ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 4, Winter 1987

SPECTRUM OF DOMAINS IN RIEMANNIAN MANIFOLDS

BY

HAROLD DONNELLY

1. Introduction

Let M^n be a complete Riemannian manifold, of dimension n, with Ricci curvature bounded below by $-(n-1)c$, $c \geq 0$. Suppose that Ω is a domain, with smooth boundary, contained in M. We consider the Laplacian $\Delta = d^*d$ acting on $L^2\Omega$ with Dirichlet boundary conditions. If Ω is relatively compact, then Δ has pure point spectrum. This means that there exists an orthonormal basis of $L^2\Omega$ consisting of eigenfunctions of Δ. Moreover, all eigenvalues are isolated and have finite multiplicity.

Our primary interest will be with noncompact domains Ω. In this case, the Laplacian may have essential spectrum. Recall that the essential spectrum consists of cluster points of the spectrum and eigenvalues of infinite multiplicity. Suppose that $d(x)$ is the distance from $x \in \Omega$ to the boundary $\partial\Omega$ of Ω. If there exists a sequence x_i, eventually leaving every compact set, with $d(x_i) > \varepsilon$, for some $\varepsilon > 0$, then Δ has essential spectrum in $L^2\Omega$. This assertion will be verified in Section 2.

Now assume that $d(x)$ approaches zero as x approaches infinity. More precisely, given $\varepsilon > 0$, there exists a compact set C with $d(x) < \varepsilon$ for $x \in \Omega - C$. Let S_x be the set of points $y \in M - \Omega$ with $d(x, y) \leq \alpha d(x)$, for a fixed constant $\alpha > 1$. We say that $\partial\Omega$ is suitably regular if $\operatorname{Vol}(S_x)$, the volume of S_x, is bounded below by a constant multiple of $d^n(x)$. This represents uniform boundary regularity in a rather generalized sense. In Section 4, we suppose that $\partial\Omega$ is suitably regular and that $d(x)$ approaches zero as x approaches infinity. Under these hypotheses, we prove that Δ has pure point spectrum.

If M^n is the Euclidean space R^n, the results of this paper are well-known [5], [8]. Some new methods are required to prove our theorems for complete Riemannian manifolds. For this purpose, we develop the machinery of [1], [2]. The author thanks C. Croke for helpful discussions concerning his work.

2. Essential spectrum

Let Ω be a noncompact domain contained in a complete Riemannian manifold M^n, whose Ricci curvature is bounded below by $-(n-1)c$, $c \geq 0$.

Received April 9, 1986.

One imposes Dirichlet boundary conditions on the Laplacian Δ of Ω. There is a simple geometric criterion which assures that Δ has essential spectrum:

THEOREM 2.1. *Suppose there exists a sequence x_i, eventually leaving every compact set, with $d(x_i) > \varepsilon$, for some $\varepsilon > 0$. Then Δ has non-empty essential spectrum.*

Proof. By hypothesis, Ω contains an infinite number of disjoint geodesic balls B_j of radius ε. Let $D(c, \varepsilon)$ be the standard ball of radius ε in the simply connected complete space of constant curvature $-c$. If λ_1 denotes the first eigenvalue of Δ with Dirichlet boundary conditions, then Cheng [4] proved that $\lambda_1(B_j) \le \lambda_1(D(c, \varepsilon))$, for all j. Since the B_j are disjoint, our conclusion follows from the minimax principle.

The special case $M = \Omega$ was established by a similar argument in [6]. The converse to Theorem 2.1 is more subtle. It will be treated presently.

3. Lower bounds on compact domains

Suppose that $\mathscr{D}$ is a relatively compact domain in a complete Riemannian manifold M. The Laplacian Δ acts on $L^2\mathscr{D}$ with Dirichlet boundary conditions. We develop the method of [1], [2] to give a lower bound for the positive operator Δ.

Let π: $U\mathscr{D} \to \mathscr{D}$ be the unit sphere bundle with its canonical measure. If $v \in U\mathscr{D}$, then $\zeta^t(v)$ denotes the geodesic flow on $U\mathscr{D}$. The symbol $l(v)$ will be the smallest value of t such that $\pi(\zeta^t(v))$ lies in $\partial\mathscr{D}$, the boundary of $\mathscr{D}$. If $\pi(\zeta^t(v))$ never reaches $\partial\mathscr{D}$, then we set $l(v) = \infty$. Let $\overline{U\mathscr{D}}$ be the subset of $v \in U\mathscr{D}$ where $l(-v)$ is finite.

For $p \in \partial\mathscr{D}$, the symbol N_p will denote the inward pointing normal vector. Define $U^+\partial\mathscr{D}$ as the bundle of inwardly pointing unit vectors. The following basic formula is well known [1]:

$$\int_{\overline{U\mathscr{D}}} f(v)\,dv = \int_{U^+\partial\mathscr{D}} \int_0^{l(u)} f(\zeta^r(u))\langle u, N_{\pi(u)}\rangle\,dr\,du. \tag{3.1}$$

Here f is any integrable function.

If n is the dimension of M, let $\alpha(n-1)$ be the volume of the unit $n-1$ sphere. For $x \in \mathscr{D}$, define

$$h(x) = \frac{n}{4\alpha(n-1)} \int_{\pi^{-1}(x)} \frac{dv}{l^2(v)}. \tag{3.2}$$

If $l(v) = \infty$, then we interpret $l^{-2}(v) = 0$. The infimum of $h(x)$, over all $x \in \mathscr{D}$, will be denoted by h_0.

Our main result for this section is:

THEOREM 3.3. *The operator $\Delta - h$, acting on $L^2\mathscr{D}$ with Dirichlet boundary conditions, is positive semi-definite.*

Proof. For $f \in C_0^\infty(\mathscr{D})$, one has

$$|\nabla f(x)|^2 = \frac{n}{\alpha(n-1)} \int_{\pi^{-1}(x)} (vf)^2 \, dv.$$

Consequently

$$\int_{\mathscr{D}} |\nabla f(x)|^2 \, dx \geq \frac{n}{\alpha(n-1)} \int_{\overline{U\mathscr{D}}} (vf)^2 \, dv$$

Using formula (3.1),

$$\int_{\mathscr{D}} |\nabla f|^2 \, dx \geq \frac{n}{\alpha(n-1)} \int_{U^+ \partial\mathscr{D}} \int_0^{l(u)} (\zeta^t(u) f)^2 \langle u, N_{\pi(u)} \rangle \, dt \, du.$$

Since f vanishes on $\partial\mathscr{D}$, we may apply Lemma 6.1 to the interior integral:

$$\int_{\mathscr{D}} |\nabla f|^2 \, dx \geq \frac{n}{4\alpha(n-1)} \int_{U^+ \partial\mathscr{D}} \int_0^{l(u)} f^2(\zeta^t(u)) t^{-2} \langle u, N_{\pi(u)} \rangle \, dt \, du.$$

Applying (3.1) again,

$$\int_{\mathscr{D}} |\nabla f|^2 \, dx \geq \frac{n}{4\alpha(n-1)} \int_{\overline{U\mathscr{D}}} f^2(v) l^{-2}(-v) \, dv.$$

Since $f(v) = f(\pi v)$, we may integrate over the fiber to obtain

$$\int_{\mathscr{D}} |\nabla f|^2 \, dx - \int_{\mathscr{D}} h(x) f^2(x) \, dx \geq 0.$$

Theorem 3.3 now follows from the minimax principle. Using the spectral theorem, we deduce:

COROLLARY 3.4. *The first eigenvalue of Δ, acting with Dirichlet boundary conditions, is greater than or equal to h_0.*

It may be interesting to compare Corollary 3.4 with the corresponding result in [1], [2]. Our main improvement is to eliminate the hypothesis that every geodesic in $\mathscr{D}$ intersects $\partial\mathscr{D}$. The point is that Lemma 6.1 only requires the vanishing of f at one endpoint, that is $f(0) = 0$. Croke used a one dimensional lemma which demanded vanishing at both endpoints.

4. Pure point spectrum

Suppose that Ω is a noncompact domain in a complete Riemannian manifold M^n, with Ricci curvature bounded from below by $-(n-1)c$. We consider the Laplacian Δ, acting on $L^2\Omega$, with Dirichlet boundary conditions. The purpose of this section is to provide geometric criteria which insure that Δ has pure point spectrum.

Let $d(x)$ be the distance from $x \in \Omega$ to $\partial\Omega$, the boundary of Ω. We assume that $d(x)$ approaches zero as x leaves sufficiently large compact sets. In addition, we impose the following condition:

DEFINITION 4.1. If $x \in \Omega$, then define S_x to be the set of points $y \in M - \Omega$ with $d(x, y) \leq \alpha d(x)$, for a fixed constant $\alpha > 1$. We say that Ω has sufficiently regular boundary if $\mathrm{Vol}(S_x) \geq A_1 d^n(x)$. Here $d(x, y)$ is the geodesic distance from x to y and $\mathrm{Vol}(S_x)$ is the volume of S_x. The symbol A_1 denotes a positive constant.

Suppose $h(x, \Omega)$ is defined as in formula (3.2). Using our geometric hypotheses, we may deduce:

PROPOSITION 4.2. $h(x, \Omega) \geq A_2 d^{-2}(x)$

Proof. Choose geodesic polar coordinates (t, ω) centered at x. Since the Ricci curvature of M is bounded below by $-(n-1)c$, we may apply the volume comparison theorem:

$$\begin{aligned}\mathrm{Vol}(S_x) &\leq \int_{\overline{U}(x,\,\alpha d)} \int_0^{\alpha d(x)} \left(\frac{\sin h(\sqrt{c}\, t)}{\sqrt{c}} \right)^{n-1} dt\, dw \\ &\leq A_3 d^n(x) \,\mathrm{Vol}(\overline{U}(x, \alpha d))\end{aligned}$$

Here $\overline{U}(x, \alpha d)$ is the portion of $\pi^{-1}(x) \cap \overline{U}$ with $l(-v) \leq \alpha d(x)$. Since $\partial\Omega$ is sufficiently regular, $\mathrm{Vol}(S_x) \geq A_1 d^n(x)$, we deduce that $\mathrm{Vol}(\overline{U}(x, \alpha d)) \geq A_1/A_3$.

Furthermore,

$$h(x) \geq \frac{n}{4\alpha(n-1)} \mathrm{Vol}(\overline{U}(x, \alpha d))\alpha^{-2} d^{-2}(x) \geq A_2 d^{-2}(x)$$

We now proceed to our main result:

THEOREM 4.3. *Suppose that $d(x)$ approaches zero as x approaches infinity and $\partial\Omega$ is sufficiently regular. Then the Laplacian Δ, acting on $L^2\Omega$ with Dirichlet boundary conditions, has pure point spectrum.*

Proof. Let C be a compact set in M and suppose that $\Omega - C$ is a domain with smooth boundary. Consider a relatively compact domain $\mathscr{D}$ contained in $\Omega - C$. It follows easily from the definitions that $h(x, \mathscr{D}) \geq h(x, \Omega)$. By Proposition 4.2, $h(x, \mathscr{D}) \geq A_2 d^{-2}(x)$. Therefore $h_0(\mathscr{D}) \geq \inf(A_2 d^{-2}(x))$, where the infimum is for $x \in \Omega - C$.

By Corollary 3.4, the first eigenvalue of $\mathscr{D}$ is greater than or equal to $\inf(A_2 d^{-2}(x))$, $x \in \Omega - C$. The minimax principle implies that the spectrum of Δ, with Dirichlet boundary conditions in $L^2(\Omega - C)$, is bounded below by $\inf(A_2 d^{-2}(x))$, $x \in \Omega - C$. Since $d(x)$ approaches zero as x approaches infinity, our conclusion now follows from the decomposition principle [6].

The criterion of sufficiently regular boundary is readily verified in various geometric situations. As an illustration, suppose that M has bounded geometry. This means that the curvature of M is bounded and its injectivity radius is bounded from below. For each $Z \in \partial\Omega$, assume there is a ball $B(y, \gamma) \subset M - \Omega$, centered at y with radius γ, and tangent to $\partial\Omega$ at z. The constant γ is independent of z. Under these circumstances, we say that $\partial\Omega$ admits tangential balls.

PROPOSITION 4.4. *Assume that $d(x)$ approaches zero as x approaches infinity. If M has bounded geometry and $\partial\Omega$ admits tangential balls, then $\partial\Omega$ is sufficiently regular.*

Proof. Let $x \in \Omega$ with $d(x)$ sufficiently small. This only excludes x from some compact set. Suppose that z is a contact point, on $\partial\Omega$, for a geodesic of minimum length $d(x)$, from x to $\partial\Omega$. Assume that $B(y, \gamma) \subset M - \Omega$ is tangent to $\partial\Omega$ at z. Let $w \in M - \Omega$ lie on a minimizing geodesic from z to y and satisfy

$$d(w, z) = (\alpha - 1)d(x)/2,$$

for a fixed constant $\alpha > 1$. By the triangle inequality

$$B(w, (\alpha - 1)d(x)/2) \subset B(y, \gamma).$$

Using the triangle inequality again gives

$$S_x \supset B(w, (\alpha - 1)d(x)/2).$$

Since M has bounded geometry, and $d(x)$ approaches zero as x approaches infinity, $\mathrm{Vol}(S_x) \geq A_1 d^n(x)$. Thus $\partial\Omega$ is sufficiently regular.

5. Trace class heat kernel

Assume that Ω is a noncompact domain which satisfies the hypotheses of Theorem 4.3. We have shown that the Laplacian Δ has pure point spectrum.

Under additional conditions, our methods lead to a stronger result. One may show that the heat kernel is trace class and give an upper bound for its trace.

Define $g(x) = A_2 d^{-2}(x)$, where A_2 is the constant of Proposition 4.2. Then one has:

PROPOSITION 5.1. *The operator* $\Delta - g(x)$, *acting on* $L^2\Omega$ *with Dirichlet boundary conditions, is positive semidefinite.*

Proof. Let $\mathscr{D}$ be any relatively compact subdomain of Ω. Clearly $h(x, \mathscr{D}) \geq h(x, \Omega)$. It follows from Theorem 3.3 and Proposition 4.2 that $\Delta - g(x)$, acting on $L^2\mathscr{D}$ with Dirichlet boundary conditions, is positive semi-definite. Since $\mathscr{D}$ is arbitrary, Proposition 5.1 follows from the minimax principle.

Consider the heat operator $e^{-t\Delta}$ of Ω. This operator is positively improving. Therefore, its trace is well-defined as an extended real number. By Proposition 5.1, $\Delta \geq \Delta/2 + g/2$, meaning that the difference is positive definite. Using the spectral theorem,

$$\mathrm{Tr}(e^{-t\Delta}) \leq \mathrm{Tr}(e^{-t\Delta/2 - tg/2}).$$

By the inequality of [9], this implies that

$$\mathrm{Tr}(e^{-t\Delta}) \leq \mathrm{Tr}(e^{-tg/4} e^{-t\Delta/2} e^{-tg/4}) \tag{5.2}$$

The same argument shows that (5.2) is valid for any compact subdomain $\mathscr{D}$ of Ω.

Suppose that M has bounded geometry. We may state:

THEOREM 5.3. *Let* Ω *be a noncompact domain in a manifold* M *having bounded geometry. Assume that* Ω *satisfies the hypotheses of Theorem* 4.3. *Then*

$$\mathrm{Tr}(e^{-t\Delta}) \leq B_1 t^{-n/2} \int_\Omega e^{-tg(x)/2}\, dx.$$

If the integral converges, this means that $e^{-t\Delta}$ *is trace class.*

Proof. Let $K(t, x, y)$ be the smoothing kernel representing $e^{-t\Delta}$. Duhamel's principle implies that K is less than or equal to the heat kernel of M. Since M has bounded geometry [3],

$$K(t, x, x) \leq B_1 t^{-n/2},$$

where B_1 is independent of x. Applying formula (5.2) to compact domains $\mathscr{D}$, contained in Ω, shows that $\mathrm{Tr}(e^{-t\Delta})$ is uniformly bounded above by the integral of Theorem 5.3. Using the minimax principal, the main result follows.

6. Appendix

The purpose of this appendix is to establish the following elementary result in one real variable:

LEMMA 6.1. *Let f be a continuously differentiable function vanishing at zero. For any real number $a > 0$,*

$$\int_0^a (f'(x))^2\,dx \geq \frac{1}{4}\int_0^a f^2(x)x^{-2}\,dx.$$

Proof. Define $g(x) = x^{-1/2}f(x)$. Calculating $g'(x)$ using the product rule, we check that

$$f'(x) = x^{1/2}g'(x) + x^{-1}f(x)/2.$$

Taking the square of each side gives

$$(f'(x))^2 \geq x^{-2}f^2(x)/4 + gg'$$

Integrating from 0 to a yields

$$\int_0^a (f'(x))^2\,dx \geq \frac{1}{4}\int_0^a x^{-2}f^2(x) + g^2(a)/2.$$

Here we used the hypothesis that f is a differentiable function with $f(0) = 0$. The lemma follows since $g^2(a) \geq 0$.

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PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA