ON THE DE RHAM DECOMPOSITION THEOREM[1]

BY

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Introduction

Let M be a d-dimensional riemann manifold with positive-definite metric, and let Φ be the holonomy group of M at $m \in M$. If Φ preserves a proper subspace M_m^1 of the tangent space M_m, and if M is complete and simply connected, then de Rham's decomposition theorem asserts that M is isometric to the direct product of the maximal integral manifolds for the distributions obtained by parallel translation of M_m^1 and $(M_m^1)^\perp$ over M. We will extend this result to the case where the riemann metric on M is not necessarily positive-definite, but is nondegenerate on M_m^1.

If the holonomy group of M satisfies this last condition, i.e., Φ preserves a nondegenerate proper subspace of M_m, we say it is *nondegenerately reducible.*

The de Rham theorem for the positive-definite case has been given two proofs: de Rham's original proof [6], and recently one by K. Nomizu (cf. S. Kobayashi and K. Nomizu [4]). Our proof is entirely different from theirs. In particular, our proof does not involve "piecing together" local isometries into a global one. The approach we have adopted is the following. By the holonomy theorem of Ambrose-Singer [1], the condition of reducibility of the holonomy group should be reflected in the parallel translation of curvature. Since parallel translation of curvature determines the manifold up to local isometry [2], we obtain the local product structure of the manifold by a simple analysis of the curvature form. This information about parallel translation of curvature plus simple connectivity and completeness enable us to conclude the desired global product structure, thanks to the theorem of Ambrose-Hicks [3, Theorem 1, p. 244].

The paper is divided into five sections. The first two sections set up the notation and the necessary machinery. In the third section, we prove the local part of the theorem, and in the fourth, the global part. In the last section, we produce an example, originally due to R. A. Holzsager, showing that the nondegenerate reducibility assumption cannot be removed from the decomposition theorem. We also sketch a proof of this theorem in the positive-definite case which we believe to be simple, although not elementary because it makes use of the Main Theorem of [2].

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1. Preliminaries

We shall recall briefly certain definitions and conventions used in Ambrose [2] and Hicks [3]. Let $(M, \langle\ ,\ \rangle)$ be a C^∞ riemann manifold of dimension d; $\langle\ ,\ \rangle$ in the sequel will not be assumed to be positive-definite. M_m will denote the tangent space to $m \in M$. On each M_m, among all bases, $\langle\ ,\ \rangle_m$ singles out a special class, the set of all orthonormal bases, i.e., $(e_1, \cdots, e_d)$ $(e_i \in M_m)$ is orthonormal if and only if $\langle e_i, e_j\rangle_m = \pm\delta_{ij}$, the number of minus signs depending on the signature of the metric. We have the bundle of orthonormal frames $F(M)$, with natural projections $\pi : F(M) \to M$, fibre $\mathrm{PO}(d)$ which is the group of pseudo-orthogonal matrices leaving $\langle\ ,\ \rangle_m$ invariant on each M_m. A point $b \in F(M)$ will be written as $b = (m, e_1, \cdots, e_d)$, $m \in M$.

The unique riemannian connection is specified by the set of one-forms (ω_{ij}) defined on $F(M)$ taking value in $\mathfrak{PD}(d)$, which is the Lie algebra of $\mathrm{PO}(d)$. $\mathfrak{PD}(d)$ will be considered as a subalgebra of the full matrix algebra $\mathrm{gl}(n)$. On $F(M)$, we have the usual Cartan equations:

$$d\omega_i = -\textstyle\sum_j \omega_{ij}\,\omega_j, \qquad d\omega_{ij} = -\sum_k \omega_{ik}\,\omega_{kj} + \Omega_{ij}. \tag{1.1}$$

The one-forms (ω_i) are defined as follows: If $f \in F(M)_b$, $b = (m, e_1, \cdots, e_d)$, then $d\pi(f) = \sum_i \omega_i(f)e_i$, and the two-forms Ω_{ij} are the curvature forms of this riemannian connection. Define horizontal vector fields $\{E^i\}$ by $\omega_j(E^i) = \delta_{ij}$.

From this point on, we will assume M to be complete in the sense that the riemann connection is complete, i.e., all geodesics can be infinitely extended in terms of the parameter. This completeness assumption allows one to define two maps:

$$\exp_m : M_m \to M \quad \text{and} \quad \mathrm{Exp}_b : M_m \to B(M), \quad \forall\, m \in M, \quad \forall\, b \in \pi^{-1}(m).$$

Let $\theta_i^b = \omega_i \circ d\,\mathrm{Exp}_b$, $\theta_{ij}^b = \omega_{ij} \circ d\,\mathrm{Exp}_b$, $\Theta_{ij}^b = \Omega_{ij} \circ d\,\mathrm{Exp}_b$; then from (1.1)

$$d\theta_i^b = -\textstyle\sum_j \theta_{ij}^b\,\theta_j^b, \qquad d\theta_{ij}^b = -\sum_k \theta_{ik}^b\,\theta_{kj}^b + \Theta_{ij}^b. \tag{1.2}$$

Later on, b will be fixed, and we will often write θ_i, θ_{ij}, Θ_{ij} instead of θ_i^b, θ_{ij}^b, Θ_{ij}^b. Unless there is danger of confusion, this convention will hold in what follows.

Concerning these θ_i, θ_{ij}, Θ_{ij}, the following basic lemmas were proved in Ambrose [2]. Before stating them, we fix our notation: Let

$$b = (m, e_1, \cdots, e_d) \in F(M),$$

and let $x_1, \cdots, x_d$ denote the dual basis of $e_1, \cdots, e_d$ on M_m. Then, define

$z = (\sum_i x_i^2)^{1/2}$, and if $\operatorname{Exp}_b p = (n, f_1, \cdots, f_d)$, we write $m(p)$ for n and $e_i(p)$ for f_i, i.e., $\operatorname{Exp}_b p = (m(p), e_1(p), \cdots, e_d(p))$.

LEMMA 1. (1) *If t is the tangent vector to the ray $\gamma(\nu) = \nu \sum_i c_i e_i$ at any point on the ray, then*

$$d \operatorname{Exp}_b t = \sum_j c_j E^j(b).$$

(2) *If t is a tangent vector to M_m at p, then*

$$d \exp_m t = \sum_i \theta_i(t) e_i(p).$$

(3) *If t is a tangent vector to M_m at p, then*

$$H\, d \operatorname{Exp}_b t = \sum_i \theta_i(t) E^i,$$

where H denotes the operation of taking the horizontal component of a tangent vector on $F(M)$.

LEMMA 2. *Let $p = \sum_i c_i e_i \in M_m$ be such that $\sum_i c_i^2 = 1$. Let $\gamma(s) = sp$. Let T be the field of tangent vectors to γ. So $T = \sum_i c_i \partial/\partial x_i$. Let t be any constant field on M_m, i.e., $t = \sum a_i(\partial/\partial x_i)$, $a_i \in R$. Then along γ*

(1) $\theta_i(zt)(0) = 0$, $T\theta_i(zt)(0) = a_i$,

(2) $T\theta_i(zt) = a_i + \sum_j c_j \theta_{ij}(zt)$,

(3) $T\theta_{ij}(zt) = \Theta_{ij}(T, zt)$,

(4) $T^2\theta_i(zt) = \sum c_j \Theta_{ij}(T, zt)$.

We now introduce the curvature transformation on M_m, $\forall\, m$.

DEFINITION. Let $m \in M$, $x, y \in M_m$. Choose any

$$b = (m, e_1, \cdots, e_d) \in F(M)$$

over m, and choose $\bar{x}, \bar{y} \in F(M)_b$ so that $d\pi(\bar{x}) = x$, $d\pi(\bar{y}) = y$. Then R_{xy} is defined to be that endomorphism of M_m for which, relative to the base $e_1, \cdots, e_d$,

$$R_{xy}\, e_j = -\sum_i \Omega_{ij}(\bar{x}, \bar{y}) e_i .$$

It is now classical that because Ω is horizontal and equivariant, R_{xy} so defined is independent of the choice of b and of $\bar{x}, \bar{y}$ (Ambrose-Singer [1]). We shall subsequently need the following lemma which is well known.

LEMMA 3. *R_{xy} has the following properties:*

(1) $R_{xy} + R_{yx} = 0$.

(2) $\langle R_{xy}\, z, w\rangle + \langle z, R_{xy}\, w\rangle = 0$.

(3) $R_{xy}\, z + R_{yz}\, x + R_{zx}\, y = 0$.

(4) $\langle R_{xy}\, z, w\rangle = \langle R_{zw}\, x, y\rangle$.

2. Statement of the theorem

Let M be a simply connected, complete, C^∞ riemann manifold. We denote the holonomy group of M based at $b \in F(M)$ by Φ_b, or if there is no danger of confusion, by Φ (Ambose-Singer [1]). Now suppose at a point, $m \in M$, a proper subspace M_m^1 of M_m is left invariant by Φ, and suppose also that $\langle\ ,\ \rangle_m$, when restricted to M_m^1, is nondegenerate, i.e., Φ is nondegenerately reducible, in the terminology of the Introduction. Then we know that, if we denote the orthogonal complement of M_m^1 by M_m^2, $M_m = M_m^1 \oplus M_m^2$, that $\langle\ ,\ \rangle_m$ when restricted to M_m^2 is nondegenerate, and that M_m^2 is also left invariant by Φ. We say M_m^1 and M_m^2 reduce Φ. Furthermore, M_m is clearly isometric to $M_m^1 \oplus M_m^2$ when M_m^1 and M_m^2 are given the induced metric.

Now we shall define two distributions T_1 and T_2 on M (distributions in the sense of Chevalley). Given $n \in M$, then $T_1(n) \subseteq M_n$ is defined as follows: Join n to m by a (broken C^∞) curve γ. $T_1(n)$ is then the parallel translate of M_m^1 to n along γ. Similarly, $T_2(n)$ is the parallel translate of M_m^2 to n along γ. By definition of Φ, it is easily seen that parallel translation of M_m^1 and M_m^2 to n is independent of γ. So T_1, T_2 are well-defined C^∞ distributions. Because parallel translation respects the riemann metric, T_1, T_2 are seen to be orthogonal at all points, and the restriction of $\langle\ ,\ \rangle_n$ to $T_i(n)$, $i = 1, 2$, is nondegenerate.

We shall show that T_1, T_2 are both involutive distributions. For this and for later purposes, it is convenient to introduce the holonomy bundle $H(M)$ which is a subbundle of $F(M)$. Precisely, we take a point

$$b = (m, e_1, \cdots, e_d) \in F(M)$$

so that $e_1, \cdots, e_r$ span $T_1(m)$ and $e_{r+1}, \cdots, e_d$ span $T_2(m)$. Let $H(M)$ be the subset of $F(M)$ which can be joined to b by a broken C^∞ horizontal curve. Then the Holonomy Reduction Theorem states that $H(M)$ is a subbundle of $F(M)$ and that $F(M)$ can be reduced to $H(M)$ in a connection-preserving manner, i.e., $H(M)$ and $F(M)$ have the same horizontal subspaces (Ambrose-Singer [1], Nomizu [5]). The structure group of $H(M)$ is, of course, Φ, i.e., the holonomy group of M based at b.

From now on we shall work in $H(M)$ exclusively, so we describe $H(M)$ in greater detail. If $c = (n, f_1, \cdots, f_d) \in H(M)$, then $f_1, \cdots, f_r$ span $T_1(n)$, and $f_{r+1}, \cdots, f_d$ span $T_2(n)$. Furthermore, Φ is contained in the direct product $\mathrm{PO}(r) \times \mathrm{PO}(d-r) \subseteq \mathrm{PO}(d)$. Thus, if ψ is the Lie algebra of Φ, in terms of matrices this says that $(a_{ij}) \in \psi$ implies $a_{ij} = 0$ for $i \geqq r+1$, $j \leqq r$ and for $i \leqq r, j \geqq r+1$. Hence we obtain

LEMMA 4. *As an algebra of endomorphisms, ψ leaves the distributions T_1, T_2 invariant (at each point of M). In particular, ω_{ij}, Ω_{ij}, θ_{ij}, Θ_{ij} all vanish for*

$$i \geqq r+1, \quad j \leqq r \qquad \textit{and} \qquad i \leqq r, \quad j \geqq r+1.$$

We now prove T_1 and T_2 are involutive. Take $n \in M$, and a neighborhood U of n so that there is a cross-section χ of U into $\pi^{-1}(U) \subseteq H(M)$, i.e., $\chi : U \to \pi^{-1}(U)$, $\pi \circ \chi =$ identity. (We still denote the natural projection $H(M) \to M$ by π.) By definition of the ω_i's and by the above remarks on

$H(M)$, T_1 on U is described by

$$(*) \qquad \omega_i \circ d\chi = 0, \qquad i = r+1, \cdots, d.$$

To show T_1 is involutive, it suffices to show the ideal generated by $\omega_{r+1} \circ d\chi, \cdots, \omega_d \circ d\chi$ is closed under exterior differentiation. We now apply $d\chi$ to the first Cartan structure equation of (1.1), and we obtain

$$\begin{aligned} d\omega_i \circ d\chi &= -\textstyle\sum_{j=1}^{j=d} (\omega_{ij} \circ d\chi)(\omega_j \circ d\chi) \qquad (i \in \{r+1, \cdots, d\}) \\ &= -\textstyle\sum_{j=1}^{j=r} (\omega_{ij} \circ d\chi)(\omega_j \circ d\chi) - \textstyle\sum_{j=r+1}^{j=d} (\omega_{ij} \circ d\chi)(\omega_j \circ d\chi) \\ &= -\textstyle\sum_{j=r+1}^{j=d} (\omega_{ij} \circ d\chi)(\omega_j \circ d\chi) \qquad \text{(Lemma 4).} \end{aligned}$$

So T_1 is involutive. Similarly T_2 is involutive. Thus through each point n of M passes a unique maximal integral submanifold $M^1(n)$ of T_1 and a unique maximal integral submanifold $M^2(n)$ of T_2. Now, by our assumption that Φ is nondegenerately reducible, each integral manifold of T_1 and T_2 is equipped with a (possibly non-positive-definite) riemann metric. We give these submanifolds of M this natural riemann structure. Since T_1 and T_2 are obtained by parallel translation of M_m^1 and M_m^2 respectively along arbitrary curves, we see that each such integral manifold is totally geodesic. By this we mean that a geodesic in the submanifold is also a geodesic in M. In particular, since M is complete, each such integral manifold of T_1 and T_2 is complete.

We now pause to say a few words about the canonical riemann structure of a direct product of riemann manifolds. Let $(M^1, \langle\ ,\ \rangle^1)$ and $(M^2, \langle\ ,\ \rangle^2)$ be two riemann manifolds, and let $N = M^1 \times M^2$. Then we give N the product metric, i.e., if $v \in N_n$, and $v = (v^1, v^2)$, $n = (m^1, m^2)$, then

$$\| v \|^2 = \langle v^1, v^1 \rangle_{m_1}^1 + \langle v^2, v^2 \rangle_{m_2}^2$$

by definition. Thus, $M_{m_1}^1$ and $M_{m_2}^2$ considered as subspaces of N_n are defined to be orthogonal. Since the holonomy group $\bar{\Phi}$ of N is clearly reduced by the tangent subspaces to M^1 and M^2 at each point, we apply the Holonomy Reduction Theorem as above. So we shall only consider, from now on, the holonomy bundle $H(N)$ rather than the frame bundle $F(N)$ in case N is a direct product. Actually it is easy to see that $H(N)$ itself is the direct product $H(M^1) \times H(M^2)$.

Suppose we denote the exponential maps of N, M^1, M^2 by $\overline{\exp}$, $\exp^1$, $\exp^2$, and $\overline{\mathrm{Exp}}$, Exp^1, Exp^2, respectively. A little thought will show that $\overline{\exp} = (\exp^1, \exp^2)$ and $\overline{\mathrm{Exp}} = (\mathrm{Exp}^1, \mathrm{Exp}^2)$. For example,

$$\overline{\exp}\left(\textstyle\sum_{a=1}^{d} c_a e_a\right) = \left(\exp^1\left(\textstyle\sum_{\alpha=1}^{r} c_\alpha e_\alpha\right),\ \exp^2\left(\textstyle\sum_{i=r+1}^{d} c_i e_i\right)\right),$$

where we have assumed $\dim M^1 = r$, $\dim M^2 = d - r$. From this it follows that each geodesic of N is a product of geodesics of M^1, M^2, and vice versa. This remark will be important in the following sections.

We give one final definition. We say two riemann manifolds M and N are *isometric* if and only if there exists a diffeomorphism $\phi : M \rightarrow N$ such that

$$\langle u, v\rangle_m = \langle d\phi(u), d\phi(v)\rangle_{\phi(m)}, \qquad \forall\, m \in M \quad \text{and} \quad \forall\, u, v \in M_m .$$

We can now state the main result of this paper.

THE DECOMPOSITION THEOREM OF DE RHAM. *Let M be a complete simply connected manifold whose metric may be indefinite. Suppose the holonomy group Φ of M at m is nondegenerately reducible; let M_m^1 and $M_m^2 = (M_m^1)^{\perp}$ be the subspaces of M_m left invariant by Φ. Then M is naturally isometric to the direct product of the maximal integral manifolds of the distributions T_1 and T_2, obtained by parallel translation of M_m^1 and M_m^2 over M. More precisely, let M^i be the integral manifold of T_i through m, $i = 1, 2$. Then there exists an isometry $\phi : M^1 \times M^2 \rightarrow M$ which maps (M^1, m) identically onto M^1 and (m, M^2) identically onto M^2.*

3. Local isometry[2]

In this section and the next, we adopt a special convention on subscripts and superscripts. Small Greek indices $\alpha, \beta, \gamma, \cdots$ will run from 1 to r, small Latin indices $i, j, \cdots$ will run from $r + 1$ to d, and capital Latin indices $A, B, C, D, \cdots$ will be allowed to vary from 1 to d.

We now fix our notation for the rest of this section. We fix $m \in M$, $b = (m, e_1, \cdots, e_d) \in H(M)$, and the orthogonal decomposition $M_m = M_m^1 \oplus M_m^2$ so that $e_1, \cdots, e_r$ span M_m^1 and $e_{r+1}, \cdots, e_d$ span M_m^2. M^1 will be the maximal integral manifold of T_1 through m, and M^2 that of T_2 through m also. Note that $T_1(m) = M_m^1$, $T_2(m) = M_m^2$. We proceed to prove the existence of a neighborhood U of m in M isometric to a neighborhood of (m, m) in $M^1 \times M^2$.

We begin with an important observation:

$$(3.1) \qquad \Omega_{ij}(E^A, E^\alpha) \equiv 0, \qquad \Omega_{\alpha\beta}(E^A, E^i) \equiv 0.$$

(The ω_{AB} and Ω_{AB} are, of course, considered as forms defined on $H(M)$ only.)

To prove (3.1) we need the full force of our assumption of the nondegenerate reducibility of Φ. Take $c = (n, f_1, \cdots, f_d) \in H(M)$, and we must show $\Omega_{ij}(E^A, E^\alpha)(c) = \Omega_{\alpha\beta}(E^A, E^i)(c) = 0$. For convenience, we shall write R_{AB} for $R_{d\pi(E^A(c))\, d\pi(E^B(c))} \equiv R_{f_A f_B}$. By definition

$$R_{A\alpha} f_j = \textstyle\sum_D \Omega_{Dj}(E^A, E^\alpha)(c) f_D .$$

Since $(f_1, \cdots, f_d)$ is orthonormal, it suffices to show

$$(3.1)' \qquad \langle R_{A\alpha} f_j, f_i\rangle = 0.$$

By (3) of Lemma 3, it is equivalent to proving

$$\langle R_{\alpha j} f_A, f_i\rangle + \langle R_{jA} f_\alpha, f_i\rangle = 0.$$

[2] Conversations with J. Simons have greatly influenced the presentation of materials in this section.

But (4) of Lemma 3 implies $\langle R_{\alpha j} f_A , f_i\rangle = \langle R_{Ai} f_\alpha , f_j\rangle$. So $(3.1)'$ is again equivalent to

$$\langle R_{Ai} f_\alpha , f_j\rangle + \langle R_{jA} f_\alpha , f_i\rangle = 0.$$

Now by Lemma 4, the curvature transformation leaves invariant $T_1(n)$. Since $f_\alpha \in T_1(n)$ and $f_i , f_j \in T_2(n)$, and $T_1(n)$ is orthogonal to $T_2(n)$, we see that each term of the above is zero. Hence $\Omega_{ij}(E^A, E^\alpha)(c) = 0$; similarly, $\Omega_{\alpha\beta}(E^A, E^i)(c) = 0$. This proves $(3.1)'$ and hence (3.1).

PROPOSITION 1. *Let t be a constant vector field on M_m with $t = \sum_\alpha a_\alpha \, \partial/\partial X_\alpha$; then $\theta_i(t) = 0$. Similarly, if $t^1 = \sum_j a_j \, \partial/\partial X_j$, then $\theta_\beta(t^1) = 0$.*

Proof. We use the same notation as in Lemma 2. By (4) of that lemma, $T^2\theta_i(zt) = \sum_A c_A \, \Theta_{iA}(T, zt)$. By (1) and (3) of Lemma 1, we have

$$\begin{aligned} T^2\theta_i(zt) &= \textstyle\sum_A c_A \, \Omega_{iA}\left(\sum_B c_B \, E^B, \sum_D \theta_D(zt)E^D\right) \\ &= \textstyle\sum_{ABD} c_A \, c_B \, \Omega_{iA}(E^B, E^D)\theta_D(zt) \\ &= \textstyle\sum_{jBD} c_j \, c_B \, \Omega_{ij}(E^B, E^D)\theta_D(zt) && \text{(by Lemma 4)} \\ &= \textstyle\sum_{jkl} c_j \, c_k \, \Omega_{ij}(E^k, E^l)\theta_l(zt) && \text{(by (3.1))}. \end{aligned}$$

$$T^2\theta_i(zt) - \textstyle\sum_{jkl} c_j \, c_k \, \Omega_{ij}(E^k, E^l)\theta_l(zt) = 0. \tag{3.2}$$

We now claim that (3.2) implies $\theta_i(t) = 0$. For (1) of Lemma 2 implies that $\theta_i(zt) = T\theta_i(zt) = 0$ at the origin. Thus (3.2) is a system of linear homogeneous differential equations in $\theta_i(zt)$, $i \in \{r + 1, \cdots , d\}$ with zero initial data along each ray $\gamma(s)$. By uniqueness of solution $\theta_i(zt) = 0$ at all points. So $\theta_i(t) = 0$; similarly $\theta_\alpha(t^1) = 0$, Q.E.D.

PROPOSITION 2. *Let $t = \sum_\alpha a_\alpha \, \partial/\partial X_\alpha$ be a constant vector field on M_m. Suppose p, $p_1 \in M_m$ are such that $p = \sum_A c_A \, e_A$, $p_1 = \sum_\alpha c_\alpha \, e_\alpha$ (i.e., p_1 is the projection of p on M_m^1); then $\theta_\gamma(t)(p) = \theta_\gamma(t)(p_1)$. Similarly, if $t^1 = \sum_i a_i \, \partial/\partial X_i$, and $p_2 = \sum_j c_j \, e_j$, then $\theta_k(t^1)(p) = \theta_k(t^1)(p_2)$.*

Proof. We prove only the first statement. It clearly suffices to prove

$$(\partial/\partial X_j)\theta_\gamma(t) = 0, \qquad \forall j \in \{r + 1, \cdots , d\}. \tag{$*$}$$

Consider the following C^∞ map of the unit square into $H(M)$,

$$\rho : [0, 1] \times [0, 1] \to H(M) \quad \ni \quad \rho = \mathrm{Exp}_b \circ h,$$

where $h : [0, 1] \times [0, 1] \to M_m$ is defined by $h(u, v) = \sum_\alpha u(a_\alpha \, e_\alpha) + v(e_j)$. Clearly, $dh(\partial/\partial u) = t$ and $dh \, (\partial/\partial v) = \partial/\partial X_j$. Now evaluate both sides of the first structure equation (1.2) on $(\partial/\partial u, \partial/\partial v)$; we get, using $[\partial/\partial u, \partial/\partial v] = 0$,

$$\begin{aligned} &t\theta_\alpha(\partial/\partial X_i) - (\partial/\partial X_i)\theta_\alpha(t) \\ &\qquad = \textstyle\sum_A \theta_\alpha(t)\cdot\theta_{\alpha A}(\partial/\partial X_i) - \sum_\alpha \theta_\alpha(\partial/\partial X_i)\theta_\alpha(t). \end{aligned} \tag{3.3}$$

By Proposition 1, the first and last terms vanish. We claim the third term also vanishes. To show this, we will prove $\theta_{\alpha A}(\partial/\partial X_i) = 0$, which is equivalent to showing $\theta_{\alpha\beta}(\partial/\partial X_i) = 0$ by virtue of Lemma 4. For this, we use (3) of Lemma 2 and the same notation as in that lemma.

$$\begin{aligned} T\theta_{\alpha\beta}(z\,\partial/\partial X_i) &= \Theta_{\alpha\beta}(T, z\,\partial/\partial X_i) \\ &= \Omega_{\alpha\beta}\left(\textstyle\sum_B c_B E^B, \sum_D \theta_D(z\,\partial/\partial X_i)E^D\right) \\ &= \Omega_{\alpha\beta}\left(\textstyle\sum_B c_B E^B, \sum_k \theta_k(z\,\partial/\partial X_i)E^k\right) && \text{(by Proposition 1)} \\ &= \textstyle\sum_{B,k} c_B\, \theta_k(z\,\partial/\partial X_i)\Omega_{\alpha\beta}(E^B, E^k) = 0 && \text{(by (3.1))}. \end{aligned}$$

So $T\theta_{\alpha\beta}(z\,\partial/\partial X_i) = 0$ along any ray, while $\theta_{\alpha\beta}(z\,\partial/\partial X_i)(0) = 0$. Hence $\theta_{\alpha\beta}(\partial/\partial X_i) = 0$. Thus (3.3) reduces to $(\partial/\partial X_i)\theta_\alpha(t) = 0$. This is precisely what we set out to prove.

We now give the geometric content of the above two propositions. By using (2) of Lemma 1 and the fact that the integral manifold of an involutive distribution is locally unique, Proposition 1 says the image under $\exp_m$ of the various slices in M_m:

$$(x_1, \cdots, x_r, c_{r+1}, \cdots, c_d), \qquad (c_1, \cdots, c_r, x_{r+1}, \cdots, x_d), \qquad c_A \in R,$$

are precisely the integral manifolds of T_1 and T_2. (Actually, the above is only true if we restrict ourselves to a neighborhood of O in M_m on which $\exp_m$ is a diffeomorphism). Proposition 2 says, furthermore, that these images of the various slices in M_m are in fact isometric to each other. They combine to say that each m has a neighborhood which is isometric to a direct product. We proceed to make these rather vague statements more precise.

First we make some comments about the various forms on $H(N)$ of a direct product $N = M^1 \times M^2$.

Convention. We shall put a bar above all the canonical maps, vector fields, and forms associated with a direct product $N = M^1 \times M^2$, e.g., $\bar\omega_{AB}$, $\bar\Omega_{AB}$, $\bar\theta_{AB}$, $\bar\Theta_{AB}$, $\bar\Phi$, $\overline{\exp}$, $\overline{\mathrm{Exp}}$. As in §2, we shall assume $\dim M^1 = r$, $\dim M^2 = d - r$. Then we have the analogue of (3.1):

$$(3.4) \qquad \bar\Omega_{ij}(\bar E^A, \bar E^\alpha) = 0, \qquad \bar\Omega_{\alpha\beta}(\bar E^A, \bar E^i) = 0.$$

Now let $\bar b = (n, \bar e_1, \cdots, \bar e_d) \in F(N)$. Let $n = (m_1, m_2)$ be fixed in N. If $\bar p = \sum_A c_A \bar e_A \in N_n$, and $\bar p^1 = \sum_\alpha c_\alpha \bar e_\alpha$, $\bar p^2 = \sum_i c_i \bar e_i$ are the projections of $\bar p$ into $M^1_{m_1}$, $M^2_{m_2}$ (considered as subspaces of N_n), let $\bar t = \sum_\alpha a_\alpha\, \partial/\partial \bar X_\alpha$, $\bar t^1 = \sum_i a_i\, \partial/\partial \bar X_i$ be arbitrary constant vector fields on N_n; then by using remarks in §2 about the exponential maps, it is immediate that

$$(3.5) \qquad \bar\theta_\alpha(\bar t)(\bar p) = \bar\theta_\alpha(\bar t)(\bar p^1), \qquad \bar\theta_i(\bar t^1)(\bar p) = \bar\theta_\alpha(\bar t)(\bar p^2),$$

$$(3.6) \qquad \bar\theta_\alpha(\bar t^1) = \bar\theta_i(t) = 0.$$

We now resume the proof of local isometry. We have $m \in M$, integral manifolds M^1, M^2 of T^1, T^2 through m with the induced riemann structure. As above, let $N = M^1 \times M^2$, $\bar{m} = (m, m)$. If $b = (m, e_1, \cdots, e_d) \in H(M)$ as before, then $(e_1, \cdots, e_r)$ is a frame in M^1, and $(e_{r+1}, \cdots, e_d)$ is a frame in M^2. So, we consider $\bar{b} = (\bar{m}, \bar{e}_1, \cdots, \bar{e}_d) \in H(N)$, where each $\bar{e}_A = e_A$, but $\bar{b}$ is now regarded as a tangent vector to N at n. Denote by I the canonical isometry $I : M_m \to N_{\bar{m}}$ such that $I(e_A) = \bar{e}_A$. Finally, since we have observed that M^1 and M^2 are totally geodesic submanifolds of M, it is then clear that $\exp_m$ maps $M_m^1 = \operatorname{Span}(e_1, \cdots, e_r)$ and $M_m^2 = \operatorname{Span}(e_{r+1}, \cdots, e_d)$ into M^1 and M^2 respectively.

Now it is well known that $\exp_m$ is nonsingular at the origin O of M_m, and so there is a neighborhood U of O on which $\exp_m$ is diffeomorphic. We may well assume $U = U^1 \times U^2$, where U^2 is a neighborhood of O in M_m^i ($i = 1, 2,$). We assert that

$$f \equiv \overline{\exp}_{\bar{m}} \circ I \circ (\exp_m \mid U)^{-1} \text{ is an isometry of } (\exp_m U) \text{ onto } (\exp_m U^1, \exp_m U^2) \subseteq N. \tag{3.7}$$

(Clearly, for the above to make sense, one has to make certain obvious identifications of M^1 as a subset of N, M_m^1 as subspace of N_n, etc.)

We now prove (3.7). f is a diffeomorphism by choice, so we prove df preserves the metric at each tangent space. We convert U into a new manifold $(U, \langle\ ,\ \rangle')$ by defining the new metric:

$$\| t \|' = \left\| \textstyle\sum_A \theta_A(t)(p)e_A(p) \right\| = \left\| \textstyle\sum_A \theta_A(t)(p)e_A \right\|$$

for $p \in U$, $t \in (M_m)_p$.

Then (2) of Lemma 1 shows $(U, \langle\ ,\ \rangle')$ is isometric to $\exp_m U$ under $\exp_m$. Similarly we define a new manifold structure on $U^1 \times U^2$ by

$$\begin{aligned} \| t \|'' &= \left\| \textstyle\sum_\alpha \bar{\theta}_\alpha(t)(p^1)\bar{e}_\alpha(p^1) \right\| + \left\| \textstyle\sum_i \bar{\theta}_i(t)(p^2)\bar{e}_i(p^2) \right\| \\ &= \left\| \textstyle\sum_\alpha \bar{\theta}_\alpha(t)(p^1)\bar{e}_\alpha \right\| + \left\| \textstyle\sum_i \bar{\theta}_i(t)(p^2)\bar{e}_i \right\| \end{aligned}$$

for $(p^1, p^2) \in U^1 \times U^2$, $t \in (N_n)_{(p^1, p^2)}$. Again (2) of Lemma 1 and (3.5), (3.6) show that $\exp_{\bar{m}}$ is an isometry of $(U^1 \times U^2, \langle\ ,\ \rangle'')$ with $\exp_{\bar{m}}(U^1 \times U^2)$. It follows that f is an isometry, by Propositions 1 and 2.

We summarize what we have proved so far in

PROPOSITION 3. *Suppose M is any riemann manifold such that its restricted holonomy group is nondegenerately reducible. Then each point of M has a neighborhood which is isometric to a direct product. The explicit product structure is exhibited in* (3.7).

COROLLARY. *With notation as above, let $p = \sum_A a_A e_A$ be in U. Then*

$$\operatorname{Exp}_b p = \operatorname{Exp}_{b_1}\left(\textstyle\sum_i a_i e_i(n_1)\right) = \operatorname{Exp}_{b_2}\left(\textstyle\sum_\alpha a_\alpha e_\alpha(n_2)\right), \tag{3.8}$$

$$\begin{aligned} &\Omega_{\alpha\beta}(E^\gamma(\operatorname{Exp}_b p), E^\delta(\operatorname{Exp}_b p)) \\ &= \Omega_{\alpha\beta}(E^\gamma(\operatorname{Exp}_b p^1), E^\delta(\operatorname{Exp}_b p^1)) \\ &= \Omega_{\alpha\beta}(E^\gamma(\operatorname{Exp}_{b_1}(\textstyle\sum_i a_i e_i(n_1))), E^\delta(\operatorname{Exp}_{b_1}(\textstyle\sum_i a_i e_i(n_1)))), \end{aligned} \tag{3.9}$$

where

$$b_1 = \operatorname{Exp}_b p^1 = (n_1, e_1(n_1), \cdots, e_d(n_1)),$$

$$b_2 = \operatorname{Exp}_b p^2 = (n_2, e_1(n_2), \cdots, e_d(n_2)).$$

Remark. Note that (3.8) includes as a special case

$$\exp_m p = \exp_{n_1}(\textstyle\sum_i a_i e_i(n_1)) = \exp_{n_2}(\textstyle\sum_\alpha a_\alpha e_\alpha(n_2)). \tag{3.10}$$

Proof. Simply observe that the corresponding statements are true for N because of the splitting of $\overline{\operatorname{Exp}}$, and then apply (3.7).

Remark. In Proposition 3, we have not assumed M to be complete nor M simply connected, so we get only a local theorem. In the following section, we will see that as a result of these assumptions, we will get a global product structure for M.

4. Global isometry

Our convention on subscripts and superscripts set up in §3 still holds.

For the rest of this section, we fix the following: a complete, simply-connected riemann manifold M, the holonomy group Φ of M is nondegenerately reducible, $H(M)$ is the holonomy bundle. Further, for $m \in M$, we have $b = (m, e_1, \cdots, e_d) \in H(M)$, a decomposition $M_m = M_m^1 \oplus M_m^2$ such that M_m^1 and M_m^2 reduce Φ, $M_m^1 = \operatorname{Span}(e_1, \cdots, e_r)$, $M_m^2 = \operatorname{Span}(e_{r+1}, \cdots, e_d)$. If a point $p \in M_m$, p^1, p^2 will always denote the projections of p onto M_m^1 and M_m^2 respectively. Thus, if $p = \sum_A a_A e_A$, then $p^1 = \sum_\alpha a_\alpha e_\alpha$ and $p^2 = \sum_i a_i e_i$. This notation will be used without further comment.

We find it convenient to introduce an abbreviation. We will let $\Omega(A, B)$ stand for $\Omega_{AB}(E^A, E^B)$. Then the following proposition, which is the basic result of this section, can be stated as follows:

PROPOSITION 4. *Let $p \in M_m$, $p = \sum_A a_A e_A$. Let σ be the geodesic $\sigma(t) = \exp_m tp$, $t \in [0, 1]$, and σ_i the geodesic $\sigma_i(t) = \exp_m tp^i$, where $i = 1, 2$, $t \in [0, 1]$. Then*

$$\begin{aligned} \Omega(\alpha, \beta)(\operatorname{Exp}_b p) &= \Omega(\alpha, \beta)(\operatorname{Exp}_b p^1), \\ \Omega(i, j)(\operatorname{Exp}_b p) &= \Omega(i, j)(\operatorname{Exp}_b p^2). \end{aligned} \tag{4.1}$$

Remark. It will be noted that we will actually prove more than the above.

Proof. We will prove the first statement only. We fix our notation: Let $n = \sigma(1) = \exp_m p$, $n_1 = \exp_m p^1$, $\operatorname{Exp}_b tp = (\sigma(t), e_1(t), \cdots, e_d(t))$. For brevity, we write, sometimes, $b(t)$ for $\operatorname{Exp}_b tp$. We now construct a one-parameter family of singly broken geodesics τ_s, $s \in [0, 1]$, $\tau_s : [0, 1] \to M$ as follows. The first segment of τ_s is $\sigma \mid [0, s]$, σ being the same σ as in Proposition 4. The second segment of τ_s is $\tau_s' : [0, 1 - s] \to M$ where $\tau_s'(t) \equiv \tau_s(s + t) = \exp_{\sigma(s)} t(\sum_\alpha a_\alpha e_\alpha(s))$.

Consider these three statements for a number $s \in [0, 1]$:

(A) $\Omega_{\alpha\beta}(\operatorname{Exp}_b p^1) = \Omega_{\alpha\beta}(\operatorname{Exp}_{b(s)} p^1(s))$ *where* $p^1(s) = \sum_\alpha a_\alpha e_\alpha(s)$.

(B) $\exp_{n_1} s(\sum_i a_i\, e_i(n_1)) = \exp_{\sigma(s)} p^1(s)$ *where*
$\mathrm{Exp}_b\, p^1 = (n_1, e_1(n_1), \cdots, e_d(n_1))$.

(C) $\mathrm{Exp}_{b(s)}\, p^1(s) = \mathrm{Exp}_{b_1}\, s(\sum_i a_i\, e_i(n_1))$ *where* $b_1 = \mathrm{Exp}_b\, p^1$.

We define a set $\mathfrak{S} \subseteq [0, 1]$:

$$\mathfrak{S} = \{u : u \in [0, 1] \ni r \leqq u \text{ implies } \tau_r \text{ has properties (A), (B), (C)}\}.$$

Clearly $0 \in \mathfrak{S}$, so $\mathfrak{S}$ is nonempty. We would like to show $1 \in \mathfrak{S}$, i.e., $\mathfrak{S} = [0, 1]$. This fact, as a special case, would prove Proposition 4. To show $\mathfrak{S} = [0, 1]$, we will show $\mathfrak{S}$ is both open and closed in [0, 1], which is clearly sufficient. The proof is broken into two parts.

(i) $\mathfrak{S}$ *is closed.*

Proof of (i). Let $w_k \to w$, $w_k \in \mathfrak{S}$, $\forall\, k$. We must show $w \in \mathfrak{S}$, i.e., τ_w has properties (A), (B), and (C). To do this, we have to introduce a positive-definite riemann metric g on M. Assume this is done; define

$$B(x, \delta) = \{t \in M_x : g(t, t) < \delta\}.$$

Since the image of τ_w is a compact set, it is well known that there exists $\delta > 0$ such that $\exp_x$ is a diffeomorphism on $B(x, \delta)$, $\forall\, x \in$ the image of τ_w. Pick such a δ. Recall we denoted the second segment of τ_w by τ_w', i.e., $\tau_w'(t) = \tau_w(w + t)$, $0 \leqq t \leqq 1 - w$. Let

$$\mathrm{Exp}_{b(w)}\, t(\textstyle\sum_\alpha a_\alpha\, e_\alpha(w)) = (\tau_w'(t), e_1^\star(t), \cdots, e_d^\star(t)), \quad t \in [0, 1 - w].$$

So $(\tau_w'(0), e_1^\star(0), \cdots, e_d^\star(0)) = (\tau_w(w), e_1(w), \cdots, e_d(w))$.

Let $T(M)$ be the tangent bundle of M with natural projection $\bar{\pi} : T(M) \to M$. Then the positive-definite riemann metric g defines, in a natural way, a C^∞ real-valued function $g : \bar{\pi}^{-1}(\tau_w') \to \mathbf{R}$. Define the compact set $C \subseteq \bar{\pi}^{-1}(\tau_w')$ by

$$C = \{(x, q)\colon \text{if } x = \tau_w'(t),\ 0 \leqq t \leqq 1 - w, \text{ then } q = \textstyle\sum_A \eta_A\, a_A\, e_A^\star(t), \text{ where each } \eta_A \in [-1, 1]\}.$$

On this C, g attains a maximum, call it L. Choose $\mu \in \mathbf{Z}$ so large that $L/\mu < \delta/2 < \delta$. We now define a finite sequence of points

$$\{y_0, y_1, \cdots, y_{\mu-1}, y_\mu\}$$

on τ_w' as follows:

$$
\begin{aligned}
y_0 &= \sigma(w) = \tau_w(w) = \tau_w'(0),\\
&\vdots\\
y_l &= \exp_{\sigma(w)}(l/\mu)(1 - w)(\textstyle\sum_\alpha a_\alpha\, e_\alpha(w)) = \tau_w'((l/\mu)(1 - w)),\\
&\vdots\\
y_\mu &= \exp_{\sigma(w)}(1 - w)(\textstyle\sum_\alpha a_\alpha\, e_\alpha(w)) = \tau_w(1) = \tau_w'(1 - w).
\end{aligned}
\tag{4.2}
$$

We observe that, by choice of μ and σ, each

$$y_l \in \exp_{y_{l-1}} B(y_{l-1}, \sigma) \cap \exp_{y_{l+1}} B(y_{l+1}, \sigma).$$

Furthermore, from the elementary theory of solutions of differential equations, if we consider the geodesic τ_w' as integral curve of the vector field τ_{w_*}' (recall that $\tau_{w_*}'(t)$ is the tangent to τ_w' at $\tau_w(t)$), we see easily that

$$\begin{aligned} y_{l+1} &= \exp_{y_l}(1/\mu)(1-w)\left(\textstyle\sum_\alpha a_\alpha e_\alpha^\star((l/\mu)(1-w))\right), \\ y_{l-1} &= \exp_{y_l}(-1/\mu)(1-w)\left(\textstyle\sum_\alpha a_\alpha e_\alpha^\star((l/\mu)(1-w))\right). \end{aligned} \tag{4.3}$$

Denote by $B(t) \subseteq M_{\tau_w'(t)}$, $0 \leqq t \leqq 1-w$, the "box":

$B(t) = \{(\tau_w'(t), q) : q = (1/\mu)\left(\sum_A \eta_A a_A e_A(t)\right)$ where each $\eta_A \in [-1, 1]\}$.

By choice of δ and μ, we have this crucial fact:

(4.4) $\exp_{\tau_w'(t)}$ *is a diffeomorphism on* $B(t)$, $0 \leqq t \leqq 1-w$.

We now start toward the proof of (i). First we claim that there exists $Q \in \mathbf{Z}$ such that $\{\tau_{w_k}(w_k)\} \subseteq \exp_{y_0} B(0)$ for all $k \geqq Q$. This is so because $((-1/\mu)\sigma_*(w)) = (-1/\mu)(\sum_A a_A e_A(w)) = (-1/\mu)(\sum_A a_A e_A^\star(0)) \in B(0)$. So we pick such a $k \geqq Q$ and fix τ_{w_k}'. Just as in (4.2) we define on τ_{w_k}' a finite sequence of points $(x_{-1}, x_0, x_1, \cdots, x_\mu)$ as follows:

$$\begin{aligned} x_{-1} &= \sigma(w_k) = \tau_{w_k}(w_k) = \tau_{w_k}'(0), \\ x_0 &= \exp_{\sigma(w_k)}(w-w_k)\left(\textstyle\sum_\alpha a_\alpha e_\alpha(w_k)\right), \\ &\vdots \\ x_l &= \exp_{\sigma(w_k)}\{(w-w_k)\textstyle\sum_\alpha a_\alpha e_\alpha(w_k) + (l/\mu)(1-w)\sum_\alpha a_\alpha e_\alpha(w_k)\}, \\ &\vdots \\ x_\mu &= \exp_{\sigma(w_k)}\{(w-w_k)\textstyle\sum_\alpha a_\alpha e_\alpha(w_k) + (1-w)\sum_\alpha a_\alpha e_\alpha(w_k)\} \\ &= \exp_{\sigma(w_k)}(1-w_k)\textstyle\sum_\alpha a_\alpha e_\alpha(w_k) = \tau_{w_k}(1). \end{aligned} \tag{4.5}$$

Just as in (4.3), we prove easily, by letting

$$\mathrm{Exp}_{b(w_k)}\, t\left(\textstyle\sum_\alpha a_\alpha e_\alpha(w)\right) = (\tau_{w_k}'(t), e_1^{\star\star}(t), \cdots, e_d^{\star\star}(t))$$

that if $l = 1, \cdots, \mu$,

$$\begin{aligned} x_{l+1} &= \exp_{x_l}(1/\mu)(1-w)\left(\textstyle\sum a_\alpha e_\alpha^{\star\star}((l/\mu)(1-w))\right), \\ x_{l-1} &= \exp_{x_l}(-1/\mu)(1-w)\left(\textstyle\sum a_\alpha e_\alpha^{\star\star}(l/\mu)(1-w)\right). \end{aligned} \tag{4.6}$$

We claim

$$\begin{aligned} x_{-1} &= \exp_{y_0}\{-(w-w_k)\textstyle\sum_A a_A e_A^\star(0)\}, \\ x_0 &= \exp_{y_0}\{-(w-w_k)\textstyle\sum_i a_i e_i^\star(0)\}. \end{aligned} \tag{4.7}$$

For, by (4.4) and Proposition 3, $\exp_{y_0}(B(0))$ has a product structure. Then the second follows from (4.5) and a trivial variant of (3.10). The first is obvious, proving (4.7). Now (4.7) and (3.9) together imply that

$$(4.8)\quad \Omega(\alpha, \beta)(b(w)) = \Omega(\alpha, \beta)(\mathrm{Exp}_{b(w_k)}(w - w_k)\textstyle\sum_\alpha a_\alpha e_\alpha^{\star\star}(0)).$$

Next, still considering the product structure on $\exp_{y_0}(B(0))$, by (4.3) and (4.6) one sees easily that

$$x_1 = \exp_{y_1}\{-(w - w_k)\textstyle\sum_i a_i e_i^\star((1/\mu)(1 - w))\}.$$

In other words

$$x_1 = \exp_{y_0}\{(1/\mu)(1 - w)(\textstyle\sum_\alpha a_\alpha e_\alpha^\star(0)) - (w - w_k)\sum_i a_i e_i^\star(0)\}.$$

So, using (3.9) (or a variant of it) and (4.8), we have

$$\Omega(\alpha, \beta)(\mathrm{Exp}_{b(w)}(1/\mu)(\textstyle\sum_\alpha e_\alpha^\star(0)))$$
$$= \Omega(\alpha, \beta)(\mathrm{Exp}_{\bar b}(1/\mu)(\textstyle\sum_\alpha a_\alpha e_\alpha((w - w_k) + (1/\mu)(1 - w)))),$$

where $\bar b = (x_0, e_1^{\star\star}(w - w_k), \cdots, e_d^{\star\star}(w - w_k))$. Observing that

$$\mathrm{Exp}_{\bar b}(1/\mu)(\textstyle\sum_\alpha a_\alpha e_\alpha((w - w_k) + (1/\mu)(1 - w)))$$
$$= \mathrm{Exp}_{b(w_k)}\{(w - w_k)\textstyle\sum_\alpha a_\alpha e_\alpha^{\star\star}(0) + (1/\mu)(1 - w)\sum_\alpha a_\alpha e_\alpha^{\star\star}(0)\}$$

we obtain, combining the above,

$$(4.9)\quad \Omega(\alpha, \beta)(\mathrm{Exp}_{b(w)}(1/\mu)(\textstyle\sum_\alpha a_\alpha e_\alpha^\star(0))) = \Omega(\alpha, \beta)(b(w_k, 1/\mu)),$$

where we have introduced the abbreviation:

$$b(w_k, \nu/\mu)$$
$$\equiv \mathrm{Exp}_{b(w_k)}\{(w - w_k)\textstyle\sum_\alpha a_\alpha e_\alpha^{\star\star}(0) + (\nu/\mu)(1 - w)\sum_\alpha a_\alpha e_\alpha^{\star\star}(0)\}$$

for $\nu \in \mathbf{Z}$. (Geometrically, $b(w_k, \nu/\mu)$ is nothing other than the frame over x_ν obtained by parallel-translating $(e_1^{\star\star}(0), \cdots, e_d^{\star\star}(0))$ along τ'_{w_k}.) It is now clear that we can pass from x_1 to x_2, from y_1 to y_2 and obtain equality on $\Omega(\alpha, \beta)$ of type (4.9). This is done essentially as above, by observing that we have a product structure on $\exp_{y_1}(B((1/\mu)(1 - w)))$, and then applying (3.9) and (3.8). Repeating this operation, we obtain, after μ-steps

$$\Omega(\alpha, \beta)(\mathrm{Exp}_{b(w)}(\mu/\mu)(\textstyle\sum_\alpha e_\alpha^\star(0))) = \Omega(\alpha, \beta)(b(w_k, \mu/\mu)).$$

This expression, when we refer back to our definitions, becomes

$$(4.10)\quad \begin{aligned}&\Omega(\alpha, \beta)(\mathrm{Exp}_{b(w)}(1 - w)\textstyle\sum_\alpha e_\alpha^\star(0))\\ &\qquad = \Omega(\alpha, \beta)(\mathrm{Exp}_{b(w_k)}(1 - w_k)\textstyle\sum_\alpha a_\alpha e_\alpha^{\star\star}(0)).\end{aligned}$$

Now τ_{w_k} has property (A) by hypothesis. So (4.10) says τ_w also has property (A). To finish the proof of (i), we need only to show τ_w has properties (B) and (C). We first show τ_w has property (B). So we must show $\tau_w(1)$ lies on $\exp_{n_1} s(\sum_i a_i e_i(n_1))$. In the same manner that one proves (4.7), one gets by iterating the same process μ-times

$$(4.11)\quad \begin{aligned} \tau_{w_k}(1) = x_\mu &= \exp_{y_\mu}\{-(w - w_k)\textstyle\sum_i a_i e_i^\star(1-w)\} \\ &= \exp_{\tau_w(1)}\{-(w-w_k)\textstyle\sum_i a_i e_i^\star(1-w)\}. \end{aligned}$$

The relation (4.11) is true for every τ_{w_k} if $K \geqq Q$, so all such $\tau_{w_k}(1)$ lie in the geodesic $\gamma : [0, 1] \to M$ such that

$$\gamma(t) = \exp_{\tau_w(1)}\{-(w - w_Q)\textstyle\sum_i a_i e_i^\star(1-w)\}.$$

By assumption, all such $\tau_{w_k}(1)$ lie on $\exp_{n_1} s(\sum_i a_i e_i(n_1))$. Now observe that

$$\tau_{w_k}(1) \in \exp_{\tau_w(1)} B(1 - w)$$

if $K \geqq Q$ (recall that by choice, $\tau_{w_k}(w_k) \in \exp_{y_0} B(0)$). Since there exists only one geodesic in $\exp_{\tau_w(1)} B(1 - w)$ that joins $\tau_w(1)$ to any other point of $\exp_{\tau_w(1)} B(1 - w)$, we see that

$$\gamma(t) \subseteq \exp_{n_1} s(\textstyle\sum_i a_i e_i(n_1))$$

set-theoretically. But $\gamma(0) = \tau_w(1)$, so $\tau_w(1) = \exp_{n_1} w(\sum_i a_i e_i(n_1))$. This shows τ_w has property (B).

Finally to show τ_w has property (C), we merely have to note that on each $\exp_{y_l} B((l/\mu)(1 - w))$ there is a product structure. Each τ_{w_k} for $K \geqq Q$ has property (C). Then an obvious application of (3.8) plus the fact that τ_w has been shown to have property (B) yield the desired conclusion. The proof of (i) is complete.

(ii) *$\mathfrak{S}$ is open.*

Proof of (ii). If $w \in \mathfrak{S}$, we must show there exists $\varepsilon > 0$ such that $(w + \varepsilon) \in \mathfrak{S}$. Using notation as in (i), what we have to do is to choose an $l \in \mathbf{R}$ so small that $\exp_{\tau_w(t)}$ is a diffeomorphism on $B(t)$, where $B(t)$ is defined in essentially the same way as in the proof of (i); cf. discussion preceding (4.4). Then take $\varepsilon < l$, and we have to show each $\tau_{w+t\varepsilon}$ for $t \in [0, 1]$ has properties (A), (B), (C). This can be done in the same manner as in (i) with trivial modifications, so we leave out the details.

Corollary. *With notation as in the preceding proposition,*

$$(4.12)\quad \operatorname{Exp}_b p = \operatorname{Exp}_{b_1}(\textstyle\sum_i a_i e_i(n_1)) = \operatorname{Exp}_{b_2}(\textstyle\sum_\alpha a_\alpha e_\alpha(n_2)),$$

where we denote by b_1, b_2, the elements of $H(M)$:

$$b_1 = \operatorname{Exp}_b(\textstyle\sum_\alpha a_\alpha e_\alpha) = (n_1, e_1(n_1), \cdots, e_d(n_1)),$$

$$b_2 = \operatorname{Exp}_b(\textstyle\sum_i a_i e_i) = (n_2, e_1(n_2), \cdots, e_d(n_2)).$$

In particular,

$$(4.13)\quad \exp_m p = \exp_{n_1}(\textstyle\sum_i a_i e_i(n_1)) = \exp_{n_2}(\textstyle\sum_\alpha a_\alpha e_\alpha(n_2)).$$

Proof. (4.13) and (4.12) merely state that $\tau_1 = \sigma$ has properties (B) and (C).

We now deduce a slightly more general fact from Proposition 4. Consider a singly broken geodesic γ in M, such that the first segment of γ is

$$\sigma : [0, 1] \to M, \qquad \sigma(t) = \exp_m tp$$

(notation as in Proposition 4). Let the second segment τ of γ be represented by

$$\tau : [0, 1] \to M, \qquad \tau(t) = \exp_n t(\textstyle\sum_A c_A e_A(1)),$$

where $\operatorname{Exp}_b p = (n, e_1(1), \cdots, e_d(1))$ as usual. Corresponding to γ, we have for M^1, M^2 respectively, singly broken geodesics γ_1, γ_2, defined as follows: The first segment of γ_1 is

$$\sigma_1 : [0, 1] \to M^1, \qquad \sigma_1(t) = \exp_m tp^1,$$

the second segment of γ_1 is τ_1,

$$\tau_1 : [0, 1] \to M^1, \qquad \tau_1(t) = \exp_{n_1} t(\textstyle\sum_\alpha c_\alpha e_\alpha(n_1)),$$

where $\operatorname{Exp}_b p^1 = (n_1, e_1(n_1), \cdots, e_d(n_1))$ as usual. γ_2 is defined similarly.

We assert that Proposition 4 implies that

$$(4.14)\qquad \begin{aligned} \Omega(\alpha, \beta)(\operatorname{Exp}_{b(1)} \textstyle\sum_A c_A e_A(1)) &= \Omega(\alpha, \beta)(\operatorname{Exp}_{b_1} \textstyle\sum_\alpha c_\alpha e_\alpha(n_1)), \\ \Omega(i, j)(\operatorname{Exp}_{b(1)} \textstyle\sum_A c_A e_A(1)) &= \Omega(i, j)(\operatorname{Exp}_{b_2} \textstyle\sum_i c_i e_i(n_2)), \end{aligned}$$

where $b(1) = \operatorname{Exp}_b p$, $b_1 = \operatorname{Exp}_b p^1$, $b_2 = \operatorname{Exp}_b p^2$. To prove (4.14), first note that we can join n_1 to $\sigma(1) = n$ by $\rho : [0, 1] \to M$, with

$$\rho(t) = \exp_{n_1} t(\textstyle\sum_i a_i e_i(n_1)).$$

This is (4.13). So consider the singly broken geodesic σ' with first segment equal to ρ, second segment equal to τ. Let

$$\operatorname{Exp}_{b_1}(\textstyle\sum_i a_i e_i(n_1)) = b'(1) = (n, e'(1), \cdots, e'_d(1)).$$

Exactly the same technique used to prove Proposition 4 proves, for the geodesics σ' and τ_1, the following:

$$\Omega(\alpha, \beta)(\operatorname{Exp}_{b'(1)} \textstyle\sum_A a_A e'_A(1)) = \Omega(\alpha, \beta)(\operatorname{Exp}_{b_1} \textstyle\sum_\alpha c_\alpha e_\alpha(n_1)).$$

But according to (4.12), $b'(1) = b(1)$, and according to (4.1),

$$\Omega(\alpha, \beta)(\operatorname{Exp}_b p) = \Omega(\alpha, \beta)(\operatorname{Exp}_b p^1).$$

Combining these three facts, one has the first statement of (4.14). The second follows similarly.

The main idea of our proof of the de Rham theorem is, of course, to be able to apply the Ambrose-Hicks theorem [2], [3, Theorem 1]. To do that, we need to know the meaning of "parallel translation of curvature is the same along finitely broken geodesics in two different manifolds." A precise and complete definition of this term has been very nicely done in Hicks [3]. So we only explain the term in the special case of our M and $N = M^1 \times M^2$ and only for singly broken geodesics. For complete details, we refer to Hicks [3].

We continue our convention about putting bars above maps, forms, and fields of the product manifold $N = M^1 \times M^2$. So we have chosen

$$b = (m, e_1, \cdots, e_d) \in H(M), \qquad \bar{b} = (\bar{m}, \bar{e}_1, \cdots, \bar{e}_d) \in H(N)$$

(with $\bar{m} = (m, m)$), the canonical isometry $I : M_m \to N_{\bar{m}}$, $I(e_A) = \bar{e}_A$. All these were previously defined in §3 and will be fixed once and for all.

Consider, as before, the once-broken geodesic γ, with first segment equal to σ, $\sigma(t) = \exp_m tp$, second segment equal to τ, $\tau(t) = \exp_n t(\sum_A c_A e_A(1))$. All parameters run from 0 to 1, i.e., $t \in [0, 1]$, where

$$\operatorname{Exp}_b p = (n, e_1(1), \cdots, e_d(1)).$$

The isometry I of M_m to $N_{\bar{m}}$ gives us a corresponding geodesic $\bar{\gamma}$ in N as follows. The first segment $\bar{\sigma}$ of $\bar{\gamma}$ is $\bar{\sigma}(t) = \overline{\exp}_{\bar{m}} t\bar{p}$ (i.e., $\bar{p} = \sum_A a_A \bar{e}_A$) and the second segment $\bar{\tau}$ of $\bar{\gamma}$ is $\bar{\tau}(t) = \overline{\exp}_{\bar{n}} t(\sum_A c_A \bar{e}_A(1))$, where $\overline{\operatorname{Exp}}_{\bar{b}} \bar{p} = (\bar{n}, \bar{e}_1(1), \cdots, \bar{e}_d(1))$. We now say that parallel translation of curvature is the same in M and N, with respect to b and $\bar{b}$, if for every such γ and $\bar{\gamma}$

$$\begin{aligned} \Omega_{AB}(E^C, E^D)&(\operatorname{Exp}_{b(1)}(\textstyle\sum c_A e_A(1))) \\ &= \bar{\Omega}_{AB}(\bar{E}^C, \bar{E}^D)(\overline{\operatorname{Exp}}_{\bar{b}(1)}(\textstyle\sum c_A \bar{e}_A(1))), \end{aligned} \tag{4.15}$$

where clearly $b(1) = \operatorname{Exp}_b p$, $\bar{b}(1) = \overline{\operatorname{Exp}}_{\bar{b}} \bar{p}$.

We assert that (4.14) already implies

(4.16) *Parallel translation of curvature is the same in M and N with respect to b and $\bar{b}$ along singly broken geodesics.*

Proof of (4.16). We first consider the Ω of M. By Lemma 3, $\Omega_{AB}(E^C, E^D)$ is completely determined by $\Omega_{AB}(E^A, E^B)$, i.e., knowing the latter for all A, B, implies we know the former for all A, B, C, D. This is classical, and we shall not give a proof. So we only need to consider $\Omega(A, B)$. Now (3.1) is trivially equivalent to

$$\Omega(E^\alpha, E^i) = 0, \qquad \forall\, \alpha \in \{1, \cdots, r\}, i \in \{r+1, \cdots, d\}.$$

So, for M, all we need to know is the parallel translation of $\Omega(\alpha, \beta)$ and $\Omega(i, j)$. By virtue of (3.4), similarly, we only need consider $\bar{\Omega}(\alpha, \beta)$ and $\bar{\Omega}(i, j)$. Now if we identify (M^1, m) and (m, M^2) in the canonical way, account being taken of the splitting of $\overline{\exp}$ and $\overline{\operatorname{Exp}}$ (as remarked in §2), one sees without difficulty that the content of (4.14) is precisely (4.16), Q.E.D.

It is now clear that the technique employed in proving (4.14), which is the heart of (4.16), can be extended immediately to cover the case of geodesics with any number of breaks. One finally reaches the conclusion:

(4.17) *Parallel translation of curvature of M and N with respect to b and $\bar{b}$ is the same along finitely broken geodesics.*

Proof of the de Rham Decomposition Theorem. Take the universal covering $\tilde{M}^\alpha$ of M^α, $\alpha = 1, 2$. Let $\tilde{N} = \tilde{M}^1 \times \tilde{M}^2$. We first show $\tilde{N}$ isometric to M. To apply the Ambrose-Hicks theorem, we need to know parallel translation of torsion and curvature to be the same in $\tilde{N}$ and M (with respect to $\tilde{b}$ and b, where $\tilde{b}$ is now any element that covers $\bar{b}$ in N). Torsion being zero for riemann connections, the first part is trivial. Since N and $\tilde{N}$ are locally isometric, parallel translation of curvature is the same in N and $\tilde{N}$, relative to $\bar{b}$ and $\tilde{b}$. So, the second part is taken care of by (4.17). Then, according to Ambrose-Hicks, there is a connection-preserving diffeomorphism of $\tilde{N}$ onto M, such that $(d\phi)_{\tilde{b}} = I$. But since ϕ induces the isometry I at $\bar{m}$, we see that ϕ is actually a global isometry. The fact that $(d\phi)_{\tilde{b}} = I$ also shows immediately that ϕ maps $(\tilde{M}^1, m)$ onto $M^1 \subseteq M$, $(m, \tilde{M}^2)$ onto $M^2 \subseteq M$. But ϕ being 1:1, $\phi \mid (\tilde{M}^1, m)$ and $\phi \mid (m, \tilde{M}^2)$ are in fact global isometries. Hence M^α ($\alpha = 1, 2$) is already simply connected to start with, and $\phi : M^1 \times M^2 \approx M$, Q.E.D.

5. Concluding remarks

(1) It is natural to ask the following question: Does there exist a similar decomposition theorem for a general affinely connected manifold? We shall formulate precisely one version of this problem. Let M be a d-dimensional manifold with a torsionless connection such that its holonomy group Φ at m preserves a *pair* of supplementary proper subspaces M_m^1, M_m^2 of M_m. Then define as before the holonomy bundle $H(M)$, distributions T_1, T_2, and their maximal integral manifolds M^1, M^2. We say two manifolds are CP-diffeomorphic if there is a diffeomorphism between them that preserves the connection. Then the problem can be stated as follows:

(5.1) *With M as above, is M locally CP-diffeomorphic to $M^1 \times M^2$?*

We remark that since the global part of our proof is independent of our connection being riemannian (inasmuch as the theorem of Ambrose-Hicks does not depend on that fact), once we have established a local product structure for M, simple-connectivity plus completeness will always give us a global result.

(2) It may be interesting to note that there exists a necessary and sufficient condition on the curvature form for (5.1) to admit an affirmative answer (which is what one expects).

($*$) *M is CP-locally-diffeomorphic to $M^1 \times M^2$ if and only if $\Omega(E^\alpha, E^i) \equiv 0$ on $H(M)$, for all $\alpha \in \{1, \cdots, r\}$, for all $i \in \{r+1, \cdots, d\}$.*

(3) We turn to examples in which the answer to (5.1) is negative. In the case where the connection is not riemannian, H. Ozeki has shown that the answer to (5.1) is negative. He constructed a torsionlsss linear connection on the plane R^2 with reducible holonomy; but R^2 does not admit a local product structure in that connection. The case where the connection is riemannian was first suggested to us by J. A. Wolf. One naturally tries to prove the condition in $(*)$. It can be seen that the proof of (3.1), which corresponds to $\Omega(E^\alpha, E^i) \equiv 0$, breaks down precisely when Φ is not nondegenerately reducible. So one may suspect that without the nondegenerate reducibility condition, the answer to (5.1) is negative too. That this is indeed the case is shown by the following example. (We are indebted to Holzsager for this.) Equip R^2 with the Lorentz metric: Relative to the canonical coordinates,

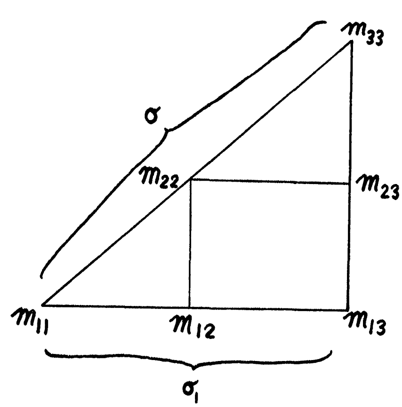
$$g_{11}(x_1, x_2) = \cos x_1, \qquad g_{22}(x_1, x_2) = -\cos x_1,$$

$$g_{12}(x_1, x_2) = g_{21}(x_1, x_2) = \sin x_1.$$

At the origin O of R^2, the null directions are $(1, 1)$, $(-1, 1)$. (We have identified R_0^2 with R^2.) Let $S^1 = \mathrm{Span}(1, 1)$, $S^2 = \mathrm{Span}(-1, 1)$. R^2 being simply connected, the holonomy group Φ lies in the identity component of $\mathrm{PO}(2)$, and so each element of Φ has determinant $+1$. Since $\Phi \subseteq \mathrm{PO}(2)$, Φ either (a) leaves invariant S^1, S^2, or (b) flips S^1 onto S^2 and vice versa. If $A \in \mathrm{PO}(2)$ does (b), it has determinant -1. So Φ leaves invariant S^1, S^2.

To finish our counterexample, one can either compute $\Omega(\bar{S}^1, \bar{S}^2) \neq 0$ directly ($\bar{S}^1$, $\bar{S}^2$ being horizontal lifts of $(1, 1)$ and $(-1, 1)$), or else observe that $M^1 \times M^2$ is flat (M^i being the null geodesic tangent to S^i, $i = 1, 2$) while R^2 in that metric is not.

(4) The rather complicated proof of Proposition 4 can be considerably simplified in the case of a positive-definite metric. We sketch it here.



For convenience, we define the "α-projection" of a geodesic σ: If $\sigma(t) = \exp_x t(\sum_A a_A e_A)$, then the α-projection of σ is the geodesic σ_1, $\sigma_1(t) = \exp_x t(\sum_\alpha a_\alpha e_\alpha)$. (As usual, A runs through 1 to d, while α runs only from 1 to r; cf. beginning of §3 for all notation.)

Now, given a geodesic segment σ, let σ_1 be its α-projection. We wish to show parallel translation of $\Omega(\alpha, \beta)$ ($\equiv \Omega_{\alpha\beta}(E^\alpha, E^\beta)$) to be the same along σ and σ_1. Let B be a sphere of radius twice the length of σ, and let $\delta > 0$ be chosen so that $\exp_x$ is a diffeomorphism on every ball of radius δ, for all $x \in B$. Divide σ into two pieces of equal length ("two" for simplicity) so that each piece is of length $< \delta/2 < \delta$. Call these $(m_{11}\, m_{22})$, $(m_{22}\, m_{33})$ as indicated in the diagram. Divide σ_1 into two equal pieces also; call these $(m_{11}\, m_{12})$, $(m_{12}\, m_{13})$. Now because of local product structure, (cf. (3.7)), m_{22} is joinable to m_{12} by a unique geodesic $(m_{12}\, m_{22})$, and furthermore, parallel translation of $\Omega(\alpha, \beta)$ along $(m_{11}\, m_{22})$ is the same as along $(m_{11}\, m_{12})$, and is the same as along $(m_{11}\, m_{12})$ followed by $(m_{12}\, m_{22})$. Take the α-projection of $(m_{22}\, m_{33})$; call it $(m_{22}\, m_{23})$. Again because of local product structure, m_{23} is joinable to m_{13} by a unique geodesic $(m_{13}\, m_{23})$, and parallel translation of $\Omega(\alpha, \beta)$ from m_{12} to m_{23} along $(m_{12}\, m_{22})$ through $(m_{22}\, m_{23})$ is the same as along $(m_{12}\, m_{13})$ through $(m_{13}\, m_{23})$. Combined with the above, parallel translation of $\Omega(\alpha, \beta)$ from m_{11} to m_{23} is the same along $(m_{11}\, m_{22})$ and $(m_{22}\, m_{23})$ as along σ_1. Again, m_{33} is joinable to m_{23} by a unique geodesic $(m_{23}\, m_{33})$, and parallel translation of $\Omega(\alpha, \beta)$ along $(m_{22}\, m_{33})$ is the same as along $(m_{22}\, m_{23})$ and $(m_{23}\, m_{33})$. Thus together with the last statement, we have parallel translation of $\Omega(\alpha, \beta)$ along σ is the same as along σ_1, and the same as along σ_1, then up through $(m_{13}\, m_{23})$ and $(m_{23}\, m_{33})$.

It will be observed that $(m_{13}\, m_{23})$, $(m_{23}\, m_{33})$ together form an unbroken geodesic.

Note that in the above, we had to use the notion of the global riemann distance on M to make sure we were legally operating entirely inside the compact set B (and thus we had a uniform lower bound on the size of the local product neighborhood via the choice of δ). We do not know how to do this, or to say something similar to this, when the metric is indefinite.

(5) In conclusion, we list here a number of results the proofs of which either are immediate from our main theorem, or can be modelled closely after the corresponding facts in the positive-definite case. Hence, we omit the details. (We are indebted to the referee who pointed out to us most of these results.)

We will have to assume throughout, except for (B) below, that M satisfies the following condition:

(ES) *If M_m^0 is the maximal subspace of M_m on which the holonomy group of M acts trivially, then the metric, $\langle\ ,\ \rangle_m$ on M_m^0 is nondegenerate.*

We have not been able to settle just how restrictive is this condition (ES).

We hope to be able to return to this question in a later publication. We have to make one more definition.

DEFINITION. A subspace M_m^1 of M_m is an irreducible subspace if and only if (i) $\langle\ ,\ \rangle_m$ on M_m^1 is nondegenerate; (ii) the holonomy group Φ preserves M_m^1 and preserves no proper subspace of M_m^1 on which $\langle\ ,\ \rangle_m$ is nondegenerate. If $M_m = M_m^1$, we say Φ *acts irreducibly on* M_m and that M is an *irreducible manifold.*

(A) With notation and hypothesis as in the de Rham theorem (end of §2), M_m admits a decomposition into mutually orthogonal subspaces which is unique up to order: $M_m = M_m^0 \oplus M_m^1 \oplus \cdots \oplus M_m^p$, such that M_m^0 is the maximal subspace on which Φ acts trivially and M_m^i, $1 \leqq i \leqq p$, are irreducible. Then M is isometric to a direct product $M_0 \times M_1 \times \cdots \times M_p$, where each M_i, $0 \leqq i \leqq p$, is the maximal integral manifold for the distribution obtained by parallel-translating M_m^i over M. Moreover, M_0 is flat, and each M_i, $1 \leqq i \leqq p$, is irreducible. Finally Φ is the direct product $\Phi_1 \times \cdots \times \Phi_p$, where each Φ_i is the holonomy group of M_i and Φ_i acts trivially on M_m^j for $j \neq i$.

(B) If M is symmetric, then each factor M_i is symmetric. For, each factor being totally geodesic, parallel translation of curvature is still constant within each factor.

(C) If M is real analytic, each M_i is real analytic, and the isometry between M and $M_0 \times M_1 \times \cdots \times M_p$ is also analytic. Cf. Hicks [3, Theorem 4].

(D) If M is indefinite-metric kahlerian, then so is each factor. Cf. J. Hano and Y. Matsushima, Nagoya Math. J., vol. 11 (1957), pp. 77–92, Theorem 1.

(E) If M is homogeneous, then each factor is homogeneous. Cf. K. Nomizu, Nagoya Math. J., vol. 9 (1955), pp. 43–56, Theorem 3.

(F) Suppose M is such that each Φ_i, $1 \leqq i \leqq p$, leaves invariant no proper subspace whatsoever; then if Φ lies inside the isotropy group at each point, M is symmetric. Cf. K. Nomizu, Nagoya Math. J., vol. 9 (1955), pp. 57–66 and vol. 11 (1957), pp. 111–114. Again we leave open the question of whether this is true without the above assumption on the Φ_i.

Note. After the completion of this paper, M. Berger called to our attention the paper by Kashiwabara [7] which also contains an extension of the de Rham Theorem. Using homotopy methods, he proved the following result: *If* (5.1) has an affirmative answer, and M is connected and simply connected, then M is globally CP-diffeomorphic to $M^1 \times M^2$.

Added November 13, 1963. Materials in §5, (1), (3), and (5) are treated more fully in a forthcoming paper, *Some remarks on holonomy.* In particular, the assumption (ES) of (5) cannot be dropped, and a more convincing counterexample to (5.1) will also be given.

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