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BOUNDARY CONTINUITY FOR QUASIMINIMIZERS ON METRIC SPACES

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Abstract. A pointwise estimate near a boundary point is obtained for quasiminimizers of the energy integral on a doubling metric measure space admitting a Poincaré inequality. Wiener type conditions sufficient for the (Hölder) continuity of quasiminimizers at a boundary point are also given.

1. Introduction

In $\mathbf{R}^n$, the problem of minimizing the p-energy integral

$$\int_\Omega |\nabla u(x)|^p \, dx,$$

among all functions u with prescribed boundary data is, for $1 < p < \infty$, equivalent to solving the Dirichlet problem for its Euler equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0,$$

and the theory of partial differential equations can be used to obtain qualitative information about the minimizer. For non-differentiable variational integrals, there is no Euler equation and the approach of partial differential equations is not possible. Instead, variational methods such as those based on De Giorgi classes [4] have to be used; see, e.g., Giaquinta–Giusti [8].

In [9], Giaquinta and Giusti introduced the notion of quasiminimizers as a tool for a unified treatment of variational integrals, elliptic equations and systems, obstacle problems and quasiregular mappings. They proved several fundamental properties of quasiminimizers such as local Hölder continuity, the Harnack inequality and the maximum principle. The boundary behaviour

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of quasiminimizers was studied by Ziemer [22], who obtained Wiener type conditions that are sufficient for continuity at a boundary point.

In recent years, several papers have been written on Sobolev spaces and the calculus on metric measure spaces without a differentiable structure; see, e.g., Hajłasz [10], Heinonen–Koskela [13], Cheeger [2], Shanmugalingam [19], [20] and Franchi–Hajłasz–Koskela [5]. In these papers it is shown that first order Sobolev spaces can be defined without the notion of partial derivatives and a version of the p-energy integral can be considered. This theory of Sobolev spaces on abstract metric measure spaces unifies, and has applications in, several areas of analysis, such as weighted Sobolev spaces, calculus on Riemannian manifolds, subelliptic operators associated with vector fields, and potential theory on graphs.

In the setting of metric spaces, the approach with quasiminimizers is particularly useful, as the Euler equation for the p-energy integral need not exist. Local properties of quasiminimizers of the p-energy integral on metric spaces are studied in Kinnunen–Shanmugalingam [14]. In particular, it is shown that if the space is doubling and admits a Poincaré inequality, then quasiminimizers are locally Hölder continuous and satisfy the Harnack inequality and the maximum principle.

The aim of this paper is to obtain a pointwise estimate for quasiminimizers of the p-energy integral on metric measure spaces and to give sufficient conditions for their continuity and Hölder continuity at a boundary point. We show that if the measure μ on the metric space is doubling and the space admits a p_1-Poincaré inequality for some $p_1 < p$, then every quasiminimizer u of the p-energy integral with boundary data w satisfies

$$\begin{aligned}(1)\qquad \operatorname*{osc}_{B(x_0,\rho)} u \le{}& \operatorname*{osc}_{B(x_0,r_0)} w \\ &+ C_1 \operatorname*{osc}_{\Omega} w \exp\biggl(-\frac{1}{8}\int_\rho^{r_0} \exp\biggl(-C_0\biggl(\frac{r^{-p_1}\mu(B(x_0,r))}{\mathrm{C}_{p_1}(B(x_0,r)\setminus\Omega)}\biggr)^{p/(p-p_1)}\biggr)\frac{dr}{r}\biggr),\end{aligned}$$

where C_{p_1} is the p_1-capacity on X; see Theorem 2.11 and (16).

Pointwise capacitary estimates for p-harmonic functions in $\mathbf{R}^n$ were first proved by Maz′ya in [16] (for $p = 2$) and [17] (for $p > 1$), and used to obtain the sufficiency part of the Wiener criterion. Similarly, our estimate (1) implies sufficient conditions for the continuity and Hölder continuity of quasiminimizers at a boundary point; see Theorem 2.12 and the comment following it. In particular, we obtain (Hölder) continuity if the complement of Ω is p_1-fat or has a corkscrew at x_0. Due to the p_1-capacity and the exponential in the integrand, these conditions are more restrictive than the Wiener criteria for solutions of various classes of elliptic equations that have been obtained, for example, in the papers [21], [15], [17], [6], [12], [3], and [1], and which are of the form

$$\int_0^1 \left(\frac{\mathrm{C}_p(B(x_0,r) \setminus \Omega)}{r^{-p}\mu(B(x_0,r))} \right)^{1/(p-1)} \frac{dr}{r} = \infty.$$

For quasiminimizers in $\mathbf{R}^n$, there are two sufficient conditions for continuity at a boundary point, both due to Ziemer [22]. One of them, which more resembles the classical Wiener criterion, can be generalized to our setting. Namely, we prove that there exists Λ such that if

$$\int_0^1 \left(\frac{\mathrm{C}_{p_1}(B(x_0,r) \setminus \Omega)}{r^{-p_1}\mu(B(x_0,r))} \right)^{\Lambda} \frac{dr}{r} = \infty, \tag{2}$$

then every quasiminimizer of the p-energy integral on Ω with continuous boundary data is continuous at the boundary point x_0; see Theorem 2.13 and the remarks following it. At the same time, Ziemer's method, which is used to obtain the sufficient condition (2), does not give any pointwise estimates or conditions for Hölder continuity at a boundary point. From this point of view, our Theorems 2.11 and 2.12 seem to be new, even in the Euclidean setting.

2. Definitions and results

Throughout the paper, $X = (X, d, \mu)$ will be a metric space equipped with a Borel regular measure μ satisfying $0 < \mu(B) < \infty$ for all balls $B = B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ in X with $0 < r < \infty$. Later, we shall impose further restrictions on the space X.

In [13], Heinonen and Koskela introduced upper gradients as a substitute for the modulus of the usual gradient. The advantage of this new notion is that it can easily be generalized to the metric space setting. In order to give an exact definition of upper gradients we first need to introduce some terminology.

A *path* in X is a continuous map γ from an interval $I \subset \mathbf{R}$ to X, or the image $\gamma(I) \subset X$ of such a map. If γ is rectifiable, we automatically assume that it is parameterized by the arclength ds, so that $I = [0, l_\gamma]$, where l_γ is the length of γ.

The *p-modulus* of a family Γ of paths in X is

$$\operatorname{Mod}_p(\Gamma) = \inf_\rho \int_X \rho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions ρ on X such that the path integral $\int_\gamma \rho \, ds$ is ≥ 1 for all locally rectifiable paths in Γ. We say that a property holds on *p-almost every path* if the family of paths for which it does not hold has p-modulus zero.

Definition 2.1. A Borel function g on X is an *upper gradient* of a real-valued function u on X if for all rectifiable paths $\gamma : [0, l_\gamma] \to X$

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \le \int_\gamma g\,ds. \tag{3}$$

If (3) holds only for p-almost every path in X, then g is called a *p-weak upper gradient* of u.

Following Shanmugalingam [19], we define a version of Sobolev type spaces on the metric space X. Note that there are several other definitions of Sobolev type spaces on metric spaces; see, e.g., Hajłasz [10], Heinonen–Koskela [13], Cheeger [2] and Franchi–Hajłasz–Koskela [5]. However, it has been shown, e.g., in Franchi–Hajłasz–Koskela [5] and Shanmugalingam [19], that under some reasonable hypotheses (including Euclidean spaces) most of these definitions lead to the same space. From now on, p will be a fixed number with $1 < p < \infty$.

Definition 2.2. Let

$$\|u\|_{N^{1,p}} = \left(\int_X |u|^p\,d\mu\right)^{1/p} + \inf_g \left(\int_X g^p\,d\mu\right)^{1/p},$$

where the infimum is taken over all upper gradients of u or, equivalently, over all p-weak upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}} = 0$.

The space $N^{1,p}(X)$ equipped with the norm $\|\cdot\|_{N^{1,p}}$ is a Banach space and a lattice; see Shanmugalingam [19]. Corollary 3.7 in [20] shows that every $u \in N^{1,p}$ has a minimal p-weak upper gradient g_u in the sense that $g_u \le g$ holds μ-a.e. for all p-weak upper gradients of u. Note also that the proof of Lemma 1.7 in Cheeger [2] together with Lemma 2.1 in Shanmugalingam [19] shows that if $u, v \in N^{1,p}(X)$, then the function $|u|g_v + |v|g_u$ is a p-weak upper gradient of uv. The following lemma shows that, as long as $u \in N^{1,p}(X)$, the notion of p-weak upper gradient is independent of p.

Lemma 2.3. *If $u \in N^{1,p}(X)$, then for μ-a.e. $x \in X$,*

$$g_u(x) = \inf_g \tilde{g}(x),$$

where the infimum is taken over all upper gradients g of u and

$$\tilde{g}(x) = \limsup_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} g\,d\mu.$$

Proof. The inequality $\tilde{g}_u(x) \le \inf_g \tilde{g}(x)$ is immediate. Conversely, let Γ be the collection of paths on which (3) fails for g_u. By Lemma 2.1 in Shanmugalingam [19] there exists a non-negative Borel function $\rho \in L^p(X,\mu)$ such that $\int_\gamma \rho\, ds = \infty$ for all $\gamma \in \Gamma$. Then the functions $g_\varepsilon = g_u + \varepsilon\rho$ are upper gradients of u, and hence

$$\inf_g \tilde{g}(x) \le \tilde{g}_\varepsilon(x) \le \tilde{g}_u(x) + \varepsilon\tilde{\rho}(x).$$

Since $\tilde{\rho}(x)$ is finite μ-a.e. (because $\rho \in L^p(X,\mu)$), letting $\varepsilon \to 0$ shows that $\inf_g \tilde{g}(x) \le \tilde{g}_u(x)$ for μ-a.e. $x \in X$. The fact that $\tilde{g}_u(x) = g_u(x)$ μ-a.e. finishes the proof. □

Definition 2.4. The *p-capacity* of a set $E \subset X$ is

$$\mathrm{C}_p(E) = \inf_u \|u\|^p_{N^{1,p}},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \ge 1$ on E. We say that a property holds *p-quasieverywhere* (p-q.e.) if the set of points for which it does not hold has p-capacity zero.

The p-capacity is the right gauge for distinguishing between two Newtonian functions. In particular, Corollary 3.3 in Shanmugalingam [19] together with the lattice property of $N^{1,p}(X)$ shows that if $u, v \in N^{1,p}(X)$ and $u \le v$ μ-a.e. then $u \le v$ p-q.e.

In order to give a definition of quasiminimizers of the energy integral on a set $E \subset X$ we need a Newtonian space with zero boundary values. Let

$$N^{1,p}_0(E) = \{u \in N^{1,p}(X) : u = 0 \ p\text{-q.e. on } X \setminus E\}.$$

Corollary 3.9 in [19] implies that $N^{1,p}_0(E)$ equipped with the norm $\|\cdot\|_{N^{1,p}}$ is a closed subspace of $N^{1,p}(X)$. Note also that if $\mathrm{C}_p(X \setminus E) = 0$, then $N^{1,p}_0(E) = N^{1,p}(X)$. We shall therefore always assume that $\mathrm{C}_p(X \setminus E) > 0$.

Definition 2.5. Let $w \in N^{1,p}(X)$. We say that u is a *quasiminimizer of the p-energy integral* on Ω with boundary data w if $u - w \in N^{1,p}_0(\Omega)$ and there exists a constant $Q > 0$ such that for all $v \in N^{1,p}_0(\Omega)$

$$\int_{v \ne 0} g_u^p\, d\mu \le Q \int_{v \ne 0} g_{u+v}^p\, d\mu.$$

To guarantee good properties of the space $N^{1,p}$ and the quasiminimizers, we shall impose two additional conditions on the space X.

Definition 2.6. If $\lambda > 0$ and $B = B(x_0, r)$ is a ball, we let λB denote the ball $B(x_0, \lambda r)$. We say that the measure μ (or the space X) is *doubling*, if there exists $C > 0$ such that

$$\mu(2B) < C\mu(B)$$

for all balls B in X.

DEFINITION 2.7. We say that the space X admits a *weak (q,p)-Poincaré inequality* if there exist $C>0$ and $\lambda \ge 1$ such that

$$\left(\fint_B |u-u_B|^q \, d\mu\right)^{1/q} \le Cr\left(\fint_{\lambda B} g^p \, d\mu\right)^{1/p} \tag{4}$$

holds for all balls $B=B(x_0,r)$ in X and all pairs (u,g), where u is an integrable function on X and g is an upper gradient of u. Here and in what follows, we use the notation

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu.$$

By the Hölder inequality, a weak (q,p)-Poincaré inequality implies weak (q',p')-Poincaré inequalities with the same λ for all $q' \le q$ and $p' \ge p$. On the other hand, by Theorem 5.1 in Hajłasz–Koskela [11], a weak $(1,p)$-Poincaré inequality implies a weak (q,p)-Poincaré inequality for some $q>p$ and possibly a new λ. The argument in the proof of Lemma 2.3 shows that if the Poincaré inequality (4) holds for all upper gradients, then it holds for all p-weak upper gradients as well.

It is shown in Shanmugalingam [20] that the p-energy functional $v \mapsto \int_X g_{w+v}^p \, d\mu$, where $w \in N^{1,p}(X)$ is fixed and $v \in N_0^{1,p}(\Omega)$, is convex and lower semicontinuous. If X is doubling and admits a $(1,p)$-Poincaré inequality and we also assume that Ω is bounded and $\mathrm{C}_p(X \setminus \Omega) > 0$, then the p-energy functional is coercive on $N_0^{1,p}(\Omega)$. Indeed, let $v_n \in N_0^{1,p}(\Omega)$ satisfy $\|v_n\|_{N^{1,p}} \to \infty$, as $n \to \infty$. Then $g_{v_n} \le g_{w+v_n} + g_w$ and Proposition 3.2 below applied to a sufficiently large ball B together with Lemma 3.3 below implies

$$\begin{aligned}\int_X g_{w+v_n}^p d\mu &\ge \int_X g_{v_n}^p d\mu - \|w\|_{N^{1,p}}^p \\ &\ge \left(1+\frac{C\mu(B)}{\mathrm{C}_p(B\setminus\Omega)}\right)^{-1} \|v_n\|_{N^{1,p}}^p - \|w\|_{N^{1,p}}^p \to \infty,\end{aligned}$$

where $\mathrm{C}_p(B \setminus \Omega) > 0$ and C depends only on p, X and B. In Shanmugalingam [20], the coercivity of the energy functional was proved under slightly stronger assumptions. Standard arguments from functional analysis (using the local sequential weak compactness of $N_0^{1,p}(\Omega)$ and the Mazur lemma) then yield the existence of a minimizer of the p-energy integral on Ω (i.e., a function satisfying Definition 2.5 with $Q=1$); see Shanmugalingam [20].

In Kinnunen–Shanmugalingam [14], it is shown that if X is doubling and admits a weak $(1,p_1)$-Poincaré inequality for some $p_1 < p$, then quasiminimizers of the p-energy integral are locally Hölder continuous in Ω and satisfy the Harnack inequality and the maximum principle.

EXAMPLE 2.8. Using Lemma 3.2 in Shanmugalingam [20] it is not difficult to verify that every minimizer u of the integral functional $u \mapsto \int_\Omega F(g_u)\, d\mu$ with a nondecreasing function F satisfying $Ct^p \le F(t) \le C' t^p$ is a quasiminimizer of the p-energy integral.

EXAMPLE 2.9. It is shown in Cheeger [2] that for a doubling space X with a $(1,p)$-Poincaré inequality it is possible to define a "gradient" Du of $u \in N^{1,p}(X)$, which has all the properties of the usual gradient ∇u in $\mathbf{R}^n$. Moreover, for μ-a.e. $x \in X$, $|Du|$ is comparable to g_u. It is then easily verified that every minimizer u of the integral functional $u \mapsto \int_\Omega F(Du)\, d\mu$ with $C|\xi|^p \le F(\xi) \le C'|\xi|^p$ is a quasiminimizer of the p-energy integral. Note that we do not require any regularity or monotonicity of F.

Similarly, if $A(\xi)$ is a vector-valued function such that $A(\xi)\cdot\xi \ge C|\xi|^p$ and $|A(\xi)| \le C'|\xi|^{p-1}$, then every weak solution of the differential equation

$$\int_\Omega A(Du)\cdot D\varphi\, d\mu = 0 \quad \text{for all } \varphi \in N_0^{1,p}(\Omega)$$

is a quasiminimizer of the p-energy integral. Again, the function A need not satisfy any regularity or monotonicity conditions.

Next, we introduce a new capacity on X which will appear in the pointwise estimate for quasiminimizers; cf. Section 6.16 in Heinonen–Kilpeläinen–Martio [12]. Lemma 3.3 below shows that the capacities cap_p and C_p are essentially equivalent.

DEFINITION 2.10. Let $B \subset X$ be a ball and $E \subset B$. We define the capacity

$$\mathrm{cap}_p(E, 2B) = \inf_u \int_{2B} g_u^p\, d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(2B)$ such that $u \ge 1$ on E.

We are now ready to state the main results of this paper. We shall use the notation

$$M(r, r_0) = \Bigl(\operatorname*{ess\,sup}_{B(x_0,r)} u - \operatorname*{ess\,sup}_{B(x_0,r_0)} w\Bigr)_+,$$

where $0 < r \le r_0$, $a_+ = \max\{a, 0\}$, and $u \in N^{1,p}(X)$ is a quasiminimizer of the p-energy integral on Ω with the boundary data $w \in N^{1,p}(X)$. Let also

$$\gamma(p_1, r) = \frac{r^{-p_1}\mu(B(x_0,r))}{\mathrm{cap}_{p_1}(B(x_0,r)\setminus\Omega, B(x_0,2r))}.$$

Our first result is a pointwise estimate for quasiminimizers of the p-energy integral near a boundary point; cf. Maz′ya [16] and [17].

THEOREM 2.11. *Let X be a doubling metric measure space admitting a weak $(1,p_1)$-Poincaré inequality for some $p_1 < p$. Then there exist $C_0, C_1 > 0$ such that if $u \in N^{1,p}(X)$ is a quasiminimizer of the p-energy integral on Ω with the boundary data $w \in N^{1,p}(X)$, then*

$$M(\rho, r_0) \le C_1 M(r_0, r_0) \exp\biggl(-\frac{1}{8}\int_\rho^{r_0} \exp\bigl(-C_0\gamma(p_1,r)^{p/(p-p_1)}\bigr)\,\frac{dr}{r}\biggr).$$

As a consequence of Theorem 2.11 we obtain the following sufficient condition for the Hölder continuity of quasiminimizers of the p-energy integral at a boundary point.

THEOREM 2.12. *Let X be a doubling metric measure space admitting a weak $(1,p_1)$-Poincaré inequality for some $p_1 < p$. Then there exists $C_0 > 0$ such that if $w \in N^{1,p}(X)$ is Hölder continuous at $x_0 \in \partial\Omega$,*

$$\liminf_{\rho\to 0} \frac{1}{|\log\rho|}\int_\rho^1 \exp\bigl(-C_0\gamma(p_1,r)^{p/(p-p_1)}\bigr)\,\frac{dr}{r} > 0,$$

and $u \in N^{1,p}(X)$ is a quasiminimizer of the p-energy integral on Ω with the boundary data w, then u is Hölder continuous at x_0.

A sufficient condition for the continuity of quasiminimizers of the p-energy integral at a boundary point can also be obtained from the estimate in Theorem 2.11. However, this condition is more restrictive than the following condition which is obtained directly by a different method; cf. Ziemer [22].

THEOREM 2.13. *Let X be a doubling metric measure space admitting a weak $(1,p_1)$-Poincaré inequality for some $p_1 < p$. Then there exists $\Lambda > 0$ such that if $w \in N^{1,p}(X)$ is continuous at $x_0 \in \partial\Omega$,*

$$\int_0^1 \biggl(\frac{\operatorname{cap}_{p_1}(B(x_0,r)\setminus\Omega, B(x_0,2r))}{r^{-p_1}\mu(B(x_0,r))}\biggr)^\Lambda \frac{dr}{r} = \infty,$$

and $u \in N^{1,p}(X)$ is a quasiminimizer of the p-energy integral on Ω with the boundary data w, then u is continuous at x_0.

REMARK 2.14. The proof shows that Theorem 2.13 is true for all Λ satisfying

$$\Lambda > \frac{1}{p_1} + \frac{p}{\sigma(p-p_1)},$$

where σ is the exponent from the weak Harnack inequality (Theorem 4.6).

REMARK 2.15. By Lemma 3.3 below, $\gamma(p_1,r)$ is for sufficiently small r comparable to

$$\frac{\operatorname{cap}_{p_1}(B(x_0,r), B(x_0,2r))}{\operatorname{cap}_{p_1}(B(x_0,r)\setminus\Omega, B(x_0,2r))}, \tag{5}$$

which is the reciprocal of the relative capacity of $B(x_0,r)\setminus\Omega$ appearing in many Wiener criteria. Lemma 3.3 also shows that for small r

$$\gamma(p_1,r)\le\frac{Cr^{-p_1}\mu(B(x_0,r))}{\mathrm{C}_{p_1}(B(x_0,r)\setminus\Omega)}\quad\text{and}\quad\gamma(p_1,r)\le\frac{C\mu(B(x_0,r))}{\mu(B(x_0,r)\setminus\Omega)},\tag{6}$$

so that Theorems 2.11 and 2.13 yield (1) and (2) in the introduction.

Finally, (5) and (6) show that the conditions in Theorems 2.12 and 2.13 are satisfied, e.g., if one of the following conditions holds for some $C>0$ and all sufficiently small r:

(i) The complement of Ω has a corkscrew at x_0, i.e., the set $B(x_0,r)\setminus\Omega$ contains a ball with radius Cr, or, more generally,

$$\mu(B(x_0,r)\setminus\Omega)\ge C\mu(B(x_0,r)).$$

(ii) The complement of Ω is p_1-fat at x_0, i.e.,

$$\operatorname{cap}_{p_1}(B(x_0,r)\setminus\Omega,B(x_0,2r))\ge C\operatorname{cap}_{p_1}(B(x_0,r),B(x_0,2r)).$$

Note also that by Theorem 1.2 in [1] every set which is uniformly p-fat (i.e., p-fat at all its boundary points) is also uniformly p_1-fat for some $p_1<p$.

Example 2.16. Let X be a doubling metric measure space admitting a weak $(1,p_1)$-Poincaré inequality for some $p_1<p$. Assume moreover that X is geodesic and that all geodesic curves are "open" in the sense that they do not contain the first and the last point. Then every ball $B=B(z_0,r)$ in X is regular, i.e., every quasiminimizer of the p-energy integral on B with continuous boundary data is continuous up to the boundary. Indeed, let $x_0\in\partial B$ and let γ be a geodesic curve connecting z_0 and x_0. By the assumptions, x_0 cannot be the last point of γ. Hence for all sufficiently small $\rho>0$ there exists $x\in\gamma$ such that $d(x_0,x)=\rho$ and $d(z_0,x)=r+\rho$. It follows that $B(x,\rho)\subset B(x_0,2\rho)\setminus B$, i.e., the complement of B has a corkscrew at x_0.

3. Sobolev type inequalities and capacity

In this section we will prove two Sobolev type inequalities for Newtonian functions. We start by mentioning some properties of Newtonian functions which will be needed later. By χ_A we denote the characteristic function of a set A.

By Proposition 3.1 in Shanmugalingam [19], every $u\in N^{1,p}(X)$ is *absolutely continuous on p-almost every curve*, i.e., the function $u\circ\gamma:I\to\mathbf{R}$ is absolutely continuous on I for p-almost all rectifiable paths γ in X. A set $V\subset X$ is *p-path open* if for p-almost every path γ in X the set $\gamma^{-1}(V)$ is open in $\mathbf{R}$. Note that if u is absolutely continuous on p-almost every curve, then the set $\{x\in X:u(x)>k\}$ is p-path open for all k.

Let u, u_1 and u_2 be absolutely continuous on p-almost every curve in X, $u=u_1$ on V and $u=u_2$ on $X\setminus V$ for some p-path open set $V\subset X$. If g_1 and g_2 are p-weak upper gradients of u_1 and u_2, respectively, then

$g_1 + g_2\chi_{X\setminus V}$ and $g_1\chi_V + g_2$ are p-weak upper gradients of u; see Lemma 3.2 in Shanmugalingam [20]. In particular, if $u_2 = 0$, then $g_1\chi_V$ is a p-weak upper gradient of u.

Unless otherwise stated, C denotes a positive constant whose exact value is unimportant and depends only on the fixed parameters, such as X, d, μ and p. The following proposition is a generalization of the classical Sobolev inequality.

Proposition 3.1. *Let X be a doubling metric measure space admitting a weak (q,p)-Poincaré inequality. Then there exists $C > 0$ such that if $B = B(x_0, r)$ is a ball in X, $0 < r < (1/3)\operatorname{diam} X$ and $u \in N_0^{1,p}(B)$, then*

$$\biggl(\fint_B |u|^q \, d\mu\biggr)^{1/q} \le Cr\biggl(\fint_B g_u^p \, d\mu\biggr)^{1/p}.$$

Proof. The arguments are as in the proof of Lemma 2.8 in Kinnunen–Shanmugalingam [14]; cf. Theorem 13.1 in Hajłasz–Koskela [11]. Let $u_{2B} = \fint_{2B} u \, d\mu$. Then

$$\biggl(\fint_{2B} |u|^q \, d\mu\biggr)^{1/q} \le \biggl(\fint_{2B} |u - u_{2B}|^q \, d\mu\biggr)^{1/q} + |u_{2B}|.$$

By the Hölder inequality and the fact that u vanishes on $2B \setminus B$, we have

$$|u_{2B}| \le \fint_{2B} |u|\chi_B \, d\mu \le \biggl(\fint_{2B} |u|^q \, d\mu\biggr)^{1/q} \biggl(\frac{\mu(B)}{\mu(2B)}\biggr)^{1-1/q}.$$

As $r < (1/3)\operatorname{diam} X$, the set $X \setminus (3/2)B$ is nonempty and the Poincaré inequality implies that there exists a point x' on the sphere $\bigl\{x \in X : d(x, x_0) = 3r/2\bigr\}$. Then the set $2B \setminus B$ contains the ball $B' = B\bigl(x', r/2\bigr)$, $2B \subset 7B'$, and the doubling property of μ implies

$$\frac{\mu(B)}{\mu(2B)} \le 1 - \frac{\mu(B')}{\mu(2B)} \le \gamma < 1.$$

The last three estimates and the weak (q,p)-Poincaré inequality now give

$$\bigl(1-\gamma^{1-1/q}\bigr)\biggl(\fint_{2B} |u|^q \, d\mu\biggr)^{1/q} \le \biggl(\fint_{2B} |u-u_{2B}|^q \, d\mu\biggr)^{1/q} \le Cr\biggl(\fint_{2B} g_u^p \, d\mu\biggr)^{1/p}.$$

As u and g_u vanish outside B, the ball $2B$ in the above integrals can be replaced by B. $\square$

The following inequality was first proved by Maz'ya in the Euclidean setting; see, e.g., [18]. The original proof goes through in the metric space setting and is given here for the reader's convenience.

PROPOSITION 3.2. *Let X be a doubling metric measure space admitting a weak $(1,p)$-Poincaré inequality. Then there exist $C>0$ and $\lambda \ge 1$ such that if B is a ball in X, $u \in N^{1,p}(X)$ and $S = \bigl\{x \in (1/2)B : u(x)=0\bigr\}$, then*

$$\fint_B |u|^p \, d\mu \le \frac{C}{\operatorname{cap}_p(S,B)} \int_{\lambda B} g_u^p \, d\mu.$$

Proof. By splitting u into its positive and negative parts and considering them separately, we can assume that $u \ge 0$ in B. Let

$$\bar{u} = \biggl(\fint_B u^p \, d\mu \biggr)^{1/p}.$$

Denote the radius of B by r and let η be a $2/r$-Lipschitz function (i.e., a Lipschitz continuous function with Lipschitz constant $2/r$) vanishing outside B such that $0 \le \eta \le 1$ and $\eta = 1$ on $(1/2)B$. Then the function $v = \eta(1-u/\bar{u})$ is admissible in the definition of $\operatorname{cap}_p(S,B)$ and $g_v \le (1-u/\bar{u})g_\eta + g_u/\bar{u}$. Hence

$$\operatorname{cap}_p(S,B) \le \int_B g_v^p \, d\mu \le \frac{2^{2p-1}}{r^p \bar{u}^p} \int_B |u-\bar{u}|^p \, d\mu + \frac{2^{p-1}}{\bar{u}^p} \int_B g_u^p \, d\mu.$$

Next,

$$\biggl(\int_B |u-\bar{u}|^p \, d\mu \biggr)^{1/p} \le \biggl(\int_B |u-u_B|^p \, d\mu \biggr)^{1/p} + |\bar{u}-u_B| \, \mu(B)^{1/p}.$$

By Theorem 5.1 in Hajłasz–Koskela [11], X admits a weak (p,p)-Poincaré inequality with some λ, which can be used to estimate the right-hand side of the last inequality. The first term is estimated directly and for the second term we have

$$\begin{aligned} |\bar{u}-u_B| \, \mu(B)^{1/p} &= \bigl| \|u\|_{L^p(B)} - \|u_B\|_{L^p(B)} \bigr| \\ &\le \|u-u_B\|_{L^p(B)} \le Cr \biggl(\int_{\lambda B} g_u^p \, d\mu \biggr)^{1/p}. \end{aligned}$$

The last three inequalities now give

$$\operatorname{cap}_p(S,B) \le \frac{C}{\bar{u}^p} \int_{\lambda B} g_u^p \, d\mu,$$

and the proposition follows. □

The following lemma shows that the capacities cap_p and C_p are essentially equivalent and gives a precise estimate for the capacity of a ball.

LEMMA 3.3. *Let X be a doubling metric measure space admitting a weak $(1,p)$-Poincaré inequality and let $E \subset B = B(x_0,r)$ with $0 < r < (1/6)\operatorname{diam} X$. Then there exists $C>0$ such that*

$$\frac{\mu(E)}{Cr^p} \le \operatorname{cap}_p(E,2B) \le \frac{C\mu(B)}{r^p}$$

and

$$\frac{\mathrm{C}_p(E)}{C(1+r^p)} \le \mathrm{cap}_p(E,2B) \le 2^{p-1}\left(1+\frac{1}{r^p}\right)\mathrm{C}_p(E).$$

In particular, for bounded sets E, $\mathrm{C}_p(E)=0$ if and only if $\mathrm{cap}_p(E,2B)=0$ for some ball B containing E.

Proof. If v is admissible in the definition of $\mathrm{cap}_p(E,2B)$, then by Proposition 3.1 with $q=p$,

$$\mu(E) \le \int_{2B} |v|^p\,d\mu \le Cr^p \int_{2B} g_v^p\,d\mu,$$

and

$$\mathrm{C}_p(E) \le \int_X |v|^p\,d\mu + \int_X g_v^p\,d\mu \le C(1+r^p)\int_{2B} g_v^p\,d\mu.$$

Taking infimum over all admissible v yields the left inequalities in the lemma.

Conversely, let η be a $1/r$-Lipschitz function vanishing outside $2B$ such that $0\le\eta\le 1$ and $\eta=1$ on B. If $u\in N^{1,p}(X)$ is admissible in the definition of $\mathrm{C}_p(E)$, then η and $u\eta$ are admissible in the definition of $\mathrm{cap}_p(E,2B)$, and hence

$$\mathrm{cap}_p(E,2B) \le \int_{2B} g_\eta^p\,d\mu \le \frac{\mu(2B)}{r^p} \le \frac{C\mu(B)}{r^p},$$

and

$$\begin{aligned}\mathrm{cap}_p(E,2B) &\le \int_{2B} g_{u\eta}^p\,d\mu \le \int_{2B} (|u|g_\eta+\eta g_u)^p\,d\mu \\ &\le \frac{2^{p-1}}{r^p}\int_X |u|^p\,d\mu + 2^{p-1}\int_X g_u^p\,d\mu.\end{aligned}$$

Taking infimum over all $u\in N^{1,p}(X)$ finishes the proof. □

4. Proofs of the results

We shall first show that quasiminimizers satisfy a Caccioppoli type estimate on level sets, as in the De Giorgi classes; cf. Definition 4.5 below. These estimates will be used to obtain pointwise estimates for quasiminimizers in terms of their L^p-norm and then iterated to prove their (Hölder) continuity. In this section, the general constant C is allowed to also depend on the constant Q in the definition of quasiminimizers.

Proposition 4.1. *Let $u\in N^{1,p}(X)$ be a quasiminimizer of the p-energy integral on Ω with the boundary data $w\in N^{1,p}(X)$. Let $x_0\in X$ and $A(k,r)=\{x\in B(x_0,r): u(x)>k\}$. Then for $0<r_1<r_2$ and $k\ge \operatorname{ess\,sup}_{B(x_0,r_2)} w$ we have*

$$\int_{A(k,r_1)} g_u^p\,d\mu \le \frac{C}{(r_2-r_1)^p}\int_{B(x_0,r_2)} (u-k)_+^p\,d\mu. \tag{7}$$

Proof. Let η be a $1/(r_2-r_1)$-Lipschitz function vanishing outside $B(x_0,r_2)$, such that $0 \le \eta \le 1$ and $\eta = 1$ on $B(x_0,r_1)$. Let $v = -\eta(u-k)_+$. Then, as $u = w \le k$ p-q.e. on $B(x_0,r_2) \setminus \Omega$ and $\eta = 0$ outside $B(x_0,r_2)$, we have $v \in N_0^{1,p}(\Omega)$. Also, $u+v = (1-\eta)(u-k)_+ + k$ on $A(k,r_2)$ and $u+v = u$ outside $A(k,r_2)$. The set $A(k,r_2)$ is p-path open, and hence the function $(1-\eta)g_u + (u-k)_+ g_\eta + g_u \chi_{X \setminus A(k,r_2)}$ is a p-weak upper gradient of $u+v$. It then follows from the quasiminimizing property of u that

$$\begin{aligned}\int_{A(k,r_1)} g_u^p \, d\mu \le \int_{v\ne 0} g_u^p \, d\mu &\le Q \int_{v \ne 0} g_{u+v}^p \, d\mu \\ &\le Q \int_{A(k,r_2)} \bigl((1-\eta)g_u + (u-k)_+ g_\eta\bigr)^p \, d\mu \\ &\le C \int_{A(k,r_2)\setminus A(k,r_1)} g_u^p \, d\mu + \frac{C}{(r_2-r_1)^p} \int_{B(x_0,r_2)} (u-k)_+^p \, d\mu,\end{aligned}$$

where $C = 2^{p-1}Q$. Adding C times the left-hand side to both sides of the inequality yields

$$\int_{A(k,r_1)} g_u^p \, d\mu \le \theta \int_{A(k,r_2)} g_u^p \, d\mu + \frac{\theta}{(r_2-r_1)^p} \int_{B(x_0,r_2)} (u-k)_+^p \, d\mu,$$

where $\theta = C/(C+1) < 1$. The following lemma now finishes the proof. □

LEMMA 4.2. *Let $f(r)$ be a nonnegative function defined on $[R_1,R_2]$, where $R_1 \ge 0$. Suppose that for all $R_1 \le r_1 < r_2 \le R_2$,*

$$f(r_1) \le \theta f(r_2) + \frac{A}{(r_2-r_1)^\alpha} + B,$$

where $A, B \ge 0$, $\alpha > 0$ and $0 \le \theta < 1$. Then there exists $C > 0$ depending only on α and θ such that for all $R_1 \le r_1 < r_2 \le R_2$,

$$f(r_1) \le C\biggl(\frac{A}{(r_2-r_1)^\alpha} + B\biggr).$$

Proof. See, e.g., Lemma 3.1 in Chapter V in Giaquinta [7]. □

THEOREM 4.3. *Let X be a doubling metric measure space admitting a weak $(1,p)$-Poincaré inequality. Then there exists $C > 0$ such that if $u \in N^{1,p}(X)$ and the condition* (7) *holds for all $k \ge k^*$ and $0 < r_1 < r_2 \le R < (1/3)\operatorname{diam} X$, then for all $k_0 \ge k^*$*

$$\operatorname*{ess\,sup}_{B(x_0,R/2)} u \le k_0 + C\biggl(\fint_{B(x_0,R)} (u-k_0)_+^p \, d\mu\biggr)^{1/p}.$$

Proof. Let x_0 be fixed and write $B(r) = B(x_0,r)$. Let $0 < r/2 \le \rho < r \le R$, $r_1 = (\rho+r)/2$, and let η be a $2/(r-\rho)$-Lipschitz function vanishing outside $B(r_1)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on $B(\rho)$. Let $k > l \ge k^*$ and $v = \eta(u-k)_+$. Then

$$\int_{B(r)} (u-l)_+^p \, d\mu \ge \int_{A(k,\rho)} (u-l)^p \, d\mu \ge (k-l)^p \mu(A(k,\rho))$$

and the Hölder inequality yields for $q > p$

$$\begin{aligned} \int_{B(\rho)} (u-k)_+^p \, d\mu &\le \biggl(\int_{B(\rho)} (u-k)_+^q \, d\mu \biggr)^{p/q} \mu(A(k,\rho))^{1-p/q} \\ &\le \biggl(\int_{B(r_1)} v^q \, d\mu \biggr)^{p/q} \biggl(\frac{1}{(k-l)^p} \int_{B(r)} (u-l)_+^p \, d\mu \biggr)^{1-p/q}. \end{aligned} \tag{8}$$

At the same time, as $A(k,r_1)$ is p-path open, the function

$$g = (\eta g_u + (u-k) g_\eta) \chi_{A(k,r_1)}$$

is a p-weak upper gradient of v and $g_v \le g$. By Theorem 5.1 in Hajłasz–Koskela [11], X admits a weak (q,p)-Poincaré inequality for some $q > p$. Hence, by Proposition 3.1 and the assumption (7),

$$\begin{aligned} &\biggl(\int_{B(r_1)} v^q \, d\mu \biggr)^{p/q} \\ &\quad \le \frac{C r_1^p}{\mu(B(r_1))^{1-p/q}} \int_{B(r_1)} g_v^p \, d\mu \\ &\quad \le \frac{C r_1^p}{\mu(B(r_1))^{1-p/q}} \biggl(\int_{A(k,r_1)} g_u^p \, d\mu + \frac{2^p}{(r-\rho)^p} \int_{B(r_1)} (u-k)_+^p \, d\mu \biggr) \\ &\quad \le \frac{C r_1^p}{(r-\rho)^p \mu(B(r_1))^{1-p/q}} \int_{B(r)} (u-k)_+^p \, d\mu. \end{aligned} \tag{9}$$

The estimates (8) and (9) then yield

$$\begin{aligned} \int_{B(\rho)} (u-k)_+^p \, d\mu &\le \frac{C r_1^p}{(r-\rho)^p \mu(B(r_1))^{1-p/q}} \int_{B(r)} (u-l)_+^p \, d\mu \\ &\qquad \times \biggl(\frac{1}{(k-l)^p} \int_{B(r)} (u-l)_+^p \, d\mu \biggr)^{1-p/q}. \end{aligned}$$

Note that, by the doubling property of μ, the measures $\mu(B(\rho))$, $\mu(B(r_1))$ and $\mu(B(r))$ are comparable. Let $\xi = 1 - p/q > 0$ and

$$u(k,\rho) = \biggl(\fint_{B(\rho)} (u-k)_+^p \, d\mu \biggr)^{1/p}.$$

Then the last inequality can be written as

$$u(k,\rho) \le \frac{Cr}{(r-\rho)(k-l)^\xi}\, u(l,r)^{1+\xi}. \tag{10}$$

For $n = 0, 1, \dots$, let $\rho_n = R(1/2+2^{-n-1}) \le R$ and $k_n = k_0 + d(1-2^{-n}) \ge k^*$, where $d > 0$ will be chosen later. Then $\rho_0 = R$, $\rho_n \searrow R/2$ and $k_n \nearrow k_0 + d$. We now show by induction that, with a suitable d,

$$u\bigl(k_0+d, \tfrac{1}{2}R\bigr) \le u(k_n,\rho_n) \le 2^{-\mu n} u(k_0,R) \to 0, \tag{11}$$

as $n \to \infty$, where $\mu = (1+\xi)/\xi$. Indeed, (11) is trivially true for $n = 0$ and assuming (11) for $n \ge 0$, we have by (10) with $r = \rho_n$, $\rho = \rho_{n+1}$, $k = k_{n+1}$ and $l = k_n$,

$$\begin{aligned} u(k_{n+1},\rho_{n+1}) &\le \frac{CR}{(\rho_n-\rho_{n+1})(k_{n+1}-k_n)^\xi}\, u(k_n,\rho_n)^{1+\xi} \\ &\le \frac{2^{n+2}C}{(2^{-n-1}d)^\xi}\, 2^{-\mu n(1+\xi)} u(k_0,R)^{1+\xi} = 2^{-\mu(n+1)} u(k_0,R), \end{aligned}$$

provided that $d = (2^{2+\xi+\mu}C)^{1/\xi} u(k_0,R)$, where C is the same constant as in the last inequality. □

PROPOSITION 4.4. *Let X be a doubling metric measure space admitting a weak $(1,p_1)$-Poincaré inequality for some $p_1 < p$. Then there exist $C > 0$ and $\lambda \ge 1$ such that if $u \in N^{1,p}(X)$ is a quasiminimizer of the p-energy integral on Ω with the boundary data $w \in N^{1,p}(X)$, then for all $x_0 \in \partial\Omega$ and $0 < 2\lambda r \le r_0 < (1/3)\operatorname{diam} X$,*

$$M\bigl(\tfrac{1}{2}r, r_0\bigr) \le (1-2^{-n(r)-2}) M(2\lambda r, r_0),$$

where

$$n(r) = C\gamma\bigl(p_1, \tfrac{1}{2}r\bigr)^{p/(p-p_1)}.$$

Proof. With x_0, r and r_0 fixed, write $M = M(2\lambda r, r_0)$ and $B = B(x_0, r)$. If $M = 0$ or $M = \infty$, there is nothing to prove. Assume that $0 < M < \infty$ and define

$$\begin{aligned} k_j &= \operatorname*{ess\,sup}_{B(x_0,r_0)} w + M(1-2^{-j}), \\ v_j &= (u-k_j)_+ - (u-k_{j+1})_+. \end{aligned}$$

Let λ be as in Proposition 3.2. Then $v_j = 0$ on $\lambda B \setminus \Omega$ and $g_u \chi_{T(k_j,k_{j+1},\lambda r)}$, with $T(k,l,r) = A(k,r) \setminus A(l,r)$, is a p-weak upper gradient of v_j in λB. Proposition 3.2 with p replaced by p_1, the doubling property of μ and the

Hölder inequality then imply

$$\begin{aligned}\int_B v_j^{p_1}\,d\mu &\le \frac{C\mu(B)}{\operatorname{cap}_{p_1}\bigl(\frac{1}{2}B\setminus\Omega,B\bigr)}\int_{\lambda B} g_{v_j}^{p_1}\,d\mu\\ &\le Cr^{p_1}\gamma\bigl(p_1,\tfrac{1}{2}r\bigr)\biggl(\int_{A(k_j,\lambda r)} g_u^p\,d\mu\biggr)^{p_1/p}\mu(T(k_j,k_{j+1},\lambda r))^{1-p_1/p}.\end{aligned}$$

Next, we estimate $\mu(A(k_{j+1},r))$ as follows:

$$\int_B v_j^{p_1}\,d\mu \ge (k_{j+1}-k_j)^{p_1}\mu(A(k_{j+1},r)) = \frac{M^{p_1}}{2^{p_1(j+1)}}\mu(A(k_{j+1},r)).$$

Proposition 4.1 and the doubling property of μ yield

$$\begin{aligned}\biggl(\int_{A(k_j,\lambda r)} g_u^p\,d\mu\biggr)^{p_1/p} &\le \frac{C}{r^{p_1}}\biggl(\int_{2\lambda B}(u-k_j)_+^p\,d\mu\biggr)^{p_1/p}\\ &\le \frac{C}{r^{p_1}}\Bigl(\operatorname*{ess\,sup}_{2\lambda B} u-k_j\Bigr)^{p_1}\mu(2\lambda B)^{p_1/p} \le \frac{CM^{p_1}}{2^{p_1 j}r^{p_1}}\mu(B)^{p_1/p},\end{aligned}$$

and putting together the last three inequalities, we get

$$\frac{\mu(A(k_{j+1},r))}{\mu(B)} \le C\gamma\bigl(p_1,\tfrac{1}{2}r\bigr)\biggl(\frac{\mu(T(k_j,k_{j+1},\lambda r))}{\mu(B)}\biggr)^{1-p_1/p}.$$

If $n\ge j+1$, then $A(k_{j+1},r)$ on the left-hand side can be replaced by $A(k_n,r)$, and the inequality remains true. We get

$$\biggl(\frac{\mu(A(k_n,r))}{\mu(B)}\biggr)^{p/(p-p_1)} \le C\gamma\bigl(p_1,\tfrac{1}{2}r\bigr)^{p/(p-p_1)}\frac{\mu(T(k_j,k_{j+1},\lambda r))}{\mu(B)},$$

and summing up over $j=0,1,\dots,n-1$, yields

$$\biggl(\frac{\mu(A(k_n,r))}{\mu(B)}\biggr)^{p/(p-p_1)} \le \frac{C\gamma\bigl(p_1,\frac{1}{2}r\bigr)^{p/(p-p_1)}}{n}. \tag{12}$$

Theorem 4.3 with k_0 and R replaced by k_n and r and the fact that $u-k_n\le 2^{-n}M$ on B give

$$\begin{aligned}\operatorname*{ess\,sup}_{B(x_0,r/2)} u &\le k_n + C\biggl(\fint_B (u-k_n)_+^p\,d\mu\biggr)^{1/p}\\ &\le \operatorname*{ess\,sup}_{B(x_0,r_0)} w + M(1-2^{-n}) + \frac{CM}{2^n}\biggl(\frac{\mu(A(k_n,r))}{\mu(B)}\biggr)^{1/p}.\end{aligned} \tag{13}$$

Using the estimate (12) we see that the last term on the right-hand side in (13) is at most $2^{-n-1}M$, whenever $n\ge n(r)=C\gamma\bigl(p_1,\frac{1}{2}r\bigr)^{p/(p-p_1)}$. Inserting the smallest integer $n\ge n(r)$ into (13) finishes the proof. $\square$

Proof of Theorem 2.11. With r_0 fixed, write $M(r) = M(r, r_0)$. We can assume that $0 < M(r_0) < \infty$. Let C and $n(r)$ be as in Proposition 4.4, $C_0 = C \log 2$, and

$$\omega(r) = \exp\bigl(-C_0 \gamma(p_1, r)^{p/(p-p_1)}\bigr) = 2^{-n(2r)}.$$

For $m = 1, 2$, we divide the interval $(0, r_0)$ into two disjoint subsets as follows:

$$I_m = \bigcup_{j=1}^{\infty} [(4\lambda)^{m-2j-1} r_0, (4\lambda)^{m-2j} r_0).$$

Then $I_1 \cup I_2 = (0, r_0)$, and hence for some m,

$$\int_\rho^{r_0} \omega(r)\, \frac{dr}{r} \le 2 \int_{(\rho, r_0) \cap I_m} \omega(r)\, \frac{dr}{r}. \tag{14}$$

For $j = 1, 2, \dots$, choose $r_j \in [(4\lambda)^{m-2j-1} r_0, (4\lambda)^{m-2j} r_0)$ so that

$$\omega(r_j) \ge \frac{1}{(4\lambda)^{m-2j-1} r_0} \int_{(4\lambda)^{m-2j-1} r_0}^{(4\lambda)^{m-2j} r_0} \omega(r)\, dr \ge \int_{(4\lambda)^{m-2j-1} r_0}^{(4\lambda)^{m-2j} r_0} \omega(r)\, \frac{dr}{r}.$$

Then, as $\omega(r) \le 1$ for all r, we have

$$\int_{(\rho, r_0) \cap I_m} \omega(r)\, \frac{dr}{r} \le \sum_{\rho \le r_j \le r_0/4\lambda} \omega(r_j) + C'. \tag{15}$$

Proposition 4.4 yields for $j = 1, 2, \dots$

$$\begin{aligned} M((4\lambda)^{m-2j-1} r_0) \le M(r_j) &\le M(4\lambda r_j)(1 - 2^{-n(2r_j)-2}) \\ &\le M((4\lambda)^{m-2j+1} r_0) \left(1 - \frac{\omega(r_j)}{4}\right). \end{aligned}$$

Iterating this estimate and using $\log(1-t) \le -t$, we obtain for $0 < \rho < r_0$,

$$M(\rho) \le M(r_0) \exp\left(-\frac{1}{4} \sum_{\rho \le r_j \le r_0/4\lambda} \omega(r_j)\right).$$

Finally, we use (14) and (15) to estimate the sum on the right-hand side. $\square$

Proof of Theorem 2.12. As $-u$ is a quasiminimizer of the p-energy integral on Ω with the boundary data $-w$, it suffices to estimate $(u(x) - w(x_0))_+$. We can assume that $w(x_0) = 0$. As w is continuous at x_0, by Theorem 4.3 we have, for some $R > 0$ and all $0 < r_0 \le R$,

$$M(r_0, r_0) \le M := \operatorname*{ess\,sup}_{B(x_0, R)} u_+ < \infty.$$

By Theorem 2.11 we have for $0 < \rho < r_0$

$$\operatorname*{ess\,sup}_{B(x_0,\rho)} u_+ \le \operatorname*{ess\,sup}_{B(x_0,r_0)} w_+ + M(\rho, r_0) \tag{16}$$

$$\le \operatorname*{ess\,sup}_{B(x_0,r_0)} w_+ + C_1 M \exp\biggl(-\frac{1}{8}\int_\rho^{r_0} \exp\bigl(-C_0\gamma(p_1,r)^{p/(p-p_1)}\bigr)\,\frac{dr}{r}\biggr).$$

By the assumptions, there exist $\alpha, \beta, C > 0$ such that for all sufficiently small ρ and r_0

$$\operatorname*{ess\,sup}_{B(x_0,r_0)} w_+ \le C r_0^\beta$$

and

$$\int_\rho^1 \exp\bigl(-C_0\gamma(p_1,r)^{p/(p-p_1)}\bigr)\,\frac{dr}{r} \ge \alpha|\log\rho|.$$

Note also that for all $0 < r_0 < 1$

$$\int_{r_0}^1 \exp\bigl(-C_0\gamma(p_1,r)^{p/(p-p_1)}\bigr)\,\frac{dr}{r} \le \int_{r_0}^1 \frac{dr}{r} = |\log r_0|.$$

Then, by (16), we have for sufficiently small ρ and r_0

$$\operatorname*{ess\,sup}_{B(x_0,\rho)} u_+ \le C r_0^\beta + C_1 M \rho^{\alpha/8} r_0^{-1/8},$$

and choosing $r_0 = \rho^{\alpha'}$ with $0 < \alpha' < \alpha$ shows that, after a redefinition on a set of measure zero, u is Hölder continuous at x_0. $\square$

In the proof of Theorem 2.13 we shall need a weak Harnack inequality for functions from the De Giorgi class, which we now introduce.

Definition 4.5. Let U be an open subset of X. The *De Giorgi class* $DG_p(U)$ consists of all functions $u \in N^{1,p}(X)$ satisfying the condition (7) for all $k \in \mathbf{R}$, all balls $B(x_0, r_2) \subset U$ and all $0 < r_1 < r_2$.

The following weak Harnack inequality is established in [14, Theorem 7.1].

Theorem 4.6. *There exist positive constants C and σ such that for every ball $B \subset X$ and every nonnegative function v such that $-v \in DG_p(B)$,*

$$\biggl(\fint_B v^\sigma\,d\mu\biggr)^{1/\sigma} \le C \inf_B v.$$

Proof of Theorem 2.13. Let $w \in N^{1,p}(X)$ be continuous at x_0 and let $u \in N^{1,p}(X)$ be a quasiminimizer of the p-energy integral on Ω with the boundary data w. Note that $u = w$ p-q.e. on $X \setminus \Omega$. Assume that u is not continuous at x_0. Replacing u by $-u$ if needed, we can assume that $\lim_{r\to 0} \operatorname{ess\,sup}_{B(x_0,r)} u >$

$w(x_0) = 0$. Then we can find positive constants k_0 and M so that for all sufficiently small $r > 0$

$$2k_0 < \operatorname*{ess\,sup}_{B(x_0,r)} u < M. \tag{17}$$

Let σ be the exponent from the weak Harnack inequality (Theorem 4.6) and let $0 < \sigma' < \sigma$ and $0 < \theta < 1$. Let

$$\omega(r) = \operatorname*{ess\,sup}_{B(x_0,10\lambda r)} u - \operatorname*{ess\,sup}_{B(x_0,r)} u \quad \text{and} \quad l(r) = \operatorname*{ess\,sup}_{B(x_0,10\lambda r)} u - \omega(r)^{\sigma'/\sigma}.$$

As $\lim_{r\to 0} \omega(r) = 0$, we have $l(r) > k_0$ and

$$\omega(r)^{\sigma'/\sigma} = \operatorname*{ess\,sup}_{B(x_0,10\lambda r)} u - l(r) \le \omega(r)^{\theta\sigma'/\sigma} \tag{18}$$

for all sufficiently small $r > 0$.

As w is continuous at x_0, we have $\operatorname{ess\,sup}_{B(x_0,r_0)} w < k_0$ for some $r_0 > 0$, and hence $(u - k_0)_+ = 0$ p-q.e. on $B(x_0, r_0) \setminus \Omega$. Proposition 3.2 then implies for sufficiently small $r > 0$

$$\begin{aligned} &\operatorname{cap}_{p_1}(B(x_0,r) \setminus \Omega, B(x_0,2r)) \fint_{B(x_0,2r)} (u-k_0)_+^{p_1}\, d\mu \\ &\quad\le C \int_{B(x_0,2\lambda r)} g_{(u-k_0)_+}^{p_1}\, d\mu \\ &\quad\le C \int_{A(l(r),2\lambda r)} g_u^{p_1}\, d\mu + C \int_{T(k_0,l(r),2\lambda r)} g_u^{p_1}\, d\mu, \end{aligned} \tag{19}$$

where $A(k,r) = \{x \in B(x_0,r) : u(x) > k\}$ and $T(k,l,r) = A(k,r) \setminus A(l,r)$. The two integrals on the right-hand side in (19) will be estimated separately. By the Hölder inequality, Proposition 4.1, and (18), we have

$$\begin{aligned} \int_{A(l(r),2\lambda r)} g_u^{p_1}\, d\mu &\le \biggl(\int_{A(l(r),2\lambda r)} g_u^{p}\, d\mu\biggr)^{p_1/p} \mu(B(x_0,2\lambda r))^{1-p_1/p} \\ &\le \frac{C\mu(B(x_0,r))}{r^{p_1}} \biggl(\fint_{B(x_0,10\lambda r)} (u-l(r))_+^p\, d\mu\biggr)^{p_1/p} \\ &\le \frac{C\mu(B(x_0,r))}{r^{p_1}}\, \omega(r)^{p_1\theta\sigma'/\sigma}. \end{aligned} \tag{20}$$

Another application of the Hölder inequality implies

(21)

$$\int_{T(k_0,l(r),2\lambda r)} g_u^{p_1}\, d\mu \le \biggl(\int_{A(k_0,2\lambda r)} g_u^{p}\, d\mu\biggr)^{p_1/p} \mu(T(k_0,l(r),2\lambda r))^{1-p_1/p}.$$

By Proposition 4.1 and (17), the integral term on the right-hand side can be estimated as follows:

$$\left(\int_{A(k_0,2\lambda r)} g_u^p\,d\mu\right)^{p_1/p} \le \frac{C}{r^{p_1}}\left(\int_{B(x_0,10\lambda r)} (u-k_0)_+^p\,d\mu\right)^{p_1/p} \le Cr^{-p_1}M^{p_1}\mu(B(x_0,r))^{p_1/p}. \tag{22}$$

In order to estimate $\mu(T(k_0,l(r),2\lambda r))$, we first observe that the function $v=\operatorname{ess\,sup}_{B(x_0,10\lambda r)} u-\max\{u,k_0\}$ is nonnegative in $B(x_0,10\lambda r)$. A simple calculation shows that $(-v-k)_+=(u-k')_+$ with $k'=k+\operatorname{ess\,sup}_{B(x_0,10\lambda r)} u$ for all $k\ge k^*=k_0-\operatorname{ess\,sup}_{B(x_0,10\lambda r)} u$. At the same time, for $k<k^*$ we have $-v-k>-v-k^*\ge 0$ on $B(x_0,10\lambda r)$. Hence, Proposition 4.1 with k replaced by k' implies that $-v$ belongs to the De Giorgi class $DG_p(B(x_0,10\lambda r))$. It then follows from the weak Harnack inequality (Theorem 4.6) that

$$\begin{aligned}\mu(T(k_0,l(r),2\lambda r))&\Bigl(\operatorname*{ess\,sup}_{B(x_0,10\lambda r)} u-l(r)\Bigr)^\sigma\\ &\le \int_{B(x_0,2\lambda r)} v^\sigma\,d\mu \le C\mu(B(x_0,2\lambda r))\Bigl(\operatorname*{ess\,sup}_{B(x_0,10\lambda r)} u-\operatorname*{ess\,sup}_{B(x_0,2\lambda r)} u\Bigr)^\sigma.\end{aligned}$$

This and (18) now yield

$$\mu(T(k_0,l(r),2\lambda r))\le C\mu(B(x_0,r))\omega(r)^{\sigma-\sigma'}.$$

Inserting this estimate, together with (20)–(22), into (19), we obtain

$$\begin{aligned}\frac{\operatorname{cap}_{p_1}(B(x_0,r)\setminus\Omega,B(x_0,2r))}{r^{-p_1}\mu(B(x_0,r))}&\fint_{B(x_0,2r)}(u-k_0)_+^{p_1}\,d\mu\\ &\le C\bigl(\omega(r)^{p_1\theta\sigma'/\sigma}+M^{p_1}\omega(r)^{(\sigma-\sigma')(1-p_1/p)}\bigr).\end{aligned}\tag{23}$$

Next, by Theorem 4.3 and (17) we have for sufficiently small $r>0$,

$$\begin{aligned}k_0^p\le\Bigl(\operatorname*{ess\,sup}_{B(x_0,r)} u-k_0\Bigr)^p&\le C\fint_{B(x_0,2r)}(u-k_0)_+^p\,d\mu\\ &\le CM^{p-p_1}\fint_{B(x_0,2r)}(u-k_0)_+^{p_1}\,d\mu,\end{aligned}$$

and hence

$$\fint_{B(x_0,2r)}(u-k_0)_+^{p_1}\,d\mu\ge CM^{p_1-p}k_0^{p_1}>0.$$

The estimate (23) with suitable choices of θ and σ' then shows that for sufficiently small $r>0$ and all $\Lambda>1/p_1+p/\sigma(p-p_1)$,

$$\left(\frac{\operatorname{cap}_{p_1}(B(x_0,r)\setminus\Omega,B(x_0,2r))}{r^{-p_1}\mu(B(x_0,r))}\right)^\Lambda\le C\omega(r).$$

The inequality $\int_0^1\omega(r)\,dr/r\le\int_1^{10\lambda}\operatorname{ess\,sup}_{B(x_0,r)} u\,dr/r<\infty$ finishes the proof. $\square$

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