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ON A SINGULAR INTEGRAL ESTIMATE FOR THE MAXIMUM MODULUS OF A CANONICAL PRODUCT

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ABSTRACT. If f is a canonical product with only real negative zeros and non-integral order ρ, $n(t, 0)$ is the zero counting function, and $B(r, f) = \sup_{0<\theta<\pi} |\log f(re^{i\theta})|$, then

$$r^{-q-1}B(r, f) \le \pi\{M\varphi(r) + MH\varphi(r)\} + \int_0^\infty \frac{\varphi(t)\,dt}{t+r},$$

where $\varphi(t) = t^{-q-1}n(t, 0)$, H is the Hilbert transform operator and M is the Hardy-Littlewood maximal operator.

1. Introduction

Let f be an entire function with zeros $\{z_n\}$, and let

$$M(r, f) = \sup_{|z|=r} |f(z)|, \quad n(r) = n(r, 0; f) = \sum_{|z_n|\le r} 1.$$

The order of f is defined by

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r},$$

and a question of long standing is to find precise lower bounds for

$$\limsup_{r\to\infty} \frac{n(r)}{\log M(r, f)}$$

in terms of ρ.

Polya [1] and Valiron [4], [5] proved, independently, that

$$\limsup_{r\to\infty} \frac{n(r)}{\log M(r, f)} \ge \begin{cases} \frac{1}{\pi}|\sin\pi\rho|, & 0 \le \rho \le 1, \\ \frac{|\sin\pi\rho|}{A_0\{1+\log\rho\}|\sin\pi\rho|+\pi}, & 1 < \rho < \infty, \end{cases} \tag{1}$$

where A_0 is a positive absolute constant. The first inequality in (1) is sharp, the constant $\frac{1}{\pi}|\sin\pi\rho|$ being best possible and achieved when all the zeros of f are on one ray and $n(r)$ is regularly varying of order ρ. In connection with the second

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inequality in (1), Shea and Wainger [2] proved the existence of a positive absolute constant A such that

$$\limsup_{r\to\infty} \frac{n(r)}{\log M(r, f)} \geq A|\sin \pi\rho|, \quad 1 < \rho < \infty \tag{2}$$

for f of order ρ, all of whose zeros lie on a single ray $\arg z = \pi$. Although the value of A obtained in [2] is not best possible, the existence of such a constant is rather remarkable. The starting point for the Shea-Wainger proof is the well-known formula of Valiron

$$\log f(z) = (-1)^q z^{q+1} \int_0^\infty \frac{n(t, 0)dt}{t^{q+1}(t+z)}, \quad q = [\rho], \quad |\arg z| < \pi \tag{3}$$

which is valid for canonical products f of non-integral order ρ, having all their zeros on the ray $\arg z = \pi$.

Writing

$$B(r, f) = \sup_{0<\theta<\pi} \left|\log f(re^{i\theta})\right|, \quad \Phi(r) = \frac{B(r, f)}{r^{q+1}}, \quad \varphi(r) = \frac{n(r, 0)}{r^{q+1}},$$

employing Valiron's formula (3), and using some rather intricate singular integral estimates they obtain

$$\Phi(r) \leq 12M\varphi(r) + \pi H^*\varphi(r) + 10 \int_0^\infty \frac{\varphi(t)}{t+r}\, dt \tag{4}$$

where

$$M\varphi(r) = \sup_{\varepsilon>0} \frac{1}{2\varepsilon} \int_{|t-r|<\varepsilon} \varphi(t)\, dt$$

is the Hardy-Littlewood maximal function and

$$H^*\varphi(r) = \frac{1}{\pi} \sup_{\varepsilon>0} \left| \int_{|t-r|>\varepsilon} \frac{\varphi(t)dt}{t-r} \right|$$

is the maximal Hilbert transform. From the inequality (4) and using the L_p-boundedness of these maximal operators, together with Tauberian arguments, they obtain, for suitable sequences $R_n \to \infty$, $\varepsilon_n \to 0$, the inequality

$$\left\{ \int_{R_n}^\infty \left(\frac{B(r, f)}{r^{q+1}} \right)^p dr \right\}^{\frac{1}{p}} \leq \left(A \sin \frac{\pi}{p} \right)^{-1} \left\{ \int_{R_n}^\infty \left(\frac{n(r, 0)}{r^{q+1}} \right)^p dr \right\}^{\frac{1}{p}} \tag{5}$$

where $p = (q + 1 - \rho)^{-1} + \varepsilon_n$. Now (2) follows immediately from (5).

This note arose in the course of examining the Shea-Wainger proof in [2] and attempting to simplify it. It turns out that the use of the maximal Hilbert transform of φ can be circumvented leading to a refinement of (4) in which the constants 12 and

10 are replaced by π and 1 respectively. Thus the value of the constant A in (2) is increased about 4 times.

2. An inequality for the maximum modulus

An examination of the integral occuring in Valiron's formula (3) suggests that it assumes the boundary values $\int_0^\infty \frac{\varphi(t)}{t+r}\,dt$ as $\theta \to 0$ and $H\varphi(r)$ as $\theta \to \pi$, where H is the Hilbert transform operator. It is then natural to expect these two terms to occur when estimating the integral. This is made precise in the following:

THEOREM 1. *If f is a canonical product with only real negative zeros and non-integral order ρ, then*

$$\Phi(r) \le \pi M\varphi(r) + \pi M(H\varphi)(r) + \int_0^\infty \frac{\varphi(t)}{t+r}\,dt \tag{6}$$

where H is the Hilbert transform.

Proof. Write

$$D_1 = (t-r)^2 + 2tr(1-\cos\theta), \quad D_2 = (t-r)^2 + 2r^2(1-\cos\theta)$$

and notice that $2D_1 \ge (1-\cos\theta)(t+r)^2$ and $D_2 \ge 2|t-r|r\sqrt{2(1-\cos\theta)}$ so that $\sqrt{D_1}D_2 \ge 2r(1-\cos\theta)|t-r|(t+r)$. Thus

$$\left|\frac{1}{D_1} - \frac{1}{D_2}\right| = \frac{2r(1-\cos\theta)|r-t|}{D_1 D_2} \le \frac{1}{\sqrt{D_1}(t+r)}. \tag{7}$$

Starting from Valiron's formula (3), if we put $\varphi(t) = 0$ for $t \le 0$ we have

$$\begin{aligned}
\left|r^{-q-1}\log f(re^{i(\pi-\theta)})\right| &= \left|\int_{-\infty}^\infty \varphi(t)\frac{t-re^{i\theta}}{D_1}\,dt\right| \\
&= \left|\int_{-\infty}^\infty \varphi(t)(t-re^{i\theta})\left(\frac{1}{D_1}-\frac{1}{D_2}\right)dt\right. \\
&\quad \left. + \int_{-\infty}^\infty \varphi(t)\frac{(t-r)}{D_2}\,dt + \int_{-\infty}^\infty \varphi(t)\frac{(r-re^{i\theta})}{D_2}\,dt\right| \\
&\le \int_{-\infty}^\infty \frac{\varphi(t)}{t+r}\,dt + \left|\int_{-\infty}^\infty H\varphi(t)\frac{2r\sin\frac{\theta}{2}}{D_2}\,dt\right| \\
&\quad + \left|\int_{-\infty}^\infty \varphi(t)\frac{2r\sin\frac{\theta}{2}}{D_2}\,dt\right|
\end{aligned} \tag{8}$$

where in the last line we have used Lemma 1.5, page 219 of [3]. We now use the fact that the Poisson integral of φ is bounded by the maximal function $M\varphi$. We indicate a short proof of this:

$$\begin{aligned}\int_{-\infty}^{\infty}\varphi(t)\frac{\varepsilon}{(t-r)^2+\varepsilon^2}\,dt &= \int_0^{\infty}[\varphi(r-t)+\varphi(r+t)]\frac{\varepsilon}{t^2+\varepsilon^2}\,dt\\ &= \int_0^{\infty}[\varphi(r-t)+\varphi(r+t)]\frac{1}{\varepsilon}\int_{t/\varepsilon}^{\infty}\frac{2x\,dx}{(1+x^2)^2}\,dt\\ &= \int_0^{\infty}\frac{2x\,dx}{(1+x^2)^2}\frac{1}{\varepsilon}\int_0^{\varepsilon x}[\varphi(r-t)+\varphi(r+t)]\,dt\\ &\le \int_0^{\infty}\frac{2x^2dx}{(1+x^2)^2}\,2M\varphi(r)\\ &= \pi M\varphi(r).\end{aligned}$$

If we put $\varepsilon = 2r\sin\frac{\theta}{2}$, then it follows from (8) that for $0<\theta<\pi$,

$$|r^{-q-1}\log f(re^{i(\pi-\theta)})| \le \pi M\varphi(r)+\pi M(H\varphi)(r)+\int_0^{\infty}\frac{\varphi(t)}{t+r}\,dt$$

and (6) follows. □

We remark that the exponent $p=2$ is, perhaps, the most convenient to use in connection with the Hilbert transform since $\|H\varphi\|_2 = \|\varphi\|_2$. But this holds for $\varphi\in L_2(0,\infty)$ which, in the present context, requires that the order ρ be smaller than $q+1/2$. If we recall that $\|M\varphi\|_p \le 2(\frac{3p}{p-1})^{1/p}\|\varphi\|_p$ (see the proof of Theorem 3.7, page 58 of [3]) we obtain:

COROLLARY 1. *If f is as in Theorem 1, and its order satisfies $q<\rho<q+1/2$ then*

$$\left\{\int_0^{\infty}\{\Phi(r)\}^2\,dr\right\}^{1/2} \le \{4\sqrt{6}+1\}\pi\left\{\int_0^{\infty}\{\varphi(r)\}^2\,dr\right\}^{1/2}.$$

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