. Since $R_n = B(\ell_n^2, \mathbb{C})$, the Paulsen system $\mathcal{S}$ of $R_n(A)$ is

$$\mathcal{S} = \left\{ \begin{pmatrix} \alpha 1_A & x \\ y* & \beta I_n \otimes 1_A \end{pmatrix},\ \alpha, \beta \in \mathbb{C},\ x, y \in R_n(A) \right\} \subset M_{n+1}(A).$$

Clearly the C^*-algebra $C^*(\mathcal{S})$ generated by $\mathcal{S}$ (inside $M_{n+1}(A)$) coincides with $M_{n+1}(A)$.

Next, we show that the C^*-envelope $C_e^*(\mathcal{S})$ of $\mathcal{S}$ is $M_{n+1}(A)$. By the universal property of $C_e^*(\mathcal{S})$, there is a surjective $*$-homomorphism $\pi : C^*(\mathcal{S}) \twoheadrightarrow C_e^*(\mathcal{S})$ such that the following diagram commutes:

$$\begin{tikzcd} C^*(\mathcal{S}) \arrow[dr, "\pi"] & \\ \mathcal{S} \arrow[u, hook] \arrow[r, hook] & C_e^*(\mathcal{S}) \end{tikzcd}$$

We let

$$p = \pi\left(\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}\right) \quad \text{and} \quad q = \pi\left(\begin{pmatrix} 0 & 0 \\ 0 & I_n \otimes 1_A \end{pmatrix}\right).$$

Then p and q are projections of $C_e^*(\mathcal{S})$ satisfying $p + q = 1$ and $pq = 0$. Thus we can decompose $C_e^*(\mathcal{S})$ in "2×2" matrix corners. Hence π is corner preserving and there exist $\pi_1, \pi_2, \pi_3, \pi_4$ such that for any $a \in A$, $b \in M_n(A)$, $x, y \in R_n(A)$,

$$\pi\left(\begin{pmatrix} a & x \\ y* & b \end{pmatrix}\right) = \begin{pmatrix} \pi_1(a) & \pi_2(x) \\ \pi_3(y)* & \pi_4(b) \end{pmatrix}.$$

The (1,2) corners of $\mathcal{S}$ and of $C^*(\mathcal{S})$ coincide, so π_2 is injective (because π extends to $C^*(\mathcal{S})$ the inclusion $\mathcal{S} \subset C_e^*(\mathcal{S})$). Similarly, π_3 is injective. On the other hand, for any $a \in A$, $x \in R_n(A)$,

$$\pi_2(ax) = \pi_1(a)\pi_2(x).$$

Thus, choosing a "good" x, this shows that π_1 is injective too. Analogously, using

$$\pi_2(xb) = \pi_2(x)\pi_4(b), \quad \text{for any } b \in M_n(A),\ x \in R_n(A),$$

the above argument yields the injectivity of π_4.

Finally, π is injective and so $C_e^*(\mathcal{S}) = M_{n+1}(A)$. By the assumption on A, $M_{n+1}(A)$ is an injective C^*-algebra. Therefore

$$I(\mathcal{S}) = M_{n+1}(A) \qquad *\text{-isomorphically}$$

and

$$I_{11}(R_n(A)) = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} I(\mathcal{S}) \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} = A.$$

This proves (1). $\square$

Remark 5.3. We acknowledge that after this paper was submitted, D. Blecher pointed out to the author a more general result: Let X be an operator space. Then for any $p, q \in \mathbb{N}^*$,

$$\mathcal{M}_l(M_{p,q}(X)) = M_p(\mathcal{M}_l(X)).$$

We outline the proof. As in [2, Paragraph 4.4.11], we can define the C^*-algebra $\mathcal{C}(X) = I(X)I(X)^*$. Using [2, Corollary 4.6.12], we note that

$$\mathcal{C}(M_{p,q}(X)) = M_p(\mathcal{C}(X)).$$

Moreover, from [4], the multiplier algebra of $\mathcal{C}(X)$ coincides with $I_{11}(X)$, i.e.,

$$\mathcal{M}(\mathcal{C}(X)) = I_{11}(X).$$

Using these two facts, we obtain

$$\begin{aligned}
\mathcal{M}_l(M_{p,q}(X)) &= \{a \in I_{11}(M_{p,q}(X)),\ aM_{p,q}(X) \subset M_{p,q}(X)\} \\
&= \{a \in \mathcal{M}(\mathcal{C}(M_{p,q}(X))),\ aM_{p,q}(X) \subset M_{p,q}(X)\} \\
&= \{a \in \mathcal{M}(M_p(\mathcal{C}(X))),\ aM_{p,q}(X) \subset M_{p,q}(X)\} \\
&= \{a \in M_p(\mathcal{M}(\mathcal{C}(X))),\ a_{ij}X \subset X,\ \forall\, i,j\} \\
&= \{a \in M_p(I_{11}(X)),\ a_{ij}X \subset X,\ \forall\, i,j\} \\
&= M_p(\mathcal{M}_l(X)).
\end{aligned}$$

The following theorem enables us to represent completely contractively a module action on an n-minimal operator space into a C^*-algebra of the form $C(\Omega, M_n)$. This constitutes the main result of this section and it generalizes the implication (i)⇔(iii) of [3, Theorem 2.2].

THEOREM 5.4. *Let A be a Banach algebra endowed with an operator space structure (resp. a C^*-algebra). Let X be an n-minimal operator space which is also a left Banach A-module. Assume that there is a net $(e_t)_t \subset \mathrm{Ball}(A)$ satisfying $e_t \cdot x \to x$, for any $x \in X$. The following are equivalent:*

(i) *X is a left h-module over A.*

(ii) *There exists a compact Hausdorff space Ω, a complete isometry $i : X \to C(\Omega, M_n)$ and a completely contractive homomorphism (resp. $*$-homomorphism) $\pi : A \to C(\Omega, M_n)$ such that*

$$i(a \cdot x) = \pi(a)i(x), \quad \textit{for any } a \in A,\ x \in X.$$

Proof. Suppose (i). We first treat the Banach algebra case. By Blecher's oplication Theorem (see [2, Theorem 4.6.2]), there is a completely contractive homomorphism $\eta : A \to \mathcal{M}_l(X)$ such that $\eta(a)(x) = a \cdot x$, for any $a \in A$, $x \in X$. Using the homomorphism θ obtained in Lemma 5.1, we obtain a completely contractive homomorphism $\sigma = \theta \circ \eta : A \to \mathcal{M}_l(I(X))$ satisfying

$$\sigma(a)(x) = a \cdot x, \quad \text{for any } a \in A,\ x \in X.$$

Moreover, $I(X)$ is an injective n-minimal operator space, so

$$I(X) = \oplus_{i\in I}^{\infty} C(\Omega_i, M_{r_i,k_i}) \quad \text{completely isometrically,}$$

where the Ω_i's are Stonean and $r_i, k_i \leq n$, for any $i \in I$. We have the completely isometric unital isomorphisms

$$\begin{aligned}\mathcal{M}_l(I(X)) &= \oplus_i^\infty \mathcal{M}_l(C(\Omega_i, M_{r_i,k_i}))\\ &= \oplus_i^\infty \mathcal{M}_l(C_{r_i} \otimes_{\min} R_{k_i} \otimes_{\min} C(\Omega_i))\\ &= \oplus_i^\infty M_{r_i}(\mathcal{M}_l(R_{k_i} \otimes_{\min} C(\Omega_i)))\\ &= \oplus_i^\infty M_{r_i}(C(\Omega_i)) \quad \text{(by Lemma 5.2)}.\end{aligned}$$

Via these identifications, the action of $\mathcal{M}_l(I(X))$ on $I(X)$ is the one inherited from the obvious left action of M_{r_i} on M_{r_i,k_i}. More precisely, for any $u = (f_i \otimes y_i)_i \in \mathcal{M}_l(I(X))$ and $x = (g_i \otimes x_i)_i \in I(X)$,

$$u(x) = (f_i g_i \otimes y_i x_i)_i.$$

For each i, let $\varphi_i : M_{r_i} \to M_n$ (resp. $\phi_i : M_{r_i,k_i} \to M_n$) be the embedding of M_{r_i} (resp. M_{r_i,k_i}) in the "upper left corner" of M_n. Hence, as at the end of the proof of Corollary 3.3, we have now a $*$-isomorphism

$$\begin{array}{cccc}\psi : & \mathcal{M}_l(I(X)) & \to & C(\Omega, M_n)\\ & (f_i \otimes y_i)_i & \mapsto & \sum_i \tilde{f}_i \otimes \varphi_i(y_i)\end{array}$$

and a complete isometry

$$\begin{array}{cccc}j : & I(X) & \to & C(\Omega, M_n)\\ & (g_i \otimes x_i)_i & \mapsto & \sum_i \tilde{g}_i \otimes \phi_i(x_i)\end{array}$$

which verify

$$j(u(x)) = \psi(u)j(x) \quad \text{for any } u \in \mathcal{M}_l(I(X)),\ x \in I(X).$$

Finally, Ω, $i = j_{|X}$ and $\pi = \psi \circ \sigma$ satisfy the desired relations. If A is a C^*-algebra, we obtain the result using the fact that a contractive homomorphism between C^*-algebras is necessarily a $*$-homomorphism. $\square$

Remark 5.5.

(1) By the above result, a C^*-algebra which acts "suitably" on an n-minimal operator space is necessarily an extension of a subhomogeneous C^*-algebra of degree $\leq n$.

(2) Suppose that A is unital and its action too (i.e., $1 \cdot x = x$ for any x in X). In the above result, we cannot expect to obtain a unital completely contractive homomorphism π, because when A is an operator algebra and $A = X$, the assumption (i) is verified (see the BRS Theorem [2, Theorem 2.3.2]). Hence this particular case leads back to the Remark 3.4.

The following theorem can be considered as an "n-minimal version" of the CES-Theorem (see [2, Theorem 3.3.1]). It is the bimodule version of Theorem 5.4, and its proof is "symmetrically" the same using the two lemmas above.

THEOREM 5.6. *Let A and B be two Banach algebras endowed with an operator space structure (resp. two C^*-algebras). Let X be an n-minimal operator space which is also a Banach A-B-bimodule. Assume that there is a net $(e_t)_t \subset \mathrm{Ball}(A)$ (resp. $(f_s)_s \subset \mathrm{Ball}(B)$) satisfying $e_t \cdot x \to x$ (resp. $x \cdot f_s \to x$), for any $x \in X$. The following are equivalent:*

(i) *X is an h-bimodule over A and B.*

(ii) *There exists a compact Hausdorff space Ω, a complete isometry $i : X \to C(\Omega, M_n)$ and two completely contractive homomorphisms (resp. $*$-homomorphisms) $\pi : A \to C(\Omega, M_n)$ and $\theta : B \to C(\Omega, M_n)$ such that*

$$i(a \cdot x \cdot b) = \pi(a)i(x)\theta(b), \quad \textit{for any } a \in A,\ b \in B,\ x \in X.$$

The following result states that if A and B are originally n-minimal operator algebras, then π and θ can be chosen completely isometric. This corollary generalizes [3, Corollary 2.10].

COROLLARY 5.7. *Let A, B and X be three n-minimal operator spaces such that A and B are approximately unital operator algebras and X is a Banach A-B-bimodule. Assume that there is a net $(e_t)_t \subset \mathrm{Ball}(A)$ (resp. $(f_s)_s \subset \mathrm{Ball}(B)$) satisfying $e_t \cdot x \to x$ (resp. $x \cdot f_s \to x$), for any $x \in X$. The following are equivalent:*

(i) *X is a left h-module over A.*

(ii) *There exists a compact Hausdorff space Ω, a complete isometry $i : X \to C(\Omega, M_n)$ and completely isometric homomorphisms $\pi : A \to C(\Omega, M_n)$ and $\theta : B \to C(\Omega, M_n)$ such that*

$$i(a \cdot x \cdot b) = \pi(a)i(x)\theta(b), \quad \textit{for any } a \in A,\ b \in B,\ x \in X.$$

Proof. By Theorem 5.6 there exists a compact Hausdorff space K_0, a complete isometry $j : X \to C(K_0, M_n)$ and completely contractive homomorphisms $\pi_0 : A \to C(K_0, M_n)$ and $\theta_0 : B \to C(K_0, M_n)$ satisfying

$$j(a \cdot x \cdot b) = \pi_0(a)i(x)\theta_0(b),$$

for any $a \in A$, $b \in B$, $x \in X$. Moreover, by Corollary 2.3, there exists a compact Hausdorff space K_A (resp. K_B) and a completely isometric homomorphism $\pi_A : A \to C(K_A, M_n)$ (resp. $\theta_B : B \to C(K_B, M_n)$). Let

$$C = C(K_A, M_n) \oplus^\infty C(K_0, M_n) \oplus^\infty C(K_B, M_n) = C(\Omega, M_n),$$

where Ω is the disjoint union of K_A, K_B and K_0. Let $i : X \to C(\Omega, M_n)$ defined by $i(x) = 0 \oplus j(x) \oplus 0$, for any $x \in X$. Thus i is a complete isometry. Let $\pi : A \to C(\Omega, M_n)$ (resp. $\theta : B \to C(\Omega, M_n)$) be defined by $\pi(a) = \pi_A(a) \oplus \pi_0(a) \oplus 0$, for any $a \in A$ (resp. $\theta(b) = 0 \oplus \theta_0(b) \oplus \theta_B(b)$, for any $b \in B$). Hence, π and θ are completely isometric homomorphisms. Finally, Ω, π, θ and i satisfy the desired relation. $\square$

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