A NOTE ON THE BOREL-CANTELLI LEMMA

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It is often desirable to find the probability that for a given sequence $E_1, E_2, \cdots$ of events an infinite number of E_n occur. If the E_n are independent events, then by the Borel-Cantelli lemma this probability is 0 or 1 according as $\sum \mathbf{P}(E_n)$ converges or diverges.

This and similar situations are perhaps more easily understood when translated into the language of random variables. Let X_n be the random variable denoting the number of $E_1, \cdots, E_n$ which occur. Then

$$\mathbf{E}X_n = \sum_{1 \leqq i \leqq n} \mathbf{P}(E_i) \quad \text{and} \quad \mathbf{E}X_n^2 = \sum_{1 \leqq i,j \leqq n} \mathbf{P}(E_i \cap E_j).$$

An infinite number of E_n occur if and only if $\lim X_n = \infty$.

It is possible to sharpen the Borel-Cantelli lemma by considering the random variables $X_n/\mathbf{E}X_n$. For if the events E_n are independent and $\sum \mathbf{P}(E_n)$ diverges, then it follows from a strong law of large numbers (cf. Loève [5, p. 238]) that $\lim X_n/\mathbf{E}X_n = 1$ with probability 1.

In somewhat greater generality, let $X_1, X_n, \cdots$ be an arbitrary sequence of random variables. If $\limsup (\mathbf{E}X_n)^2/\mathbf{E}X_n^2 = 1$, then a subsequence of $X_n/\mathbf{E}X_n$ converges to 1 with probability 1, and hence

$$\liminf \frac{X_n}{\mathbf{E}X_n} \leqq 1 \leqq \limsup \frac{X_n}{\mathbf{E}X_n}$$

with probability 1. If in addition $\lim \mathbf{E}X_n = \infty$, then clearly

$$\limsup X_n = +\infty$$

with probability 1.

The following theorem and corollaries are generalizations of these results. A somewhat different generalization of the Borel-Cantelli lemma has been given by Chung and Erdös [1, p. 180].

THEOREM. *Let $X_1, X_2, \cdots$ be a sequence of random variables, each of which has nonzero mean and positive finite second moment. Suppose in addition that* $\limsup (\mathbf{E}X_n)^2/\mathbf{E}X_n^2 > 0$. *Then*

(i) $\mathbf{P}\{\liminf X_n/\mathbf{E}X_n \leqq 1\} > 0$,

(ii) $\mathbf{P}\{\limsup X_n/\mathbf{E}X_n \geqq 1\} > 0$, *and*

(iii) $\mathbf{P}\{\limsup X_n/\mathbf{E}X_n > 0\} \geqq \limsup (\mathbf{E}X_n)^2/\mathbf{E}X_n^2$.

COROLLARY 1. *Suppose that the hypotheses of the above theorem hold and that* $\liminf X_n/\mathbf{E}X_n$ *and* $\limsup X_n/\mathbf{E}X_n$ *are constants with probability* 1. *Then*

(iv) $\mathbf{P}(\liminf X_n/\mathbf{E}X_n \leqq 1\} = 1$, *and*

(v) $\mathbf{P}\{\limsup X_n/\mathbf{E}X_n \geqq 1\} = 1$.

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Proof. Set $Y_n = X_n/\mathbf{E}X_n$. Then $\mathbf{E}Y_n = 1$ and $\liminf \mathbf{E}Y_n^2 = M < \infty$. Let c be a positive number, and let I_n be the characteristic function of the set where $Y_n < c$. Then $1 = \mathbf{E}Y_n \leqq \mathbf{E}I_n Y_n + \mathbf{E}Y_n^2/c$. Thus

$$1 - M/c \leqq \limsup \mathbf{E}I_n Y_n \leqq \mathbf{E}(\limsup I_n Y_n) \leqq \mathbf{E}(\limsup Y_n),$$

the second inequality following from Fatou's lemma. But c can be made arbitrarily large, so that $1 \leqq \mathbf{E}(\limsup Y_n)$, and consequently

$$0 < \mathbf{P}\{\limsup Y_n \geqq 1\} = \mathbf{P}\{\limsup X_n/\mathbf{E}X_n \geqq 1\}.$$

This proves (i), and (ii) may be proved in a similar manner.

We now prove (iii). Clearly $\mathbf{E}I_n Y_n \leqq c$, and hence $\mathbf{E}Y_n(1 - I_n) \geqq 1 - c$. Assume that $0 < c < 1$. Then by Schwarz's inequality

$$\begin{aligned}(1 - c)^2 &\leqq (\mathbf{E}Y_n(1 - I_n))^2 \\ &\leqq \mathbf{E}Y_n^2\, \mathbf{E}(1 - I_n)^2 = \mathbf{E}Y_n^2\, \mathbf{E}(1 - I_n) = \mathbf{E}Y_n^2\, \mathbf{P}\{Y_n \geqq c\}.\end{aligned}$$

Consequently

$$\begin{aligned}\mathbf{P}\{\limsup X_n/\mathbf{E}X_n \geqq c\} &= \mathbf{P}\{\limsup Y_n \geqq c\} \geqq \limsup \mathbf{P}\{Y_n \geqq c\} \\ &\geqq (1 - c)^2 \limsup 1/\mathbf{E}Y_n^2 = (1 - c)^2 \limsup (\mathbf{E}X_n)^2/\mathbf{E}Y_n^2.\end{aligned}$$

But c can be made arbitrarily small, so that (iii) holds.

Corollary 1 is an immediate consequence of (i) and (ii) of the theorem.

The above theorem and corollary can be translated back into statements concerning events. In the context of the second paragraph of this note, the inequality $\limsup (\mathbf{E}X_n)^2/\mathbf{E}X_n^2 > 0$ becomes

$$\limsup \frac{(\sum_{1<k<n} \mathbf{P}(E_k))^2}{\sum_{1 \leqq i,j \leqq n} \mathbf{P}(E_i \cap E_j)} > 0. \tag{1}$$

This inequality is certainly true if, for instance, $\mathbf{P}(E_i \cap E_j) \leqq c\mathbf{P}(E_i)\mathbf{P}(E_j)$ for some finite constant c and all i and j.[1]

A sequence $E_1, E_2, \cdots$ of events is called a system of recurrent events if there exist independent and identically distributed positive-integer valued random variables $Y_1, Y_2, \cdots$ such that E_k is the event that $Y_1 + \cdots + Y_j = k$ for some j. For such a system it is clear that

$$\mathbf{P}(E_i \cap E_j) = \mathbf{P}(E_i)\mathbf{P}(E_{j-i}) \qquad \text{for } 1 \leqq i < j,$$

and that, with probability 1, an infinite number of E_k occur.

COROLLARY 2. *Let $E_1, E_2, \cdots$ be a system of recurrent events. Let $m_1, m_2, \cdots$ be a strictly increasing sequence of positive integers, and let $N(n)$ denote the number of $E_{m_1}, \cdots E_{m_n}$ which occur. If $\sum \mathbf{P}(E_{m_n}) = \infty$ and*

[1] The above theorem was suggested by this special case, which was pointed out to us by R. V. Chacon.

$$\text{(2)} \qquad \limsup \frac{\left(\sum_{1 \leqq k \leqq n} \mathbf{P}(E_{m_k})\right)^2}{\sum_{1 \leqq i < j \leqq n} \mathbf{P}(E_{m_i})\mathbf{P}(E_{m_j - m_i})} > 0,$$

then, with probability 1,

$$\text{(3)} \qquad \liminf \frac{N(n)}{\sum_{1 \leqq k \leqq n} \mathbf{P}(E_{m_k})} \leqq 1 \leqq \limsup \frac{N(n)}{\sum_{1 \leqq k \leqq n} \mathbf{P}(E_{m_k})}.$$

Proof. We can easily verify that the hypotheses of Corollary 1 are satisfied by the random variables $X_n = N(n)$. To do this we need only apply the Hewitt-Savage zero-one law (cf. [3, pp. 493–494]) to the random variables Y_n.

A sufficient condition for (2) is that, for some constants $\alpha \geqq 1$ and $\beta > 0$,

$$\text{(4)} \qquad \mathbf{P}(E_{m_j - m_i}) \leqq \beta \mathbf{P}(E_{m_{[(j-i)/\alpha]}}) \qquad \text{for } 1 \leqq i \leqq j - \alpha.$$

If we suppose additionally that $\mathbf{P}(E_k)$ is asymptotically equal to a non-increasing function of k, then a sufficient condition for (2) is that there exist a positive constant α such that

$$\text{(5)} \qquad m_j - m_i \geqq m_{[(j-i)/\alpha]} \qquad \text{for } 1 \leqq i \leqq j - \alpha.$$

It is interesting to note that (5) holds if m_n is the n^{th} prime integer. To verify this, we observe that (5) is implied by

$$\text{(6)} \qquad \pi(x + y) - \pi(y) \leqq \alpha \pi(x), \qquad x \geqq 2,$$

where $\pi(x)$, as usual, denotes the number of primes not bigger than x. Now, by a weak form of the prime number theorem, there exists a positive constant c_1 such that

$$\pi(x) \geqq c_1 x / \log x, \qquad x \geqq 2.$$

On the other hand, Selberg's sieve estimate [7, p. 290] implies that for some positive constant c_2,

$$\pi(x + y) - \pi(y) \leqq c_2 x / \log x, \qquad x \geqq 2.$$

These two inequalities yield (6).

Example 1. Let E_k be the event that the simple random walk in one dimension is at the origin at time $2k$. The E_k form a system of recurrent events and $\mathbf{P}(E_k) \sim (\pi k)^{-1/2}$. If m_n is the n^{th} prime number, then $\sum_{n \geqq 1} m_n^{-1/2} = \infty$. Thus, with probability 1, the simple random walk is at the origin at time $2p$ for an infinite number of primes p. The same method shows that this result is also true for the simple random walk in two dimensions, where $\mathbf{P}(E_k) \sim (\pi k)^{-1}$.

Example 2. Let E_k be the event that the simple random walk in three dimensions hits the point $(k, 0, 0)$. Then $\mathbf{P}(E_k) \sim ck^{-1}$ for some positive constant c (cf. Itô and McKean [4]). The events E_k are not recurrent, but do satisfy the inequality

$$\mathbf{P}(E_i \cap E_j) \leqq (\mathbf{P}(E_i) + \mathbf{P}(E_j))\mathbf{P}(E_{j-i}), \qquad 1 \leqq i < j.$$

Let m_n be the n^{th} prime number. Then (2) is valid, so that by the above inequality (1) is valid also. By the Hewitt-Savage zero-one law the hypotheses of Corollary 1 are satisfied, and hence (3) holds. In particular, with probability 1, the random walk visits $(p, 0, 0)$ for an infinite number of primes p. This result was first suggested by Itô and McKean [4] and was verified by Erdös [2] and McKean [6].

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