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UNIFORM APPROXIMATION OF ASSOCIATIVE COPULAS BY STRICT AND NON-STRICT COPULAS

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Abstract. Some properties of and relationships between important classes of copulas are discussed. In particular, the class of associative copulas is shown to be compact, and the classes of strict and non-strict copulas are shown to be dense subsets of this class.

1. Preliminaries

In the context of the construction of a joint distribution function from two given marginal distribution functions, the notion of a copula was introduced by A. Sklar [24, 25], following some pioneering work of W. Hoeffding [5, 6], M. Fréchet [4] and R. Féron [3]. An excellent source for results on copulas is the recent monograph by R. B. Nelsen [17].

Recall that a two-dimensional copula (briefly, a 2-copula or, simply, a copula) is a binary operation C on the unit interval $[0,1]$, i.e., a function $C\colon [0,1]^2 \longrightarrow [0,1]$, such that for all $x, x^*, y, y^* \in [0,1]$ with $x \leq x^*$ and $y \leq y^*$

$$C(x,y) + C(x^*,y^*) \geq C(x,y^*) + C(x^*,y), \tag{1}$$

$$C(x,0) = C(0,x) = 0, \tag{2}$$

$$C(x,1) = C(1,x) = x. \tag{3}$$

The importance of copulas in probability and statistics comes from Sklar's Theorem [24], showing that the joint distribution of a random vector and the corresponding marginal distributions are necessarily linked by a copula.

In general, a copula C is neither commutative nor associative, but always non-decreasing in each component, and it satisfies the Lipschitz property with constant 1, i.e., for all points $(x_1,y_1),(x_2,y_2) \in [0,1]^2$ we have

$$|C(x_1,y_1) - C(x_2,y_2)| \leq |x_1 - x_2| + |y_1 - y_2|.$$

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As a consequence, each copula is uniformly continuous. Trivially, the functions $W, P, M\colon [0,1]^2 \longrightarrow [0,1]$ defined by

$$\begin{aligned} W(x,y) &= \max(x+y-1,0), \\ P(x,y) &= x \cdot y, \\ M(x,y) &= \min(x,y) \end{aligned}$$

are copulas, and for an arbitrary copula C we always have $W \leq C \leq M$. Therefore, the functions W and M are often referred to as the Fréchet-Hoeffding lower and upper bounds.

If C is an associative copula then $([0,1], C)$ is an I-semigroup [2, 15, 22] with neutral element 1 and annihilator 0, i.e., a topological semigroup with neutral element 1 and annihilator 0. It is well-known [7, 10, 23] that in this case C must be a triangular norm (briefly, a t-norm), i.e., $([0,1], C)$ is a fully ordered, commutative semigroup with neutral element 1 and annihilator 0 [10, 12, 19, 20, 21, 23]. In other words, a copula C is a t-norm if and only if C is associative, and a t-norm T is a copula if and only if it is Lipschitz with constant 1 (see [16]).

A copula C is said to be Archimedean if there exists a continuous, strictly decreasing function $\varphi\colon [0,1] \longrightarrow [0,\infty]$ with $\varphi(1) = 0$ (the so-called generator of the copula) such that for all $(x,y) \in [0,1]^2$

$$C(x,y) = \varphi^{[-1]}\bigl(\varphi(x) + \varphi(y)\bigr), \tag{4}$$

where $\varphi^{[-1]}\colon\ [0,\infty] \longrightarrow [0,1]$ is the pseudo-inverse of φ given by

$$\varphi^{[-1]}(x) = \begin{cases} \varphi^{-1}(x) & \text{if } x \in [0, \varphi(0)]\,, \\ 0 & \text{if } x \in \,]\varphi(0), \infty]\,. \end{cases} \tag{5}$$

A continuous, strictly decreasing function $\varphi\colon [0,1] \longrightarrow [0,\infty]$ with $\varphi(1) = 0$ is the generator of an Archimedean copula if and only if φ is a convex function [17], i.e., if $\varphi\bigl(\frac{x+y}{2}\bigr) \leq \frac{1}{2}\bigl(\varphi(x) + \varphi(y)\bigr)$ for all $x, y \in [0,1]$. A generator of an Archimedean copula is unique up to a positive multiplicative constant, and each Archimedean copula is necessarily associative. An Archimedean copula C with generator φ is called strict if $\varphi(0) = \infty$, and non-strict otherwise. Also, for each Archimedean copula C and for all $x, y \in \,]0,1[$ there is an $n \in \mathbb{N}$ such that $x_C^n < y$, where the C-powers x_C^n are defined recursively by $x_C^1 = x$ and $x_C^{n+1} = C(x, x_C^n)$.

Each copula can be represented as an ordinal sum of copulas, none of which has a non-trivial idempotent element. (This result goes back to [23]; see also [17].) More precisely, for each copula C there exists a unique (finite or countably infinite) index set A, a family of unique pairwise disjoint open subintervals $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ and a family of unique copulas $(C_\alpha)_{\alpha \in A}$ without non-trivial idempotent elements such that for all $(x,y) \in [0,1]^2$

$$(6) \quad C(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot C_\alpha\bigl(\frac{x-a_\alpha}{e_\alpha-a_\alpha}, \frac{y-a_\alpha}{e_\alpha-a_\alpha}\bigr) & \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ M(x,y) & \text{otherwise.} \end{cases}$$

In this case we shall write $C = (\langle a_\alpha, e_\alpha, C_\alpha \rangle)_{\alpha \in A}$. In particular, if C is associative then each summand C_α is necessarily an Archimedean copula.

2. The main result

The set $\mathcal{X} = [0,1]^{[0,1]^2}$ of all functions from the unit square $[0,1]^2$ into the unit interval $[0,1]$, will be equipped with the topology $\mathcal{T}_\infty$ induced by the metric $d_\infty \colon \mathcal{X}^2 \longrightarrow [0,\infty]$ given by $d_\infty(f,g) = \sup\bigl\{|f(x,y) - g(x,y)| \bigm| (x,y) \in [0,1]^2\bigr\}$ (corresponding to the uniform convergence). We shall also write $\|f-g\|$ for $d_\infty(f,g)$.

We consider the set $\mathcal{C}$ of all copulas and the subsets $\mathcal{C}_{\mathbf{c}}$ of all commutative copulas, $\mathcal{C}_{\mathbf{a}}$ of all associative copulas, $\mathcal{C}_{\mathbf{nc}}$ of all non-commutative copulas, as well as the sets $\mathcal{C}_{\mathbf{s}}$ of strict and $\mathcal{C}_{\mathbf{ns}}$ of non-strict copulas. It is well-known that the sets $\mathcal{C}$ and $\mathcal{C}_{\mathbf{c}}$ are convex sets and that $\mathcal{C}_{\mathbf{nc}}$ is not convex.

It is also evident that the set $\mathcal{C}$ is a compact subset of $\mathcal{X}$ (see, e.g., [1]). As an immediate consequence, each Cauchy sequence of commutative copulas converges uniformly to some commutative copula C, i.e., $\mathcal{C}_{\mathbf{c}}$ is also compact. Because of the convexity of $\mathcal{C}$, the set $\mathcal{C}$ is the closure of the set $\mathcal{C}_{\mathbf{nc}}$ of all non-commutative copulas. Therefore the set $\mathcal{C}_{\mathbf{nc}}$ is not closed.

Also, pointwise convergence in $\mathcal{C}$ implies uniform convergence, so all assertions in this paper remain true if we equip $\mathcal{X}$ with the product topology (which corresponds to the pointwise convergence).

LEMMA 2.1. *Each Cauchy sequence (with respect to the pointwise convergence) of associative copulas converges uniformly to some associative copula C.*

Proof. Let $(C_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}_{\mathbf{a}}$. Since $\mathcal{C}_{\mathbf{a}} \subset \mathcal{C}_{\mathbf{c}}$, $(C_n)_{n\in\mathbb{N}}$ converges uniformly to some commutative copula C. Fix $\varepsilon > 0$. There is an $n_0 \in \mathbb{N}$ such that $\|C - C_n\| \le \varepsilon/4$ for all $n \ge n_0$. Since each C_n is also Lipschitz with constant 1, we get for all $x,y,z \in [0,1]$ and all $n \ge n_0$,

$$\begin{aligned} |C(C(x,y),z) - C(x,C(y,z))| &\le |C(C(x,y),z) - C_n(C(x,y),z)| \\ &\quad + |C_n(C(x,y),z) - C_n(C_n(x,y),z)| \\ &\quad + |C_n(C_n(x,y),z) - C_n(x,C_n(y,z))| \\ &\quad + |C_n(x,C_n(y,z)) - C_n(x,C(y,z))| \\ &\quad + |C_n(x,C(y,z)) - C(x,C(y,z))| \\ &\le \tfrac{\varepsilon}{4} + |C(x,y) - C_n(x,y)| + 0 + |C_n(y,z) - C(y,z)| + \tfrac{\varepsilon}{4} \le \varepsilon. \end{aligned}$$

Hence C is associative. □

Therefore, the (non-convex) set $\mathcal{C}_{\mathbf{a}}$ of associative copulas is a compact subset of $\mathcal{X}$.

LEMMA 2.2.
(i) *For each non-strict copula C and for each $\varepsilon > 0$ there exists a strict copula C_ε such that $\|C - C_\varepsilon\| \leq \varepsilon$.*
(ii) *For each strict copula C and for each $\varepsilon > 0$ there exists a non-strict copula C_ε such that $\|C - C_\varepsilon\| \leq \varepsilon$.*

Proof. In [10, Lemma 8.7] it was shown that two continuous Archimedean t-norms (and, consequently, two Archimedean copulas C_1 and C_2) whose generators coincide on the interval $[\varepsilon, 1]$ satisfy $\|C_1 - C_2\| \leq \varepsilon$. Also, since each generator φ of an Archimedean copula is convex, for each $x_0 \in [0, 1]$ there is a subtangent $f\colon [0, 1] \longrightarrow \mathbb{R}$ of φ, i.e., a straight line passing through the point $(x_0, \varphi(x_0))$ such that $f(x) \leq \varphi(x)$ for all $x \in [0, 1]$.

Now consider an Archimedean copula C with generator φ, fix $\varepsilon > 0$, and let f be a subtangent of φ passing through $(\varepsilon, \varphi(\varepsilon))$.

If C is non-strict and if the slope of f equals k, then, because of [10, Lemma 8.7], the function $\varphi_\varepsilon\colon [0, 1] \longrightarrow [0, \infty]$ given by

$$\varphi_\varepsilon(x) = \begin{cases} \varphi(\varepsilon) - \frac{k\varepsilon^2}{x} + k\varepsilon & \text{if } x \in [0, \varepsilon[\,, \\ \varphi(x) & \text{otherwise,} \end{cases}$$

is the generator of some strict copula C_ε with $\|C - C_\varepsilon\| \leq \varepsilon$.

If C is strict then, again as a consequence of [10, Lemma 8.7], the function $\varphi_\varepsilon\colon [0, 1] \longrightarrow [0, \infty]$ given by

$$\varphi_\varepsilon(x) = \begin{cases} f(x) & \text{if } x \in [0, \varepsilon[\,, \\ \varphi(x) & \text{otherwise,} \end{cases}$$

is the generator of some non-strict copula C_ε with $\|C - C_\varepsilon\| \leq \varepsilon$. $\square$

Noting that, for any convex function $\varphi\colon [0, 1] \longrightarrow [0, \infty]$ and each $\lambda \in [1, \infty[$, the power $\varphi^\lambda\colon [0, 1] \longrightarrow [0, \infty]$ is again a convex function, the following is an immediate consequence of [10, Proposition 8.5(i)]:

LEMMA 2.3. *If C is an Archimedean copula with generator $\varphi\colon [0, 1] \longrightarrow [0, \infty]$, then for each number $\lambda \in [1, \infty[$ the function φ^λ is the generator of some Archimedean copula $C^{(\lambda)}$, and for each $\varepsilon > 0$ there is a $\lambda_0 \in [1, \infty[$ such that $\|C^{(\lambda_0)} - M\| \leq \varepsilon$.*

For example, starting from the function $\varphi\colon [0, 1] \longrightarrow [0, \infty]$ given by $\varphi(x) = 1 - x$ (which generates the Fréchet-Hoeffding lower bound W), for each $\lambda \in [1, \infty[$ the function φ^λ generates the non-strict copula $C^{(\lambda)}$ (see [17, Section 4.3, copula 4.2.2]; this copula is also called a Yager t-norm [10]), and we have $\|C^{(\lambda)} - M\| \leq \varepsilon$ if and only if $\lambda \geq -\frac{\ln 2}{\ln(1-\varepsilon)}$. For an approximation of

M by strict copulas start, e.g., with the function $\varphi\colon [0,1] \longrightarrow [0,\infty]$ given by $\varphi(x) = \frac{1-x}{x}$ (which generates a limit case of the Ali-Mikhail-Haq family). Then for each $\lambda \in [1,\infty[$ the function φ^λ generates the strict copula $C^{(\lambda)}$ (see [17, Section 4.3, copula 4.2.12]), and we have $\|C^{(\lambda)} - M\| \leq \varepsilon$ if and only if $\lambda \geq -1/(2\log_2 \frac{1+\varepsilon}{1-\varepsilon})$.

From [10, Lemmas 8.9 and 8.10] and Lemmas 2.2(i) and 2.3 it follows readily that each associative copula (which can be written as an ordinal sum of Archimedean copulas (6)) can be approximated by the ordinal sum of finitely many strict copulas:

Lemma 2.4. *If C is an associative copula, then for each $\varepsilon > 0$ there exist a number $n \in \mathbb{N}$, strict copulas $C_1, C_2, \ldots, C_n$, and numbers $b_0, b_1, \ldots, b_n \in [0,1]$ with $0 = b_0 < b_1 < \cdots < b_n = 1$ such that*

$$\|(\langle b_{i-1}, b_i, C_i\rangle)_{i\in\{1,2,\ldots,n\}} - C\| \leq \varepsilon.$$

We now show that the ordinal sum of two strict copulas can be approximated by a single strict copula.

Lemma 2.5. *If $c \in\,]0,1[$ and if C_1 and C_2 are two strict copulas and if C denotes the ordinal sum $(\langle 0, c, C_1\rangle, \langle c, 1, C_2\rangle)$, then for each $\varepsilon > 0$ there is a strict copula C_ε such that*

$$\|(\langle 0, c, C_1\rangle, \langle c, 1, C_2\rangle) - C_\varepsilon\| \leq \varepsilon.$$

Proof. Let us write $C = (\langle 0, c, C_1\rangle, \langle c, 1, C_2\rangle)$, and let $\varphi_1, \varphi_2\colon [0,1] \longrightarrow [0,\infty]$ be the convex additive generators of C_1 and C_2, respectively. Define the convex decreasing bijections $\psi_1\colon\ [0,c] \longrightarrow [0,\infty]$ and $\psi_2\colon\ [c,1] \longrightarrow [0,\infty]$ by $\psi_1(x) = \varphi_1\big(\frac{x}{c}\big)$ and $\psi_2(x) = \varphi_2\big(\frac{x-c}{1-c}\big)$. Fix an arbitrary $\varepsilon \in\,]0, \min(c, 1-c)[$ and consider the subtangent $f\colon [0,1] \longrightarrow \mathbb{R}$ of ψ_2 passing through the point $(c + \frac{\varepsilon}{2}, 0)$, and the straight line $g\colon [0,1] \longrightarrow \mathbb{R}$ connecting the two points $(c - \frac{\varepsilon}{2}, \psi_1(c - \frac{\varepsilon}{2}))$ and $(c + \frac{\varepsilon}{2}, 0)$. Finally, choose a number $x_0 \in \big]c, c + \frac{\varepsilon}{2}\big[$ such that $f(x_0) = \psi_2(x_0)$. Then the function $\varphi_\varepsilon\colon [0,1] \longrightarrow [0,\infty]$ defined by

$$\varphi_\varepsilon(x) = \begin{cases} \psi_1(x) & \text{if } x \in \big[0, c - \frac{\varepsilon}{2}\big], \\ g(x) & \text{if } x \in \big]c - \frac{\varepsilon}{2}, x_0\big], \\ \dfrac{g(x_0)}{\psi_2(x_0)} \cdot \psi_2(x) & \text{if } x \in\,]x_0, 1], \end{cases}$$

clearly is a convex decreasing bijection, and thus generates a strict copula C_ε.

Since C_ε is commutative, without loss of generality we can fix $(x,y) \in [0,1]^2$ such that $x \leq y$ and consider the following cases which, taken together, will prove $\|C - C_\varepsilon\| \leq \varepsilon$.

Case 1: $(x,y) \in \big[0, c - \frac{\varepsilon}{2}\big]^2$ or $(x, C(x,y)) \in \big]c + \frac{\varepsilon}{2}, 1\big]^2$. Here we obviously have $C_\varepsilon(x,y) = C(x,y)$.

Case 2: $(x, y) \in \left[0, c - \frac{\varepsilon}{2}\right] \times \left]c - \frac{\varepsilon}{2}, 1\right]$. In this case we obtain $C(x, y) \in \left]x - \frac{\varepsilon}{2}, x\right]$ and $C_\varepsilon(x, y) \in \left]x - \frac{\varepsilon}{2}, x\right]$ because of the convexity of ψ_1 and φ_ε. Consequently, $|C(x, y) - C_\varepsilon(x, y)| \leq \varepsilon$.

Case 3: $(x, y) \in \left]c - \frac{\varepsilon}{2}, c\right] \times \left]c - \frac{\varepsilon}{2}, 1\right]$. Taking into account (1) and the fact that c is an idempotent element of C, we obtain $|C(x, y) - C_\varepsilon(x, y)| \leq \varepsilon$ as a consequence of the inequalities

$$c - \varepsilon \leq C(c - \tfrac{\varepsilon}{2}, c - \tfrac{\varepsilon}{2}) \leq C(x, y) \leq x,$$
$$c - \varepsilon \leq C_\varepsilon(c - \tfrac{\varepsilon}{2}, c - \tfrac{\varepsilon}{2}) \leq C_\varepsilon(x, y) \leq x.$$

Case 4: $(x, C(x, y)) \in]c, 1] \times \left[0, c + \frac{\varepsilon}{2}\right]$. This implies $C(x, y) \in \left]c, c + \frac{\varepsilon}{2}\right]$ and $C_\varepsilon(x, y) \in \left]c - \frac{\varepsilon}{2}, c + \frac{\varepsilon}{2}\right]$ and, therefore, $|C(x, y) - C_\varepsilon(x, y)| \leq \varepsilon$. ☐

Combining these lemmas and using induction, we obtain the following result:

Theorem 2.6. *The set $\mathcal{C}_{\mathbf{a}}$ of all associative copulas is the closure of both the set $\mathcal{C}_{\mathbf{s}}$ of all strict copulas and the set $\mathcal{C}_{\mathbf{ns}}$ of all non-strict copulas.*

This means, in particular, that each associative copula can be approximated with arbitrary precision by strict as well as by non-strict copulas. Notice that $\mathcal{C}_{\mathbf{s}}$ and $\mathcal{C}_{\mathbf{ns}}$ are disjoint sets whose union, i.e., the set of Archimedean copulas, is a proper subset of $\mathcal{C}_{\mathbf{a}}$.

3. Concluding remarks

In addition to (and as consequences of) the results in this paper the following facts are remarkable:

1. The proof of Theorem 2.6 (which is based on Lemmas 2.2–2.5) is constructive, but for some special cases it is possible to give simpler approximating Archimedean copulas. Consider, e.g., numbers $a_0, a_1, \ldots, a_k \in [0, 1]$ such that $0 = a_0 < a_1 < \cdots < a_k = 1$ and the non-Archimedean associative copula $C = (\langle a_{i-1}, a_i, W\rangle)_{i\in\{1,2,\ldots,k\}}$, which is an ordinal sum of k copies of the Fréchet-Hoeffding lower bound W. Then we obtain a sequence of non-strict copulas $(C_n)_{n\in\mathbb{N}}$ by defining their generators $\varphi_n\colon [0, 1] \longrightarrow [0, \infty]$ as continuous convex functions satisfying $\varphi_n(1) = 0$, which are piecewise linear and whose slopes $c_{i,n}$ on the intervals $[a_{i-1}, a_i]$ satisfy $c_{1,n} \leq c_{2,n} \leq \cdots \leq c_{k,n} < 0$. It is not too difficult to see that the sequence $(C_n)_{n\in\mathbb{N}}$ converges uniformly to C if for all $i, j \in \{1, 2, \ldots, k\}$ with $i > j$ we have

$$\lim_{n\to\infty} \frac{c_{i,n}}{c_{j,n}} = 0.$$

2. Although each continuous t-norm can be approximated uniformly by strict and by nilpotent t-norms [9, 13, 18] (for a detailed proof see [10, Section 8.1]), there are Cauchy sequences of (continuous) t-norms such that the

limit function is not a t-norm [10]. Hence neither the set of all t-norms nor the set of all continuous t-norms is complete (as a normed space).

3. Taking into account the results of [10, Section 8.2] (see also [8, 14]), the convergence of Archimedean copulas is related to the convergence of their corresponding generators as follows: A sequence $(C_n)_{n\in\mathbb{N}}$ of Archimedean copulas with generators $(\varphi_n)_{n\in\mathbb{N}}$ converges to an Archimedean copula C with generator φ if and only if there is a sequence of positive constants $(c_n)_{n\in\mathbb{N}}$ such that for each $x \in]0,1]$ we have $\lim\limits_{n\to\infty} c_n \cdot \varphi_n(x) = \varphi(x)$.

4. Similarly as in Lemma 2.1, each continuous function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is the pointwise limit of a sequence of continuous t-norms is necessarily a t-norm.

5. Given two copulas C and D, consider their $*$-*product* $C * D\colon [0,1]^2 \longrightarrow [0,1]$ introduced in [1] by

$$C * D(x,y) = \int_0^1 \frac{\partial C(x,t)}{\partial t} \cdot \frac{\partial D(t,y)}{\partial t}\, dt.$$

The function $C * D$ is well-defined since the partial derivatives exist almost everywhere, and it is always a copula, i.e., the $*$-product is an operation on the set $\mathcal{C}$ of all copulas. Moreover, $(\mathcal{C}, *)$ is a non-commutative semigroup whose annihilator is the product P and whose neutral element is the minimum M [11].

As a consequence of Theorem 2.6 and [1, Theorem 2.3], for each associative copula C and for each copula D there are sequences of Archimedean and strict and non-strict copulas $(C_n)_{n\in\mathbb{N}}$, respectively, such that the sequences $(C_n)_{n\in\mathbb{N}}$ and $(C_n * D)_{n\in\mathbb{N}}$ converge uniformly to C and $C * D$, respectively.

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